International Journal of Mathematical Archive-12(2), 2021, 56-62
IJMA Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREM FOR FUZZY MAPPING AND ITS APPLICATIONS

PRASHANT NAMDEO*[1] & SUBHASHISH BISWAS[2]

[1]Research Scholar,
Department of Mathematics, Kalinga University, Raipur (C.G.), India.

[2]Supervisor,
Department of Mathematics, Kalinga University, Raipur (C.G.), India.

(Received On: 21-01-21; Revised & Accepted On: 19-02-21)

ABSTRACT

Zadeh [10] introduction of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. In this paper we study some fixed point theorems for fuzzy mapping in fuzzy metric spaces, we also give some examples in support of our main results and also give an example in support of our main result.

Key words: *Fuzzy metric space, fuzzy mapping, t-norm, quick short algorithm,*

Mathematical Subject Classification: *45H30, 54H25, 46J10, 46J15.*

INTRODUCTION

Zadeh [10] introduction of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. In the last two decades many fixed point theorems for contractions in fuzzy metric spaces and quasi fuzzy metric spaces appeared. The role of topology in logic programming has come to be recognized in recent years. In 1975, Matkowski [7] proved the following distinguished generalization of Banach's contraction principle.

Theorem 1.1: Let (X, d)be a complete fuzzy metric space and $f: X \to X$ a self map such that

$$d(fx, fy) \leq \varphi\big(d(x, y)\big)$$

for all $x, y \in X$, where $\varphi: [0, \infty) \to [0, \infty)$is a nondecreasing function satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point.

Remark 1.1 It is well known and easy to check that if $\varphi: [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, then $\varphi(t) < t$ for all $t > 0$.

Matkowski's theorem has been generalized or extended in several direction. In particular, Jachymski obtained in [5] the following nice fuzzy version of it.

Theorem 1.2: Let $(X, M,*)$be a complete fuzzy metric space, with $*$ a continuous $t-$norm of Hadžić type, and let $f: X \to X$be a self-map such that

$$M(fx, fy, \varphi(t)) \geq M(x, y, t)$$

for all $x, y \in X$and $t > 0$, where $\varphi: [0, \infty) \to [0, \infty)$is a function satisfying $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point.

Recently, Ricarte and Romaguera [8] established the following new fuzzy version of Matowski's theorem by using a type of contraction introduced in the fuzzy intuitionistic context by Huang et al.[4], and that generalizes C-contractions as defined by Hicks in [3].

Corresponding Author: Prashant Namdeo*[1]
[2]Supervisor, Department of Mathematics, Kalinga University, Raipur (C.G.), India.

Theorem 1.3: Let $(X, M, *)$be a complete fuzzy metric space and $f: X \to X$ a self-map such that

$$M(x, y, t) > 1 - t \Rightarrow M(fx, fy, \varphi(t)) > 1 - \varphi(t)$$

for all $x, y \in X$ and $t > 0$, where $\varphi: [0, \infty) \to [0, \infty)$is a nondecreasing function satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point.

In this paper we obtain a generalization of Theorem 1.3 to preordered fuzzy quasi-metric spaces which is applied to deduce, among other results, a procedure to show in a direct and easy way the existence of solution for the recurrence equations that are typically associated to Quicksort and Divide and Conquer algorithms, respectively. The key for this application is the nice fact that, for the specialization order of a fuzzy quasi-metric space, the contraction condition of Theorem 1.3 is automatically satisfied whenever the self-map f is nondecreasing for the specialization order and $\varphi: [0, \infty) \to [0, \infty)$ is any function verifying $\varphi(t) > 0$ for all > 0 .

PRELIMINARIES

In this section we recall several notions and properties which will be useful in the rest of the paper. Our basic reference for quasi-metric spaces is [6] and for fuzzy (quasi-)metric spaces they are [1, 2].

The letters $\mathbb{R}, \mathbb{N}$ and ω will denote the set of real numbers, the set of positive integer numbers and the set of non-negative integer numbers, respectively. A preorder on a nonempty set X is a reflexive and transitive binary relation $\preceq$ on X. The preorder $\preceq$ is called a partial order, or simply an order, if it is antisymmetric (i.e., condition $x \preceq y$ and $y \preceq x$, implies $x = y$).

Note that for any nonempty set X, the binary relation $\preceq^t$ defined by $x \preceq^t y$ if and only if $x, y \in X$, is obviously a preorder on X, the so-called trivial preorder on X.

A quasi-metric on a set X is a function $d: X \times X \to [0, \infty)$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X.
Given a quasi-metric d on X the function d_s defined by

$$d_s(x, y) = \max\{d(x, y), d(y, x)\}$$

for all $x, y \in X$, is a metric on X.

If d is a quasi-metric on X, then the relation $\leq_d$ on X given by

$$x \leq_d y \Leftrightarrow d(x, y) = 0,$$

is an order on X, called the specialization order of d.

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0,1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Two interesting examples are $\wedge$, and $*_L$, where, for all $a, b \in [0,1]$, $a \wedge b = \min\{a, b\}$, and $*_L$ is the well-known Lukasiewicz t-norm defined by $a *_L b = \max\{a + b - 1, 0\}$.

It seems appropriate to point out that $* \leq \wedge$ for any continuous t-norm $*$.

Definition 2.1: A KM-fuzzy quasi-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ (i.e., a function from $X \times X \times [0, \infty)$ into $[0,1]$) such that for all $x, y, z \in X$:

(KM1) $M(x, y, 0) = 0$;
(KM2) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$;
(KM3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
(KM4) $M(x, y, \_): [0, \infty) \to [0,1]$ is left continuous.

A KM-fuzzy quasi-metric $(M, *)$ on X such that for each $x, y \in X$:

$$\text{(KM5) } M(x, y, t) = M(y, x, t) \text{ for all } t > 0$$

is a fuzzy metric on X .
A simple but useful fact is that for each KM-fuzzy quasi-metric $(M, *)$ on a set X and each $x, y \in X$, the function $M(x, y, *)$ is nondecreasing.

In the following, KM-fuzzy quasi-metrics will be simply called fuzzy quasi-metrics.

If $(M, *)$ is a fuzzy quasi-metric on a set X, then the pair $(M^i, *)$ is a fuzzy metric on X where M^i is the fuzzy set in $X \times X \times [0, \infty)$ defined by $M^i(x, y, t) = \min\{M(x, y, t), M(y, x, t)\}$ for all $x, y \in X$ and $t \geq 0$.

As in the fuzzy metric case, each fuzzy quasi-metric $(M,*)$ on a set X induces a topology τ_M on X which has as a base the family of open balls $\{B_M(x,\varepsilon,t): x \in X, 0 < \varepsilon < 1, t > 0\}$, where $B_M(x,\varepsilon,t) := \{y \in X: M(x,y,t) > 1 - \varepsilon\}$.

It immediately follows that a sequence $(x_n)_{n\in\omega}$ in a fuzzy quasi-metric space $(X,M,*)$ converges to a point $x \in X$ with respect to τM if and only if $\lim_{n\to\infty} M(x,x_n,t) = 1$ for all $t > 0$.

Definition 2.2: A fuzzy (quasi-)metric space is a triple $(X,M,*)$ such that X is a set and $(M,*)$ is a fuzzy (quasi-)metric on X.

Definition 2.3: A preordered fuzzy (quasi-)metric space is $4-$tuple $(X,M,\preceq,*)$ such that $(X,M,*)$ is a fuzzy (quasi-)metric space and $\preceq$ is a preorder on X.

The notion of an ordered fuzzy (quasi-)metric space is defined in the obvious manner.

Remark 2.4: If $(M,*)$ is a fuzzy quasi-metric on X, then the relation $\leq_M$ on X given by

$$x \leq My \Leftrightarrow M(x,y,t) = 1$$

for all $t > 0$, is an order on X, called the specialization order of $(M,*)$.

It is interesting to note that, however, the relation $\leq_{M_p}$ given by

$$x \leq_{M_p} y \Leftrightarrow M(x,y,t) = 1$$

for some $t > 0$, is a preorder on X.

We conclude this section with two typical examples of fuzzy quasi-metrics induced by a given quasi-metric space.

Example 2.5: Let (X,d) be a quasi-metric space. Then the pair $(M_{(0,1)},*)$ is a fuzzy quasi-metric on X where $*$ is any continuous $t-$norm and $M_{(0,1)}$ is the fuzzy set on $X \times X \times [0,\infty)$ given by $M_{(0,1)}(x,y,t) = 1$ if $t > 0$ and $d(x,y) < t$, and $M_{(0,1)}(x,y,t) = 0$ otherwise.

Example 2.6 Let (X,d) be a quasi-metric space. Then the pair $(M_d,*)$ is a fuzzy quasi-metric on X where $*$ is any continuous $t-$norm $*$ and M_d is the fuzzy set on $X \times X \times [0,\infty)$ given by $M_d(x,y,0) = 0$ for all $x,y \in X$ and

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}$$

for all $x,y \in X$ and $t > 0$.

Remark 2.7: Let (X,d) be a quasi-metric space. It is clear that the specialization orders $\leq_d$, $\leq_{M_{(0,1)}}$ and $\leq_{M_d}$ coincide, while the preorder $\leq_{(M_{(0,1)})p}$ coincides with the trivial preorder $\preceq^t$ on X, and the preorder $\leq_{(M_{(0,1)})p}$ coincides with the specialization order $\leq_{M_d}$.

The Fixed Point Theorem and Some of Its Consequences

We start this section with the notions of fuzzy quasi-metric completeness and continuity of self-maps which will be used in our main result.

A left K-Cauchy sequence in a fuzzy quasi-metric space $(X,M,*)$ is a sequence $(x_n)_{n\in\omega}$ in X such that for each $t > 0$ and each $\varepsilon \in (0,1)$ there is $n_\varepsilon \in N$ such that $M(x_n,x_m,t) > 1 - \varepsilon$ whenever $m \geq n \geq n_\varepsilon$.

A preordered fuzzy quasi-metric space $(X,M,\preceq,*)$ will be called $\preceq-$complete if for each nondecreasing left K-Cauchy sequence $(x_n)_{n\in\omega}$ there is $x \in X$ such that $(x_n)_{n\in\omega}$ converges to x with respect to τ_{M_i}, and $x_n \preceq x$ for all $n \in \omega$.

Observe that if $(X,M,\preceq^t,*)$ is a preordered fuzzy metric space which is $\preceq^t-$complete, then $(X,M,*)$ is a complete fuzzy metric space in the usual sense .

Let $(X,M,\preceq,*)$ be a preordered fuzzy quasi-metric space. A self-map $f: X \to X$ is said to be $\preceq-$nondecreasing if condition $x \preceq y$ implies $fx \preceq fy$ for all $x,y \in X$, and it will be called $\preceq-$continuous if whenever $(x_n)_{n\in\omega}$ is a nondecreasing sequence for $\preceq$, which converges with respect to τ_{M_i} to some $x \in X$ such that $x_n \preceq x$ for all $n \in \omega$, then the sequence $(fx_n)_{n\in\omega}$ converges to fx with respect to τ_{M_i}.

Theorem 3.1: Let $(X, M, \preceq, *)$be a preordered $\preceq -$complete fuzzy quasi-metric space and $f: X \to X$a $\preceq -$continuous nondecreasing self-map such that

$$M(x, y, t) > 1 - t \Rightarrow M\big(fx, fy, \varphi(t)\big) > 1 - \varphi(t) \tag{3.1}$$

for all $x, y \in X$ with $x \preceq y$, and $t > 0$, where $\varphi: [0, \infty) \to [0, \infty)$ satisfies $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. If there is $x_0 \in X$such that $x_0 \preceq fx_0$, then $\mathrm{Fix}(f) \neq \emptyset$.

If, in addition, condition 3.1 is satisfied for each $x, y \in \mathrm{Fix}(f)$, then f has a unique fixed point.

Proof: Take $t_0 > 1$. Let $x, y \in X$ with $x \preceq y$. Since $M(x, y, t_0) > 1 - t_0$, it follows that

$$M\big(fx, fy, \varphi(t_0)\big) > 1 - \varphi(t_0). \tag{3.2}$$

Since f is nondecreasing, we deduce that $f^n x \preceq f^n y$ for all $n \in \omega$, so, by 3.1and 3.2, we immediately deduce that

$$M\big(f^n x, f^n y, \varphi^n(t_0)\big) > 1 - \varphi^n(t_0) \tag{3.3}$$

for all $x, y \in X$ with $x \preceq y$ and $n \in \omega$.

Now let $x_0 \in X$ such that $x_0 \preceq fx_0$. Put $x_n = f^n x_0$ for all $n \in \omega$. Since f is nondecreasing it follows that $(x_n)_{n\in\omega}$ is a nondecreasing sequence for $\preceq$. Now we show that $(x_n)_{n\in\omega}$ is a left $K-$Cauchy sequence in $(X, M, *)$. Choose $\varepsilon \in (0,1)$ and $t > 0$. Then there exists $n_\varepsilon \in N$ such that $\varphi^n(t_0) < \min\{\varepsilon, t\}$ for all $n \geq n_\varepsilon$. Let $m > n \geq n_\varepsilon$. Then $m = n + k$ for some $k \in N$, so, by 3.3

$$\begin{aligned} M(x_n, x_m, t) &= M\big(f^n x0, f^n f^k x_0, t\big) \\ &\geq M\left(f^n x_0, f^n f^k x_0, \varphi^n(t_0)\right) 1 - \varphi^n(t_0) \\ &> 1 - \varepsilon. \end{aligned}$$

Therefore $(x_n)_{n\in\omega}$ is a nondecreasing left K-Cauchy sequence in $(X, M, *)$. Since $(X, M, \preceq, *)$ is $\preceq -$complete there exists $z \in X$ such that $\lim_{n\to\infty} M_i(x_n, z, t) = 1$ for all $t > 0$, and $x_n \preceq z$ for all $n \in \omega$. So $\lim_{n\to\infty} M_i(x_n, fz, t) = 1$ for all $t > 0$, by $\preceq -$continuity of f. Hence $z = fz$.

Finally suppose that $u = fu$ for some $u \in X$. From our hypothesis it follows, exactly as in the first part of the proof, that

$$M\big(f^n z, f^n u, \varphi^n(t_0)\big) > 1 - \varphi^n(t_0)$$

for all $n \in \omega$. Since $f^n z = z$ and $f^n u = u$ we deduce, for $t > 0, \varepsilon \in (0, t)$ and $n \in N$ with $\varphi^n(t_0) < \varepsilon$, that

$$\begin{aligned} M(z, u, t) \geq M(z, u, \varepsilon) &\geq M\big(z, u, \varphi^n(t_0)\big) \\ &> 1 - \varphi^n(t_0) > 1 - \varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude that $M(z, u, t) = 1$ for all $t > 0$. Similarly $M(u, z, t) = 1$ for all $t > 0$, so $u = z$. This completes the proof.

Corollary 3.2: Let $(X, M, \preceq, *)$be a preordered $\preceq -$complete fuzzy quasi-metric space and $f: X \to X$ a $\preceq -$nondecreasing self-map such that

$$M(x, y, t) > 1 - t \Rightarrow M\big(fx, fy, \varphi(t)\big) > 1 - \varphi(t) \tag{3.4}$$

for all $x, y \in X$with $x \preceq y$, and $t > 0$, where $\varphi: [0, \infty) \to [0, \infty)$satisfies $0 < \varphi(t) < t$and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. If there is $x_0 \in X$such that $x_0 \preceq fx_0$, then $\mathrm{Fix}(f) \neq \emptyset$.

If, in addition, condition 3.4 is satisfied for each $x, y \in \mathrm{Fix}(f)$, then f has a unique fixed point.

Proof: We shall show that f is $\preceq -$continuous in $(X, M, \preceq, *)$. Let $(x_n)_{n\in\omega}$ be a nondecreasing sequence for $\preceq$, convergent with respect to τ_{M^i} to some $x \in X$ such that $x_n \preceq x$ for all $n \in \omega$. Given $\varepsilon \in (0,1)$ there is $n_\varepsilon \in N$ such that $M_i(x, x_n, \varepsilon) > 1 - \varepsilon$. By condition 3.4 and the fact that $0 < \varphi(\varepsilon) < \varepsilon$ it follows that $M^i(fx, fx_n, \varepsilon) > 1 - \varepsilon$ for all $n \geq n_\varepsilon$. Therefore f is $\preceq -$continuous. The conclusions follow from Theorem 3.1.

As an immediate consequence of Corollary 3.2 we obtain the following result.

Corollary 3.3: Let $(X, M, *)$be a complete fuzzy metric space and $f : X \to X_a$ self-map such that

$$M(x, y, t) > 1 - t \Rightarrow M(fx, fy, \varphi(t)) > 1 - \varphi(t)$$

for all $x, y \in X$and $t > 0$, where $\varphi: [0, \infty) \to [0, \infty)$satisfies $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point.

Proof: It is clear that $(X, M, \preceq^t, *)$ is a preordered $\preceq^t-$complete fuzzy metric space. Now the result follows from Corollary 3.2.

The contraction 3.4 is almost trivially satisfied in the case that the preorder $\preceq$ is the specialization order $\leq_M$ and f is nondecreasing for $\leq_M$. This situation, which will be crucial in our application in Section 2.4, is described in the following result.

Theorem 3.4: If the ordered fuzzy quasi-metric space $(X, M, \leq_M, *)$ is $\leq_M-$ complete and $f: X \to X$is a $\leq_M-$nondecreasing self-map such that there is $x_0 \in X$satisfying $x_0 \leq_M fx_0$, thenfhas a fixed point.

Proof: Let $\varphi: [0, \infty) \to [0, \infty)$ be any function satisfying $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then, for each $x, y \in X$ such that $x \leq_M y$, we have $fx \leq_M fy$, and thus

$$M\big(fx, fy, \varphi(t)\big) = 1 > 1 - \varphi(t)$$

for all $t > 0$. This shows that condition 3.4 is satisfied, and thus fhas a fixed point by Corollary 3.2.

We conclude this section with two examples illustrating the obtained results.

Example 3.5: Let $X = [0, \infty)$ and let M be the fuzzy set in $X \times X \times [0, \infty)$ defined as $M(x, y, 0) = 0$ for all $x, y \in X$, $M(x, y, t) = 1$ if $x \leq y$ and $t > 0$ and $M(x, y, t) = \frac{t}{t+1}$ otherwise.
It is routine to check that $(M, \wedge)$ is a fuzzy quasi-metric. Note that the specialization order $\leq M$ coincides with the usual order $\leq$ on X. Moreover, a sequence in X is left $K-$Cauchy in $(X, M, \wedge)$ if and only it is eventually constant, so $(X, M, \leq, \wedge)$ is an ordered $\leq -$complete fuzzy quasi-metric space.

Now let $\varphi: [0, \infty) \to [0, \infty)$ defined as

$$\varphi(t) = \frac{t}{2} \text{ if } 0 \leq t \leq \frac{1}{2}, \varphi(t) = 1 \text{if } t > \frac{1}{2} \text{ with } t \neq 1 \text{and } \varphi(1) = \frac{1}{2}.$$

Clearly $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. We show that if $f: X \to X$ is any $\leq -$nondecreasing self-map, the contraction condition 3.1 is satisfied for $x, y \in X$ with $x < y$ and $t > 0$. (Note that f is automatically continuous and hence $\leq$ -continuous, because τ_{M^i} is the discrete topology on X.)

Let $x \leq y$ and $t > 0$. Then $fx \leq fy$ and $\varphi(t) > 0$, so

$$M\big(fx, fy, \varphi(t)\big) = 1 > 1 - \varphi(t),$$

so condition 3.1 is trivially satisfied. If, in addition, there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point. Observe that in this example we cannot apply Corollary 3.2 because $\varphi(t) > t$ for $\frac{1}{2} < t < 1$; in fact, it is not nondecreasing.

Example 3.6: Let $X = \{a, b, c\}$ and let $(M, \wedge)$ be the fuzzy metric on X defined as $M(x, x, t) = 1$ for all $x \in X$, $M(a, b, t) = M(b, a, t) = \frac{1}{2}$ for all $t \in (0,1]$, $M(a, b, t) = M(b, a, t) = 1$ for all $t > 1$, and $M(x, y, t) = 0$ otherwise. Now define a preorder $\preceq$ on X as follows: $x \preceq x$ for all $x \in X$, $a \preceq b$, and $b \preceq a$. Obviously $\preceq$ is not an order on X. Clearly $(X, M, \preceq, *)$ is $\preceq -$complete. Let $f: X \to X$ be such that $fa = a$, $fb = a$ and $fc = c$. Then f is a $\preceq -$nondecreasing self-map with $a \preceq fa$ (also $c \preceq fc$). Let $\varphi: [0, \infty) \to [0, \infty)$ be any function such that $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Since $M\big(fa, fb, \varphi(t)\big) = M\big(a, a, \varphi(t)\big) = 1$ for all $t > 0$, we deduce that the conditions of Corollary 3.2, for $x, y \in X$ with $x \preceq y$, are satisfied. However, we cannot apply Corollary 3.3 to this example. Indeed, suppose that there exists a function $\varphi: [0, \infty) \to [0, \infty)$ satisfying $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and such that

$$M(a, c, t) > 1 - t \Rightarrow M\big(fa, fc, \varphi(t)\big) > 1 - \varphi(t)$$

for all $t > 0$. Since condition $M(a, c, t) > 1 - t$ holds for $t > 1$, we deduce that, for $t > 1$, $M\big(fa, fc, \varphi(t)\big) > 1 - \varphi(t)$, so $\varphi(t) > 1$ because $M\big(fa, fc, \varphi(t)\big) = M\big(a, c, \varphi(t)\big) = 0$. Repeating this argument, we deduce that $\varphi^n(t) > 1$ for all $n \in \omega$, which contradicts the fact that $\lim_{n\to\infty} \varphi^n(t) = 0$.

AN APPLICATION

In this section we obtain a general procedure from which we can deduce in a fast and easy fashion the existence the solution for the recurrence equations that are typically associated to Quicksort and Divide and Conquer algorithms, respectively.

Let us recall that Schellekens introduced in [9] the so-called complexity quasi-metric space in order to construct a topological foundation for the complexity analysis of programs and algorithms. In that paper, he also applied his theory to show that the existence and uniqueness of solution for the recurrence associated to Divide and Conquer algorithms. Further contributions to the study of these spaces.

The complexity (quasi-metric) space (see [9]) consists of the pair (C, d_C), where

$$C = \left\{ f: N \to (0, \infty]: \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and d_C is the quasi-metric on given by

$$d_C(f, g) = \sum_{n=1}^{\infty} 2^{-n} \left(\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right)$$

for all $f, g \in C$. (We adopt the convention that $\frac{1}{\infty} = 0$.) The elements of C are called complexity functions and d_C is said to be the complexity quasi-metric. Observe that

$$d_C(f,g) = 0 \Leftrightarrow f(n) \leq g(n) \text{ for all } n \in N,$$

and thus condition $d_C(f,g) = 0$, can be computationally interpreted as f to be 'more efficient' than g on all inputs.

In our context we shall work on the subset C_1 of C defined as

$$C_1 = \{f \in C: f(n) \geq 1 \text{ for all } n \in N\},$$

and we shall use the function Q_C introduced in [26] and defined, for each $f, g \in C$ and $t > 0$, as

$$Q_C(f,g,t) = \sum_{k=n}^{\infty} 2^{-k}\left(\left(\frac{1}{g(k)} - \frac{1}{f(k)}\right) \vee 0\right),$$

where $t \in (n-1, n], n \in N$.

The following well-known facts will be useful.

Remark 4.1: $d_C(f,g) = Q_C(f,g,t)$ for all $f, g \in C$ and $t \in (0,1]$.

Remark 4.2 : $Q_C(f,g,t) < 1$ whenever $f, g \in C_1$ and $f \neq g$.

Remark 4.3: Let $(f_n)_{k\in\omega}$ be a sequence in such that $f_n \leq f_{n+1}$ for all $n \in \omega$, and let $F \in C$ defined as $F(k) = \sup\{f_n(k): n \in \omega\}$ for all $k \in N$. Then $\lim_{n\to\infty}(d_C)^s(F, f_n) = 0$.

In the following, for any $f, g \in C$, by $f \leq g$ we mean that $f(n) \leq g(n)$ for all $n \in N$.

Now we construct a fuzzy set M_1 in $C_1 \times C_1 \times [0,\infty)$ as $M_1(f,g,0) = 0$ for all $f, g \in C_1$ and

$$M_1(f,g,t) = 1 - Q_C(f,g,t)$$

for all $f, g \in C_1$ and $t > 0$.

Note that $M_1(f,g,t) = 1$ for all $t > 0$ if and only if $f \leq g$.
Then we have:

Lemma 4.4: $(C_1, M_1, \leq_{M_1}, *_L)$ is a $\leq_{M_1}-$complete fuzzy quasi-metric space.

Proof: We first see that $(M_1, *_L)$ is a fuzzy quasi-metric on C_1 (recall that $a *_L b = \max\{a+b-1, 0\}$ for all $a, b \in [0,1]$). Indeed, conditions (KM1), (KM2) and (KM4) of Definition 1.1 are almost trivially satisfied, while condition (KM3) follows immediately from the fact shown that

$$Q_C(f,g,t+s) \leq Q_C(f,h,t) + Q_C(h,g,s)$$

for all $f, g \in C$ and $t, s > 0$. Hence $(C_1, M_1, *_L)$ is a fuzzy quasi-metric space.

Now let $(f_n)_{n\in\omega}$ be a nondecreasing left $K-$Cauchy sequence in $(C_1, M_1, \leq_{M_1}, *_L)$ Then $f_n \leq f_{n+1}$ for all $n \in \omega$, so, by Remark 7,

$$\lim_{n\to\infty} d_C(F, f_n) = d_C(f_n, F) = 0. \tag{4.1}$$

Since $f_n \leq F$ for all $n \in \omega$ (note that $F \in C_1$), we deduce that

$$M_1(f_n, F, t) = 1 \tag{4.2}$$

for all $n \in \omega$ and $t > 0$. On the other hand, and assuming that $F \neq fn$ for all $n \in \omega$, we deduce from Remark 4.1 and 4. 2 that

$$\lim_{n\to\infty} M_1(F, f_n, t) = \lim_{n\to\infty}\big(1 - Q_C(F, f_n, t)\big) = \lim_{n\to\infty}\big(1 - d_C(F, f_n, t)\big) = 1$$

4. 3

for all $t \in (0,1]$, and thus for all $t > 0$. By 4.2 and 4.3 we deduce that $(f_n)_{n\in\omega}$ converges to F with respect to τ_{M^i}

We have shown that $(C_1, M_1, \leq_{M_1}, *_L)$ is a $\leq_{M_1}-$complete fuzzy quasi-metric space.

Taking into account the preceding constructions and results, we immediately obtain the following consequence.

Theorem 4.5: If $\Phi: C_1 \to C_1$ is a $\leq_{M_1}-$nondecreasing map and there is $f_0 \in C_1$ such that $f_0 \leq_M \Phi f_0$, then Φ has a fixed point.

We finish the paper by applying Theorem 4.5 to show the existence of solution for the recurrence equations associated to Quicksort and Divide and Conquer algorithm, respectively.

Example 4.6: Consider the recurrence equation T given by $T(1) = 0$, and

$$T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1)$$

For $n > 1$.

In this case, we define a functional $\Phi: C_1 \to C_1$ as

$$\Phi f(1) = \infty, \Phi f(2) = 1, \Phi f(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} f(n-1)$$

for all $n > 2$, where $f \in C_1$.
It is clear that if $f \leq g$ then $\Phi_f \leq \Phi_g$, so by the definition of M_1, Φ is nondecreasing for $\leq_{M_1}$. Moreover, the complexity function $f_0 \in C_1$ defined as $f_0(n) = 1$ for all $n \in N$, satisfies $f_0 \leq \Phi f_0$, i.e., $f_0 \leq M_1 \Phi f_0$. Therefore, we can apply Theorem 4.5 and thus Φ has a fixed point $g_0 \in C_1$. Consequently, the function $h: N \to [0, \infty)$ given by $h(1) = 0$ and $h(n) = g_0(n)$ for all $n > 1$ is solution of the recurrence equation T.

Example 4.7: It is well known that Divide and Conquer algorithms solve a problem by recursively splitting it into sub problems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original proble. Thus, the complexity of a Divide and Conquer algorithm typically is the solution of the recurrence equation given by

$$T(1) = c \text{ and } T(n) = aT\left(\frac{n}{b}\right) + h(n),$$

where $a, b, c \in N$ with $a, b \geq 2$, n ranges over the set $\{b^p : p = 0,1,2, \dots\}$ and $h(n) < \infty$ if $n \in \{b^p : p = 0,1,2, \dots\}$.

This recurrence equation induces, in a natural way the associated functional $\Phi: C_1 \to C_1$ defined by

$$\Phi f(1) = c, \Phi f(n) = af\left(\frac{n}{b}\right) + h(n), \text{if } n \in \{b^p : p = 0,1,2, \dots\} \text{ and } \Phi f(n) = \infty \text{ if } n \notin \{b^p : p = 0,1,2, \dots\}.$$

As in Example 4.5, Φ is $\leq_{M_1}-$nondecreasing. Since the complexity function $f_0 \in C_1$ defined as $f_0(n) = 1$ for all $n \in N$, satisfies $f_0 \leq M_1 \Phi f_0$, we can apply Theorem 4.5 and thus Φ has a fixed point $g_0 \in C_1$ which is solution of the recurrence equation T.

REFERENCES

1. Cho, YJ, Grabiec, M, Radu, V., On Nonsymmetric Topological and Probabilistic Structures, Nova Science Publishers, New York (2006).
2. Gregori, V, Romaguera, S., Fuzzy quasi-metric spaces. Appl. Gen. Topol.. 5, 129–136 (2004).
3. Hicks, TL: Fixed point theory in probabilistic metric spaces. Zb. Rad. Prir.-Mat. Fak. (Novi Sad). 13, 63–72 (1983)
4. Huang, X, Zhu, C, Wen, X., On (g,φ)-contraction in intuitionistic fuzzy metric spaces. Math. Commun.. 15, 425–435 (2010).
5. Jachymski, J., On probabilistic ψ −contractions on Menger spaces. Nonlinear Anal. TMA. 73, 2199–2203 (2010).
6. Künzi, HPA: Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology. In: Aull CE, Lowen R (eds.) Handbook of the History of General Topology, pp. 853–968. Kluwer Academic, Dordrecht (2001).
7. Matkowski, J., Integrable solutions of functional equations. Diss. Math.. 127, 1-68 (1975).
8. Ricarte, LA, Romaguera, S., On ψ-contractions in fuzzy metric spaces with application to the intuitionistic setting. Iran. J. Fuzzy Syst.. 10(6), 63–72 (2013)
9. Schellekens, M., The Smyth completion: a common foundation for denotational semantics and complexity analysis. Proc. MFPS 11. 535–556 (1995)
10. Zadeh, L.A., Fuzzy sets, Inform. and Control, 8 (1965), 338-353.

Source of support: Nil, Conflict of interest: None Declared.