



## COMMUTATOR OF AUTOMORPHISMS

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### ABSTRACT

In this paper we define the commutator of automorphisms in groups. Let  $G$  be any group and  $\alpha_1, \alpha_2, \dots$  be automorphisms of  $G$ , then the commutator of automorphisms of wight 2, define as follows;

$$[\alpha_1, \alpha_2] = \alpha_1^{-1} \alpha_2^{-1} \alpha_1 \alpha_2.$$

In general, the element

$$[\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n] = [[\alpha_1, \alpha_2, \dots, \alpha_{n-1}], \alpha_n]$$

is the left norm commutator of automorphisms of wight  $n \geq 2$ , which is called a simple commutator of automorphisms of wight  $n$ .

And in particular we prove some properties of them.

**Keywords:** Automorphism, Commutator, Commutator of Automorphisms.

### 1. INTRODUCTION:

Let  $G$  be a group and consider the set  $\text{Aut}(G)$  is automorphisms of the group. In group theory [2] define the concept of commutator elements of  $G$ , as follows;

Let  $x_1, x_2, \dots$  elements of  $G$ , then the element  $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$  is the commutator of  $x_1$  and  $x_2$  of wight 2. In general, the element

$$[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$$

is the left norm commutator of wight  $n \geq 2$ , which is called a simple commutator of wight  $n$ . In this paper we are interested in the structure of commutator of automorphisms of groups. We extend the above concept for automorphisms of  $G$  in the case of 2.

### 2. COMMUTATOR OF AUTOMORPHISMS:

**Theorem (2.1):** Let  $\alpha, \beta, \delta \in \text{Aut}(G)$ , then

- (1)  $\alpha^\beta = \alpha[\alpha, \beta]$
- (2)  $[\alpha, \beta] = [\alpha, \beta]^{-1}$
- (3)  $[\alpha, \beta^{-1}] = [\beta, \alpha]^{\beta^{-1}}$  and  $[\alpha^{-1}, \beta] = [\beta, \alpha]^{\alpha^{-1}}$
- (4)  $[\alpha\beta, \delta] = [\alpha, \delta]^\beta [\beta, \delta]$
- (5)  $[\alpha, \beta\delta] = [\alpha, \delta] [\alpha, \beta]^\delta$
- (6)  $\alpha\beta = \beta\alpha[\alpha, \beta]$
- (7)  $[\alpha, \beta]^\delta = [\alpha^\delta, \beta^\delta]$
- (8)  $[\alpha, \beta^{-1}, \delta]^\beta [\beta, \delta^{-1}, \alpha]^\delta [\delta, \alpha^{-1}, \beta]^\alpha = 1$
- (9)  $[\alpha, \beta, \delta^\alpha] [\beta, \delta, \alpha^\beta] [\delta, \alpha, \beta^\delta] = 1$
- (10)  $[\alpha, \beta\delta] [\delta, \alpha\beta] [\beta, \delta\alpha] = 1$

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**Proof:** The result of 1 and 2 follows by definition.

3. We have,  $[\alpha, \beta^{-1}] = \alpha^{-1} \beta \alpha \beta^{-1}$  (1) and  $[\beta, \alpha]^{\beta^{-1}} = \beta(\beta^{-1} \alpha^{-1} \beta \alpha) \beta^{-1} = \alpha^{-1} \beta \alpha \beta^{-1}$  (2) From (1) and (2) the equality holds.

$$\begin{aligned} 4. \text{ By definition we have, } [\alpha, \delta]^{\beta} [\beta, \delta] &= (\alpha^{-1} \delta^{-1} \alpha \delta)^{\beta} (\beta^{-1} \delta^{-1} \beta \delta) \\ &= \beta^{-1} \alpha^{-1} \delta^{-1} \alpha \delta \beta \beta^{-1} \delta^{-1} \beta \delta = \beta^{-1} \alpha^{-1} \delta^{-1} \alpha \beta \delta \end{aligned} \quad (1)$$

$$\text{and } [\alpha \beta, \delta] = (\alpha \beta)^{-1} \delta^{-1} (\alpha \beta) \delta = \beta^{-1} \alpha^{-1} \delta^{-1} \alpha \beta \delta \quad (2)$$

From (1) and (2) the equality holds.

5. The result follows by previous part.

6. The equality holds by definition.

$$7. \text{ We have, } [\alpha, \beta]^{\delta} = (\alpha^{-1} \beta^{-1} \alpha \beta)^{\delta} = \delta^{-1} \alpha^{-1} \beta^{-1} \alpha \beta \delta \quad (1)$$

$$\begin{aligned} \text{and we have } [\alpha^{\delta}, \beta^{\delta}] &= (\alpha^{\delta})^{-1} (\beta^{\delta})^{-1} \alpha^{\delta} \beta^{\delta} \\ &= (\delta^{-1} \alpha \delta)^{-1} (\delta^{-1} \beta \delta)^{-1} (\delta^{-1} \alpha \delta) (\delta^{-1} \beta \delta) \\ &= \delta^{-1} \alpha^{-1} \delta \delta^{-1} \beta^{-1} \delta \delta^{-1} \alpha \delta \delta^{-1} \beta \delta = \delta^{-1} \alpha^{-1} \beta^{-1} \alpha \beta \delta \end{aligned} \quad (2)$$

So from (1) and (2) the equality holds.

$$\begin{aligned} 9. \text{ We have, } [\alpha, \beta, \delta^{\alpha}] &= [[\alpha, \beta], \delta^{\alpha}] \\ &= [\alpha, \beta]^{-1} (\delta^{\alpha})^{-1} [\alpha, \beta] \delta^{\alpha} \quad \text{by part (2)} \\ &= [\beta, \alpha] (\delta^{\alpha})^{-1} [\alpha, \beta] \delta^{\alpha} = \beta^{-1} \alpha^{-1} \beta \alpha (\alpha^{-1} \delta \alpha)^{-1} (\alpha^{-1} \beta^{-1} \alpha \beta) (\alpha^{-1} \delta \alpha) \\ &= \beta^{-1} \alpha^{-1} \beta \alpha \alpha^{-1} \delta^{-1} \alpha \alpha^{-1} \beta^{-1} \alpha \beta \alpha^{-1} \delta \alpha = \beta^{-1} \alpha^{-1} \beta \delta^{-1} \beta^{-1} \alpha \beta \alpha^{-1} \delta \alpha \end{aligned} \quad (1)$$

$$\begin{aligned} [\beta, \delta, \alpha^{\beta}] &= [[\beta, \delta], \alpha^{\beta}] = [\beta, \delta]^{-1} (\alpha^{\beta})^{-1} [\beta, \delta] \alpha^{\beta} \\ &= [\delta, \beta] (\alpha^{\beta})^{-1} [\beta, \delta] \alpha^{\beta} \quad \text{by part(2)} \\ &= \delta^{-1} \beta^{-1} \delta \beta (\beta^{-1} \alpha \beta)^{-1} (\beta^{-1} \delta^{-1} \beta \delta) (\beta^{-1} \alpha \beta) \\ &= \delta^{-1} \beta^{-1} \delta \beta \beta^{-1} \alpha^{-1} \beta \beta^{-1} \delta^{-1} \beta \delta \beta^{-1} \alpha \beta \\ &= \delta^{-1} \beta^{-1} \delta \alpha^{-1} \delta^{-1} \beta \delta \beta^{-1} \alpha \beta \end{aligned} \quad (2)$$

$$\begin{aligned} [\delta, \alpha, \beta^{\delta}] &= [[\delta, \alpha], \beta^{\delta}] = [\delta, \alpha]^{-1} (\beta^{\delta})^{-1} [\delta, \alpha] \beta^{\delta} \\ &= [\alpha, \delta] (\beta^{\delta})^{-1} [\delta, \alpha] \beta^{\delta} \quad \text{by part (2)} \\ &= (\alpha^{-1} \delta^{-1} \alpha \delta) (\delta^{-1} \beta \delta)^{-1} (\delta^{-1} \alpha^{-1} \delta \alpha) (\delta^{-1} \beta \delta) \\ &= \alpha^{-1} \delta^{-1} \alpha \delta \delta^{-1} \beta^{-1} \delta \delta^{-1} \alpha^{-1} \delta \alpha \delta^{-1} \beta \delta \\ &= \alpha^{-1} \delta^{-1} \alpha \beta^{-1} \alpha^{-1} \delta \alpha \delta^{-1} \beta \delta \end{aligned} \quad (3)$$

Therefore, by (1) and (2) and (3) we have the result.

8. The result follows by previous part.

$$10. \text{ We have, } [\alpha, \beta \delta] = \alpha^{-1} (\beta \delta)^{-1} \alpha \beta \delta = \alpha^{-1} \delta^{-1} \beta^{-1} \alpha \beta \delta \quad (1)$$

$$[\delta, \alpha\beta] = \delta^{-1}(\alpha\beta)^{-1}\delta(\alpha\beta) = \delta^{-1}\beta^{-1}\alpha^{-1}\delta\alpha\beta \quad (2)$$

$$[\beta, \delta\alpha] = \beta^{-1}(\delta\alpha)^{-1}\beta(\delta\alpha) = \beta^{-1}\alpha^{-1}\delta^{-1}\beta\delta\alpha \quad (3)$$

Therefore, by (1) and (2) and (3) the equality holds.

**Remark:** The equality of part (10) holds for  $n$  element of automorphism of  $G$ .

**Theorem (2.2):** Let  $\alpha, \beta, \delta \in \text{Aut}(G)$  such that  $[\alpha, \beta]$  commutes with  $\alpha$  and  $\beta$ . Then

(i)  $[\alpha^i, \beta^j] = [\alpha, \beta]^{ij}$ , for all integer  $i, j$ .

(ii)  $(\alpha\beta)^n = \alpha^n\beta^n[\beta, \alpha]^{\frac{n(n-1)}{2}}$ ,  $n \geq 0$ .

**Proof:** (i) we proceed by induction on  $j$ . For  $j = 2$ , by theorem (2.1), we see in [3], we have

$$[\alpha, \beta^2] = [\alpha, \beta\beta] = [\alpha, \beta][\alpha, \beta]^\beta = [\alpha, \beta][\alpha, \beta][[\alpha, \beta], \beta] = [\alpha, \beta]^2$$

Now assume the result holds for  $j$ , i.e.

$$[\alpha, \beta^j] = [\alpha, \beta]^j.$$

We proved the result for  $j+1$ ,

$$\begin{aligned} [\alpha, \beta^{j+1}] &= [\alpha, \beta^j\beta] = [\alpha, \beta][\alpha, \beta^j]^\beta = [\alpha, \beta][\alpha, \beta^j][[\alpha, \beta^j], \beta] = [\alpha, \beta][\alpha, \beta^j] \\ &= [\alpha, \beta][\alpha, \beta]^j = [\alpha, \beta]^{j+1}. \end{aligned}$$

Now we have

$$[\alpha^j, \beta^j] = [\alpha^j, \beta]^j = ([\beta, \alpha^j]^{-1})^j = ([\beta, \alpha]^{ij})^{-1} = ([\beta, \alpha]^{-1})^{ij} = [\alpha, \beta]^{ij}.$$

(ii) Use induction on  $n$ , the case  $n=1$  being obvious. For  $n=2$ , we have

$$(\alpha\beta)^2 = \alpha\beta\alpha\beta = \alpha\alpha\beta[\beta, \alpha]\beta = \alpha^2\beta[\beta, \alpha]\beta = \alpha^2\beta\beta[\beta, \alpha][[\beta, \alpha], \beta] = \alpha^2\beta^2[\beta, \alpha].$$

Assume the result holds for  $n-1$ , i.e.,

$$(\alpha\beta)^{n-1} = \alpha^{n-1}\beta^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

then we have

$$\begin{aligned} (\alpha\beta)^n &= (\alpha\beta)(\alpha\beta)^{n-1} = \alpha\beta\alpha^{n-1}\beta^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}} \\ &= \alpha\alpha^{n-1}\beta[\beta, \alpha^{n-1}]\beta^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}} \\ &= \alpha^n\beta\beta^{n-1}[\beta, \alpha^{n-1}][[\beta, \alpha^{n-1}], \beta^{n-1}][\beta, \alpha]^{\frac{(n-1)(n-2)}{2}} \\ &= \alpha^n\beta^n[\beta, \alpha^{n-1}][\beta, \alpha]^{\frac{(n-1)(n-2)}{2}} \\ &= \alpha^n\beta^n[\beta, \alpha]^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}} \\ &= \alpha^n\beta^n[\beta, \alpha]^{\frac{n(n-1)}{2}}. \end{aligned}$$

**Theorem (3.2):** Let  $\alpha, \beta, \delta$  be some elements of a group  $\text{Aut}(G)$  such that  $\alpha\beta = \beta\alpha$  and  $[\alpha, \text{Aut}(G)] = \langle [\alpha, \beta] \rangle$ ;  $\beta \in \text{Aut}(G) >$  be an abelian subgroup of  $G$ , then  $[\alpha, \beta, \delta] = [\alpha, \delta, \beta]$ .

**Proof:** We have  $[\alpha, \beta, \delta] = [[\alpha, \beta], \delta] = [\alpha, \beta]^{-1}\delta^{-1}[\alpha, \beta]\delta$

$$= \alpha^{-1}(\alpha\beta^{-1}\alpha^{-1}\beta)(\alpha\delta^{-1}\alpha^{-1}\delta)\delta^{-1}\beta^{-1}\alpha\beta\delta$$

$$= \alpha^{-1}[\alpha^{-1}, \beta][\beta^{-1}, \delta]\delta^{-1}\beta^{-1}\alpha\beta\delta.$$

Clearly  $[\alpha^{-1}, \beta] = ([\alpha, \beta]^{\alpha^{-1}})^{-1} = [\alpha, \beta^{\alpha^{-1}}]^{-1} \in [\alpha, \text{Aut}(G)]$ .

And  $[\alpha^{-1}, \delta] = [\alpha, \delta^{\alpha^{-1}}]^{-1} \in [\alpha, \text{Aut}(G)]$ .

Hence  $[\alpha^{-1}, \beta]$  commutes with  $[\alpha^{-1}, \delta]$  and therefore;

$$[\alpha, \beta, \delta] = \delta^{-1} \alpha^{-1} \delta \alpha \beta^{-1} \alpha^{-1} \beta \delta^{-1} \beta^{-1} \alpha \beta \delta$$

But  $\delta\beta = \beta\delta$ , then

$$[\alpha, \beta, \delta] = [\alpha, \delta]^{-1} \beta^{-1} [\alpha, \delta] \beta = [\alpha, \delta], \beta = [\alpha, \delta, \beta].$$

**Theorem (4.1):** Let  $\alpha, \beta \in \text{Aut}(G)$  such that  $[\alpha, \beta]$  commutes with  $\alpha$  and  $\beta$ , then

$$[\alpha^{-1}, \beta] = [\alpha, \beta]^{-1} = [\alpha, \beta^{-1}].$$

**Proof:** Since  $[\alpha, \beta]$  commutes with  $\alpha$  and  $\beta$ , it implies that

$$[[\alpha, \beta], \alpha] = 1 = [[\alpha, \beta], \beta]$$

similarly,  $[[\alpha, \beta]^{-1}, \alpha^{-1}] = 1 = [[\alpha, \beta]^{-1}, \beta^{-1}]$ .

Thus by theorem (2.1), see in [1], we have

$$[\alpha^{-1}, \beta] = ([\alpha, \beta]^{-1})^{\alpha^{-1}} = [\alpha, \beta]^{-1} [[\alpha, \beta]^{-1}, \alpha^{-1}] = [\alpha, \beta]^{-1}$$

similarly,  $[\alpha, \beta^{-1}] = ([\alpha, \beta]^{-1})^{\beta^{-1}} = [\alpha, \beta]^{-1} [[\alpha, \beta]^{-1}, \beta^{-1}] = [\alpha, \beta]^{-1}$

Therefore,  $[\alpha^{-1}, \beta] = [\alpha, \beta]^{-1} = [\alpha, \beta^{-1}]$ .

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