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PERISTALTIC FLOW OF A THIRD ORDER FLUID THROUGH A POROUS MEDIUM IN A SYMMETRIC CHANNEL

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ABSTRACT

In this paper we studied peristaltic motion of a third order fluid through a porous medium in a symmetric channel under the assumptions of long wavelength and low Reynolds number. The expressions for the velocity pressure rise and friction force over one wavelength are obtained by a regular perturbation technique. The effects of various emerging parameters on the pressure gradient, pressure rise and frictional force over one wavelength are discussed. It is observed that, the pressure gradient increases as the Darcy number or Deborah number increases. It is found that, as the Deborah number increases the pumping and magnitude of the friction force both increase. Both the pumping and friction force decrease with an increase in the Darcy number. Further, the pumping is less for Newtonian fluid $(\Gamma \to 0)$ *than that of third order fluid* $(0 < \Gamma < 1)$.

Keywords: *Darcy number, Deborah number, Peristaltic flow, Third order fluid.*

1. INTRODUCTION

In the literature, many researchers considered the peristaltic motion of Newtonian fluids. Such approach is true in ureter, but it fails to give an adequate understanding of peristalsis in blood vessels, chyme movement in intestine, sperm transport in ductus efferentes of male productive tract, in transport of spermatozoa and in cervical canal. In these body organs the fluid viscosity varies across the thickness of the duct. Also, the assumption that most of the physiological fluids behave like Newtonian fluid is not true in reality. Here, it is clear that viscoelastic rheology is the correct way of properly describing the peristaltic flow.

Flows through porous medium occur in filtration of fluids and seepage of water in river beds. Movement of underground water and oils, limestone, rye bread, wood, the human lung, bile duct, gall bladder with stones, and small blood vessels are some important examples of flow through porous medium. Another example is the seepage water under a dam which is very important Rathy [11]. Several works have been published by using the generalized Darcy's law (Scheidegger, [12]), where the convective acceleration and viscous- stress are taken into account by Yamamoto and Iwamura [16]. The net flow of compressible viscous liquids induced by traveling waves in porous media has been studied by Aarts and Ooms [1]. The effects of porous boundaries on peristaltic transport through a porous medium in a two- dimensional channel have been studied by El Shehawey and Husseny [2] in a fixed frame. The peristaltic transport in a cylindrical tube through a porous medium has been studied by El Shehwey and El Sebaei [3]. The non-linear and magneto-hydrodynamic flow effect in peristaltic transport through a porous medium was studied by Mekheimer and Al- Arabi [10]. Mekheimer [9] investigated the non linear peristaltic flow through a porous medium in an inclined planar channel. El - Shehawey et al. [5] studied peristaltic transport in an asymmetric channel through a porous medium.

Siddiqui et al. [14] investigated peristaltic motion of a third order fluid in a planer channel. Siddiqui and Schwarz [15] discussed the peristaltic flow of a second order fluid in a tube. Hayat et al. [6] studied the peristaltic flow of a third order fluid in a cylindrical tube. Hayat and Ali [7] investigated the peristaltic flow of a MHD third grade fluid in a cylindrical tube. Recently, Hayat et al. [8] discussed the peristaltic transport of a third order fluid under the effect of a magnetic field.

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In all the above studies the flow is considered through non-porous medium. But, flow through porous medium finds many applications in physiological and mechanical problems. Hence, an attempt is made to model the peristaltic flow of a third order fluid through a porous medium in a symmetric channel under the assumptions of long wavelength and zero Reynolds number. The expressions for the velocity pressure rise and friction force over one wavelength are obtained by a regular perturbation technique. The effects of Deborah number, Darcy number and porosity on the flow characteristics are studied in detail.

2. MATHEMATICAL FORMULATION

An incompressible third order fluid through a porous medium in a two-dimensional symmetrical channel of width $2a$ is considered. A rectangular coordinate system $(\overline{X},\overline{Y})$ is chosen in such a way that $\overline{X}$ -axis lies along the centre line of the channel and $\overline{Y}$ -axis normal to it. It is assumed that an infinite wave train is traveling with velocity c along the channel walls. The channel is taken to be symmetric about its centerline, as given in Shapiro (1969). Fig. 1 depicts the physical model of the problem.

The geometry of the wall surface is given by

$$\overline{Y} = \pm\overline{H}(\overline{X},\bar{t}) = \pm a \pm b\sin\frac{2\pi}{\lambda}(\overline{X} - c\bar{t}) \tag{2.1}$$

where b - the wave amplitude, λ - the wavelength and $\bar{t}$ - the time.

A Brinkman-extended Darcy equations is employed to study the third order flow through the porous medium (Alazmi and Vafai, 2001).

The equations governing the flow field are

$$\mathrm{div}\overline{\mathrm{V}} = 0 \tag{2.2}$$

$$\rho\frac{d\overline{\mathrm{V}}}{d\bar{t}} = \mathrm{div}\overline{T} - \frac{\mu}{k}\overline{\mathrm{V}} \tag{2.3}$$

where d/d$\bar{t}$ is the material derivative, $\overline{\mathrm{V}}$ - the velocity, μ - the viscosity, k -the permeability of the porous medium, ρ - the current density and $\overline{T}$ - the Cauchy stress tensor.

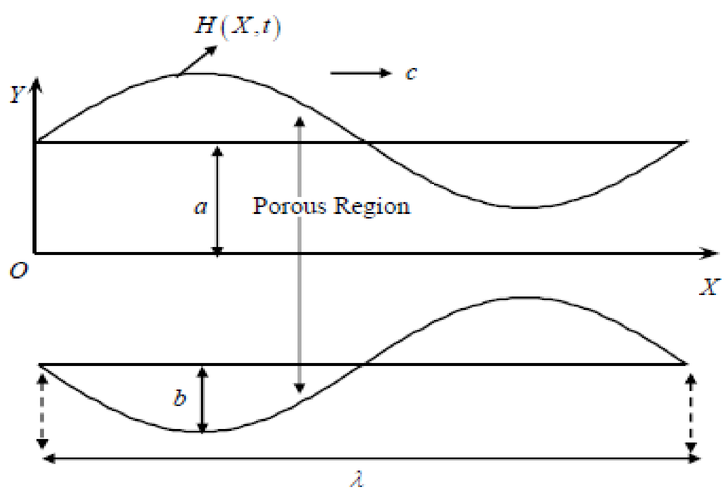

Fig. 1: Physical model

The constitutive equation for $\overline{T}$ in a third order fluid is

$$\overline{T} = -\overline{P}\,\overline{I} + \frac{1}{\varepsilon}\overline{S}, \tag{2.4}$$

where $\overline{P}$ is the pressure, $\overline{I}$ is the identity tensor, ε is the porosity and the extra stress tensor $\overline{S}$ is given by

$$\bar{S} = \mu \bar{A}_1 + \alpha_1 \bar{A}_1^2 + \beta_2(\bar{A}_2\bar{A}_1 + \bar{A}_1\bar{A}_2) + \beta_3(tr\bar{A}_1^2)\bar{A}_1 \tag{2.5}$$

in which $\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are the material constants and the Rivlin-Ericksen tensors $(\bar{A}_n)$ are given through the following relations

$$\bar{A}_1 = \left(grad\bar{V}\right) + \left(grad\bar{V}\right)^T$$

$$\bar{A}_n = \frac{d}{d\bar{t}}\bar{A}_{n-1} + \bar{A}_{n-1}(grad\bar{V}) + (grad\bar{V})^T\bar{A}_{n-1}, \quad n>1 \tag{2.6}$$

For two-dimensional flow in a fixed frame the velocity field is given by

$$\bar{\mathrm{V}} = \left[\bar{U}(\bar{X},\bar{Y},\bar{t}),\ \bar{V}(\bar{X},\bar{Y},\bar{t}), 0\right] \tag{2.7}$$

in which $\bar{U}$ and $\bar{V}$ are the velocity components in the $\bar{X}$ and $\bar{Y}$ directions respectively. It is also assumed that for the flow under consideration there is no motion of the wall in the longitudinal direction. This assumption constrains the deformation of the wall; it does not necessarily imply that the channel is rigid against longitudinal motions, but is a convenient simplification that can be justified by a more complete analysis. This assumption implies that there is no-slip condition, $\bar{U}$ = 0, at the wall.

In view of Equation (2.7), the Equations (2.2) and (2.3) can be written as

$$\frac{\partial \bar{U}}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} = 0 \tag{2.10}$$

$$\rho\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{U} = -\frac{\partial \bar{p}(\bar{X},\bar{Y},\bar{t})}{\partial \bar{X}} + \frac{1}{\varepsilon}\frac{\partial \bar{S}_{\bar{X}\bar{X}}}{\partial \bar{X}} + \frac{1}{\varepsilon}\frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{Y}} - \frac{\mu}{k}\bar{U} \tag{2.11}$$

$$\rho\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{V} = -\frac{\partial \bar{p}(\bar{X},\bar{Y},\bar{t})}{\partial \bar{Y}} + \frac{1}{\varepsilon}\frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{X}} + \frac{1}{\varepsilon}\frac{\partial \bar{S}_{\bar{Y}\bar{Y}}}{\partial \bar{Y}} - \frac{\mu}{k}\bar{V} \tag{2.12}$$

where

$$\begin{aligned}
\bar{S}_{\bar{X}\bar{X}} = {} & 2\mu\bar{U}_{\bar{X}} + \alpha_1\left(2\bar{U}_{\bar{X}\bar{t}} + 2\bar{U}\bar{U}_{\bar{X}\bar{X}} + 2\bar{V}\bar{U}_{\bar{X}\bar{Y}} + 4\bar{U}^2_{\bar{X}} + 2\bar{V}^2_{\bar{X}} + 2\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}}\right) \\
& + \alpha_2\left(4\bar{U}^2_{\bar{X}} + \bar{U}^2_{\bar{Y}} + \bar{V}^2_{\bar{X}} + 2\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}}\right) \\
& + \beta_1\ [2\bar{U}_{\bar{X}\bar{t}\bar{t}} + 2\bar{U}_{\bar{t}}\bar{U}_{\bar{X}\bar{X}} + 4\bar{U}\bar{U}_{\bar{X}\bar{X}\bar{t}} + 2\bar{V}_{\bar{t}}\bar{U}_{\bar{X}\bar{Y}} + 4\bar{V}\bar{U}_{\bar{X}\bar{Y}\bar{t}} + 12\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{t}} \\
& + 6\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{t}} + 2\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{t}} + +4\bar{U}_{\bar{Y}\bar{t}}\bar{V}_{\bar{X}} + 14\bar{U}\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{X}} + 12\bar{V}\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}} \\
& + 8\bar{U}^3_{\bar{X}} + 6\bar{U}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}} + 6\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}} + 4\bar{V}\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}\bar{Y}} + 6\bar{U}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{X}} \\
& + 2\bar{U}^2\bar{U}_{\bar{X}\bar{X}\bar{X}} + 2\bar{U}\bar{V}\bar{U}_{\bar{X}\bar{X}\bar{Y}} + 2\bar{V}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{X}} + 2\bar{V}^2\bar{U}_{\bar{X}\bar{Y}\bar{Y}} + 2\bar{V}\bar{V}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}} \\
& + 2\bar{V}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}} + 8\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}} + 2\bar{U}\bar{V}_{\bar{X}\bar{X}}\bar{U}_{\bar{Y}} + 4\bar{U}\bar{V}\bar{U}_{\bar{X}\bar{X}\bar{Y}}] \\
& + \beta_2\ [8\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{t}} + 8\bar{U}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}\bar{X}} + 8\bar{V}\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}} + 16\bar{U}^2_{\bar{X}} + 2\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{t}} + 2\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{t}} \\
& + 2\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{t}} + 2\bar{U}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{X}} + +2\bar{U}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{X}} + 2\bar{V}\bar{U}_{\bar{Y}}\bar{V}_{\bar{Y}\bar{Y}} + 2\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}} \\
& + 2\bar{U}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}} + 2\bar{U}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}} + 2\bar{V}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}} + 2\bar{V}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}} + 4\bar{V}_{\bar{X}}\bar{U}^2_{\bar{X}} \\
& + 4\bar{U}_{\bar{X}}\bar{U}^2_{\bar{Y}} + 8\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}} + 2\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}\bar{t}}] \\
& + \beta_3\left(2\bar{U}^3_{\bar{X}} + 4\bar{U}_{\bar{X}}\bar{U}^2_{\bar{Y}} + 8\bar{U}_{\bar{X}}\bar{V}^2_{\bar{Y}}4\bar{U}_{\bar{X}}\bar{V}^2_{\bar{X}} + 8\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}}\right)
\end{aligned} \tag{2.13}$$

$$
\begin{aligned}
\bar{S}_{\bar{X}\bar{Y}} = {} & \mu\left(\bar{U}_{\bar{Y}}+\bar{V}_{\bar{X}}\right)+\alpha_1\left(\bar{V}_{\bar{X}t}+\bar{U}_{\bar{Y}t}+\bar{U}\bar{U}_{\bar{X}\bar{X}}+\bar{V}\bar{V}_{\bar{X}\bar{Y}}+\bar{V}\bar{U}_{\bar{Y}\bar{Y}}+\bar{U}\bar{V}_{\bar{X}\bar{X}}+2\bar{U}_{\bar{X}}\bar{U}_{\bar{Y}}+2\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}}\right)\\
& +\alpha_2\left(2\bar{U}_{\bar{X}}\bar{V}_{\bar{X}}+2\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}}\right)\\
& +\beta_1\ [\bar{V}_{\bar{X}tt}+\bar{U}_{\bar{Y}tt}+\bar{U}_t\bar{U}_{\bar{X}\bar{Y}}+\bar{U}\bar{U}_{\bar{X}\bar{Y}t}+\bar{V}_t\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}\bar{V}_{\bar{X}\bar{Y}t}+2\bar{V}\bar{U}_{\bar{X}\bar{Y}t}+\bar{V}_t\bar{U}_{\bar{X}\bar{Y}}\\
& +\bar{U}_t\bar{V}_{\bar{X}\bar{X}}+2\bar{U}\bar{V}_{\bar{X}\bar{X}t}+2\bar{V}_{\bar{Y}}\bar{V}_{\bar{X}t}+4\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}t}+\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}t}+4\bar{U}_{\bar{X}t}\bar{U}_{\bar{Y}}+3\bar{U}_{\bar{X}}\bar{U}_{\bar{Y}t}\\
& +5\bar{V}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}}+4\bar{U}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{X}}+4\bar{U}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{X}}+4\bar{U}^2_{\bar{X}}\bar{U}_{\bar{Y}}+4\bar{V}^2_{\bar{X}}\bar{U}_{\bar{Y}}+4\bar{V}_{\bar{X}}\bar{U}^2_{\bar{Y}}\\
& +2\bar{V}_{\bar{Y}}\bar{U}_{\bar{Y}\bar{Y}}+\bar{U}\bar{V}_{\bar{X}\bar{X}}\bar{V}_{\bar{Y}}+4\bar{V}_{\bar{X}}\bar{V}^2_{\bar{Y}}+4\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}}+2\bar{V}\bar{V}_{\bar{Y}}\bar{V}_{\bar{X}\bar{X}}+4\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}\bar{Y}}\\
& +\bar{U}\bar{U}_{\bar{Y}\bar{Y}t}+\bar{U}^2\bar{V}_{\bar{X}\bar{X}\bar{X}}+\bar{U}\bar{V}\bar{U}_{\bar{X}\bar{Y}\bar{Y}}+3\bar{V}\bar{U}_{\bar{X}}\bar{U}_{\bar{Y}\bar{Y}}+5\bar{U}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}}+\bar{U}^2\bar{U}_{\bar{X}\bar{X}\bar{Y}}\\
& +\bar{U}\bar{V}\bar{V}_{\bar{X}\bar{Y}\bar{Y}}+2\bar{V}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{X}}+\bar{V}^2\bar{V}_{\bar{X}\bar{Y}\bar{Y}}+\bar{U}\bar{V}\bar{V}_{\bar{X}\bar{Y}\bar{Y}}+\bar{U}\bar{V}_{\bar{X}\bar{X}}\bar{U}_{\bar{Y}}]\\
& +\beta_2\ [2\bar{U}_{\bar{X}}\bar{V}_{\bar{X}t}+2\bar{U}_{\bar{X}}\bar{U}_{\bar{Y}t}+2\bar{U}\bar{U}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}}+2\bar{V}\bar{U}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}\bar{U}_{\bar{X}}\bar{U}_{\bar{Y}\bar{Y}}\\
& +2\bar{U}\bar{U}_{\bar{X}}\bar{V}_{\bar{X}\bar{X}}+2\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}t}+2\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}t}++2\bar{V}_{\bar{X}}\bar{U}_{\bar{X}t}+2\bar{U}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}\bar{X}\bar{Y}}\\
& +2\bar{U}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{X}}+2\bar{V}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}}+2\bar{U}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}\bar{U}_{\bar{Y}}\bar{V}_{\bar{Y}\bar{Y}}+2\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}\bar{Y}}\\
& +2\bar{U}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}\bar{X}}+2\bar{U}\bar{V}_{\bar{Y}}\bar{V}_{\bar{X}\bar{X}}+6\bar{V}_{\bar{X}}\bar{U}^2_{\bar{X}}+2\bar{V}_{\bar{Y}}\bar{U}_{\bar{Y}t}+2\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}t}+2\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}\bar{Y}}\\
& +2\bar{U}\bar{V}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}}+2\bar{V}\bar{V}_{\bar{Y}}\bar{U}_{\bar{Y}\bar{Y}}+2\bar{U}^2_{\bar{X}}\bar{U}_{\bar{Y}}6\bar{V}^2_{\bar{X}}\bar{U}_{\bar{Y}}+2\bar{V}^3_{\bar{X}}+6\bar{V}_{\bar{X}}\bar{U}^2_{\bar{Y}}+6\bar{V}^2_{\bar{Y}}\bar{U}_{\bar{Y}}\\
& +2\bar{V}^2_{\bar{Y}}\bar{V}_{\bar{X}}+2\bar{U}_{\bar{Y}}\bar{V}_{\bar{Y}t}]\\
& +\beta_3\ [\bar{U}^2_{\bar{X}}\bar{U}_{\bar{Y}}+2\bar{U}^3_{\bar{Y}}+6\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}}+4\bar{V}^2_{\bar{Y}}\bar{U}_{\bar{Y}}+6\bar{V}_{\bar{X}}\bar{U}^2_{\bar{Y}}+\bar{V}_{\bar{X}}\bar{U}^2_{\bar{X}}+2\bar{V}^3_{\bar{X}}\\
& +4\bar{V}^2_{\bar{Y}}\bar{V}_{\bar{X}}]
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
\bar{S}_{\bar{Y}\bar{Y}} = {} & 2\mu\bar{V}_{\bar{Y}}+\alpha_1\left(2\bar{V}_{\bar{Y}t}+2\bar{U}\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}\bar{V}_{\bar{Y}\bar{Y}}+2\bar{U}^2_{\bar{Y}}+4\bar{V}^2_{\bar{Y}}+2\bar{V}_{\bar{X}}2\bar{U}_{\bar{Y}}\right)\\
& +\alpha_2\left(\bar{U}^2_{\bar{Y}}+4\bar{V}^2_{\bar{Y}}+\bar{V}^2_{\bar{X}}+2\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}}\right)\\
& +\beta_1\ [2\bar{V}_{\bar{Y}t}+2\bar{U}_t\bar{V}_{\bar{X}\bar{Y}}+4\bar{U}\bar{V}_{\bar{X}\bar{Y}t}+2\bar{V}_t\bar{V}_{\bar{Y}\bar{Y}}+4\bar{V}\bar{U}_{\bar{Y}\bar{Y}t}+2\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}t}+2\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}t}\\
& +2\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}t}+6\bar{U}_{\bar{Y}t}\bar{U}_{\bar{Y}}+12\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}t}+4\bar{U}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{X}}+6\bar{V}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}}+8\bar{V}^3_{\bar{Y}}\\
& +6\bar{U}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}}+6\bar{V}\bar{U}_{\bar{Y}}\bar{U}_{\bar{Y}\bar{Y}}+12\bar{U}\bar{V}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}}+14\bar{V}\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}\bar{Y}}+2\bar{U}^2\bar{V}_{\bar{X}\bar{X}\bar{Y}}\\
& +4\bar{U}\bar{V}\bar{V}_{\bar{X}\bar{Y}\bar{Y}}+2\bar{U}\bar{V}_{\bar{X}}\bar{V}_{\bar{Y}\bar{Y}}+2\bar{V}^2\bar{V}_{\bar{Y}\bar{Y}\bar{Y}}+2\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}\bar{Y}}\\
& +8\bar{U}_{\bar{Y}}\bar{V}_{\bar{Y}}\bar{U}_{\bar{X}}+2\bar{U}\bar{U}_{\bar{X}\bar{Y}}\bar{V}_{\bar{X}}]\\
& +\beta_2[8\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}t}+8\bar{U}\bar{V}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}}+8\bar{V}\bar{V}_{\bar{Y}}\bar{V}_{\bar{Y}\bar{Y}}++16\bar{V}^3_{\bar{Y}}+2\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}t}+2\bar{U}_{\bar{Y}t}+2\bar{V}_{\bar{X}}\bar{V}_{\bar{X}t}\\
& +2\bar{V}\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}\bar{Y}}+2\bar{U}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{X}}+2\bar{V}\bar{U}_{\bar{Y}}+\bar{V}_{\bar{X}\bar{X}}+2\bar{V}\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}\bar{Y}}+2\bar{U}\bar{V}_{\bar{X}}\bar{U}_{\bar{X}\bar{Y}}+2\bar{U}\bar{U}_{\bar{Y}}\bar{U}_{\bar{X}\bar{Y}}\\
& +2\bar{V}\bar{U}_{\bar{Y}}\bar{U}_{\bar{Y}\bar{Y}}+4\bar{V}_{\bar{Y}}\bar{U}^2_{\bar{Y}}+4\bar{V}^2_{\bar{X}}\bar{V}_{\bar{Y}}+8\bar{U}_{\bar{Y}}\bar{V}_{\bar{Y}}\bar{U}_{\bar{X}}+2\bar{V}\bar{V}_{\bar{X}}\bar{V}_{\bar{X}\bar{Y}}+2\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}t}]\\
& +\beta_3\left(8\bar{V}^3_{\bar{Y}}+2\bar{U}^2_{\bar{X}}\bar{V}_{\bar{Y}}+4\bar{V}^2_{\bar{X}}\bar{V}_{\bar{Y}}+4\bar{U}^2_{\bar{Y}}\bar{V}_{\bar{Y}}+8\bar{U}_{\bar{Y}}\bar{V}_{\bar{X}}\bar{U}_{\bar{Y}}\right)
\end{aligned} \tag{2.15}
$$

In the above equations the subscripts indicate the partial derivatives. In the fixed coordinate system $\left(\bar{X},\bar{Y}\right)$, the motion is unsteady because of the moving boundary. However, if observed in a coordinate systems $\left(\bar{x},\bar{y}\right)$ moving with the speed c, it can be treated as steady because the boundary shape appears to be stationary.

The transformations between the two frames are given by

$$\bar{x}=\bar{X}-c\bar{t},\ \ \bar{y}=\bar{Y},\ \bar{u}=\bar{U}-c,\ \bar{v}=\bar{V}, \tag{2.16}$$

where ($\bar{u}, \bar{v}$) are components of the velocity in the moving co-ordinate system.

The boundary conditions for the velocity (in a fixed frame) are

$$\frac{\partial \bar{U}}{\partial \bar{Y}} = 0 \qquad \text{at} \qquad y = 0 \tag{2.17}$$

$$\bar{U} = 0 \qquad \text{at} \qquad y = h \tag{2.18}$$

Upon making use of transformations (2.16) in Equations (2.10)-(2.15) and introducing the following non-dimensional quantities

$$x = \frac{2\pi \bar{x}}{\lambda},\ y = \frac{\bar{y}}{a}, u = \frac{\bar{u}}{c}, v = \frac{\bar{v}}{c}, h = \frac{\bar{h}(\bar{x})}{a},\ p = \frac{2\pi a^2}{\lambda \mu c}\bar{p}(\bar{x}),\ \ S = \frac{a}{\mu c}\bar{S}(\bar{x}) \tag{2.19}$$

and introducing the stream function ψ (x,y) , defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\delta \frac{\partial \psi}{\partial x}, \tag{2.20}$$

we see that continuity Equation (2.10) is identically satisfied, and the Equations (2.11) - (2.15) reduce to

$$\delta \operatorname{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial \psi}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{\delta}{\varepsilon}\frac{\partial S_{xx}}{\partial x} + \frac{1}{\varepsilon}\frac{\partial S_{xy}}{\partial y} - \frac{1}{Da}\left(\frac{\partial \psi}{\partial y} + 1\right) \tag{2.21}$$

$$-\delta^3 \operatorname{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial \psi}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{\delta^2}{\varepsilon}\frac{\partial S_{xy}}{\partial x} + \frac{\delta}{\varepsilon}\frac{\partial S_{yy}}{\partial y} - \frac{\delta}{Da}v \tag{2.22}$$

where

$$\begin{aligned}
S_{xx} = {} & 2\delta \frac{\partial^2 \psi}{\partial x \partial y} + \lambda_1 \left(2\delta^4 \frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x^2 \partial y} - 2\delta^2 \frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x \partial y^2} + 4\delta^4 \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - 2\delta^3 \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + 2\delta^6 \left(\frac{\partial^2 \psi}{\partial x^2}\right)^2 \right) \\
& + 2\delta \frac{\partial^2 \psi}{\partial x \partial y} + \lambda_2 \left(\delta^4 \left(\frac{\partial^2 \psi}{\partial x^2}\right)^2 + 4\delta^2 \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - 2\delta^2 \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{\partial^2 \psi}{\partial x^2}\right)^2 \right) \\
& + \gamma_1\ [2\delta^5 \left(\frac{\partial \psi}{\partial y}\right)^2 \frac{\partial^4 \psi}{\partial x^3 \partial y} + 2\delta^5 \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y}\frac{\partial^3 \psi}{\partial x^2 \partial y} - 4\delta^3 \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}\frac{\partial^4 \psi}{\partial x^2 \partial y^2} \\
& - 2\delta^4 \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}\frac{\partial^3 \psi}{\partial x^2 \partial y} - 6\delta^4 \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^3 \psi}{\partial x \partial y^2} + 2\delta^5 \left(\frac{\partial \psi}{\partial x}\right)^2 \frac{\partial^4 \psi}{\partial x \partial y^3} \\
& + 2\delta^2 \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y}\frac{\partial^3 \psi}{\partial x \partial y^2} - 12\delta^4 \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y}\frac{\partial^3 \psi}{\partial x^2 \partial y} - 12\delta^3 \frac{\partial^2 \psi}{\partial x \partial y}\frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial x \partial y^2} \\
& - 6\delta^5 \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^3 \psi}{\partial x^2 \partial y} + 6\delta^6 \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^3 \psi}{\partial x^3} + 4\delta^3 \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^3 \psi}{\partial y^3} \\
& + 2\delta^4 \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y^2}\frac{\partial^3 \psi}{\partial x^3} + 2\delta^3 \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}\frac{\partial^3 \psi}{\partial x^2 \partial y} + 8\delta^3 \frac{\partial^2 \psi}{\partial x \partial y}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + 8\delta^3 \left(\frac{\partial \psi}{\partial x \partial y}\right)^3]
\end{aligned}$$

$$+\gamma_2\ [8\delta^4\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}+8\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}+16\delta^3\left(\frac{\partial^3\psi}{\partial x\partial y}\right)^3+4\delta^5\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial x^2}\right)^2$$

$$-8\delta^3\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}+2\delta^2\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}+2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2}$$

$$-2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^3}+2\delta^6\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3}-2\delta^5\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2}$$

$$+2\delta^3\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^2\partial y}-2\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2}+2\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}+4\delta\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2]$$

$$+\gamma_3[8\delta^3\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2-8\delta^3\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial x\partial y}+4\delta\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+4\delta^4\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial x^2}\right)^2+8\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^3] \tag{2.23}$$

$$S_{xy}=\left(\frac{\partial^2\psi}{\partial y^2}+\delta^3\frac{\partial^2\psi}{\partial x^2}\right)+\lambda_1[2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}-2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x\partial y^2}+4\delta^4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2-2\delta^3\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial y^2}+2\delta^6\left(\frac{\partial^2\psi}{\partial x^2}\right)^2]$$

$$+\lambda_2[-2\delta^3\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\partial y}+4\delta\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial y^2}+2\delta^3\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\partial y}-2\delta\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial x\partial y}]$$

$$+\gamma_1\ [\delta^4\left(\frac{\partial\psi}{\partial y}\right)^2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}-\delta^6\left(\frac{\partial\psi}{\partial x}\right)^2\frac{\partial^4\psi}{\partial x^4}$$

$$+\delta^6\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^3}\frac{\partial^3\psi}{\partial x^2\partial y}-2\delta^3\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}\frac{\partial^4\psi}{\partial x\partial y^3}-\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}+\delta^5\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\frac{\partial^4\psi}{\partial x^3\partial y}$$

$$+\delta^5\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^3}-\delta^3\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial y^3}+\delta^2\left(\frac{\partial\psi}{\partial x}\right)^2\frac{\partial^4\psi}{\partial y^4}+5\delta^5\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x^2\partial y}$$

$$+\delta^4\left(\frac{\partial\psi}{\partial y}\right)^2\frac{\partial^4\psi}{\partial x^2\partial y^2}+3\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial y^3}+4\delta^5\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^3}+2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}$$

$$-4\delta^2\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}-\delta^2\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^3}+2\delta^5\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^3}-4\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2}$$

$$+4\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\frac{\partial^2\psi}{\partial y^2}+4\delta^2\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^2\psi}{\partial x^2}+4\delta^4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\frac{\partial^2\psi}{\partial x^2}+4\delta^5\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\frac{\partial^2\psi}{\partial x\partial y}]$$

$$+\gamma_2[2\delta^3\frac{\partial^2\psi}{\partial y^2}\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}-2\delta^5\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x^2\partial y}-2\delta\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x\partial y^2}\frac{\partial^3\psi}{\partial y^2}+8\delta^2\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2$$

$$+2\delta^4\frac{\partial^3\psi}{\partial x\partial y^2}\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi}{\partial x}-2\delta^6\left(\frac{\partial^3\psi}{\partial x^2}\right)^3+6\delta^4\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\frac{\partial^3\psi}{\partial y^2}-10\delta\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^3\psi}{\partial x^2}$$

$$-2\delta^3\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}+2\delta^5\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3}\frac{\partial^2\psi}{\partial x\partial y}+2\delta^2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}+6\delta\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2$$

$$-2\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}+2\delta^3\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}-2\delta^6\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^3}\frac{\partial^3\psi}{\partial x\partial y}+2\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}$$

$$-2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^2\partial y}+2\delta^2\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}+2\left(\frac{\partial^2\psi}{\partial y^2}\right)^3+2\delta^5\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x^2\partial y}$$

$$-2\delta^4 \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2}] +\gamma_3\ [2\delta^3\left(\frac{\partial^2\psi}{\partial x^2}\right)^3 +4\delta^2\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 +6\delta^2\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^2\psi}{\partial x^2}$$

$$+2\left(\frac{\partial^2\psi}{\partial y^2}\right)^3 +2\delta^3\left(\frac{\partial^2\psi}{\partial x^2}\right)^3 +4\delta^3\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 +6\delta^2\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\frac{\partial^2\psi}{\partial y^2}$$

$$+4\delta^4\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 + 4\delta^4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2] \tag{2.24}$$

$$S_{yy} = \lambda_1\left(-2\delta^3\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y} +2\delta^2\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x\partial y^2} +4\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 -2\delta^2\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} +2\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\right)$$

$$-2\delta\frac{\partial^2\psi}{\partial x\partial y} +\lambda_2\left(\delta^4\left(\frac{\partial^2\psi}{\partial x^2}\right)^2 +4\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 -2\delta^2\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right)$$

$$+\gamma_1[-2\delta^5\left(\frac{\partial\psi}{\partial y}\right)^2\frac{\partial^4\psi}{\partial x^3\partial y} -2\delta^5\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^2\partial y} -2\delta^4\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\frac{\partial^4\psi}{\partial x^2\partial y^2}$$

$$+2\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^2\partial y} +2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2} -\delta^3\left(\frac{\partial\psi}{\partial x}\right)^2\frac{\partial^4\psi}{\partial x\partial y^3} -2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^3}$$

$$-12\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x\partial y^2} +12\delta^4\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y} -4\delta^4\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x^2\partial y}$$

$$+2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial y^3} -6\delta\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial y^3} -4\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^3} +6\delta^2\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}$$

$$-8\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\frac{\partial^2\psi}{\partial y^2} -8\delta^3\left(\frac{\partial\psi}{\partial x\partial y}\right)^3]$$

$$+\gamma_2[-8\delta^3\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^2\partial y} +8\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial x^2\partial y} +16\delta^3\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^3 +4\delta^4\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial x^2}\right)^2$$

$$+8\delta^4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\frac{\partial^2\psi}{\partial y^2} +2\delta^2\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2} -2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x\partial y^2} +2\delta\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial y^3}$$

$$+\delta^6\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3} -2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial x^2\partial y} +2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^2\partial y} +2\delta^3\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^3\psi}{\partial y^3}$$

$$-2\delta^4\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x^3} +4\delta^2\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2] +\gamma_3[-16\delta^3\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^3$$

$$+8\delta^3\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial x\partial y} -4\delta\frac{\partial^2\psi}{\partial x\partial y^2}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2 +4\delta^5\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial x^2}\right)^2] \tag{2.25}$$

where, $\delta = \frac{2\pi a}{\lambda}$ is the wave number, $\mathrm{Re} = \frac{\rho ac}{\mu}$ the Reynolds number, $Da = \frac{k}{a^2}$ is the Darcy number, $M = B_0 a\sqrt{\frac{\sigma}{\mu}}$ is the Hartman number *M* and the material coefficients λ_i (i = 1,2), γ_i (i = 1,2,3) , are given by

$$\lambda_1 = \frac{\alpha_1 c}{\mu\alpha},\ \lambda_2 = \frac{\alpha_2 c}{\alpha\mu},\ \gamma_1 = \frac{\beta_1 c^2}{\beta\alpha^2}, \gamma_2 = \frac{\beta_2 c^2}{\beta\alpha^2},\ \gamma_3 = \frac{\beta_3 c^2}{\beta\alpha^2} \tag{2.26}$$

Eliminating p from Equations (2.21) and (2.22), we have the following vorticity transport equation

$$\operatorname{Re}\delta\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi\right]=\left[\frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial}{\partial x^2}\right)S_{xy}\right]+\frac{\delta}{\varepsilon}\left[\frac{\partial^2}{\partial x\partial y}\left(S_{xx}-S_{yy}\right)\right]-\frac{1}{Da}\left(\frac{\partial^2\psi}{\partial y^2}+\delta^2\frac{\partial^2\psi}{\partial x^2}\right) \quad (2.27)$$

in which $\nabla^2=\left(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$.

Equation (2.27) is non-linear and thus the closed form solution for arbitrary values of all parameters is not possible. Here, we carryout the analysis for the case of long wavelength. This is a valid assumption especially for the flow of chyme in the small intestine. For long wavelength approximation, Equations (2.21) - (2.25) and Equation (2.27) takes the form

$$\frac{\partial^4\psi}{\partial y^4}=-2\Gamma\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\psi}{\partial y^2}\right)^3+\frac{\varepsilon}{Da}\frac{\partial^2\psi}{\partial y^2} \quad (2.28)$$

$$\frac{\partial p}{\partial x}=\frac{1}{\varepsilon}\frac{\partial Sxy}{\partial y}-\frac{1}{Da}\left(\frac{\partial\psi}{\partial y}+1\right), \quad (2.29)$$

$$\frac{\partial p}{\partial y}=0, \quad (2.30)$$

$$S_{xy}=\frac{\partial^2\psi}{\partial y^2}+2\Gamma\left(\frac{\partial^2\psi}{\partial y^2}\right)^3, \quad (2.31)$$

$$S_{xy}=S_{yy}=0, \quad (2.32)$$

where $\Gamma=\gamma_2+\gamma_3$ is the Deborah number.

The Equation (2.30) indicates that $p\neq p(y)$ and so $\frac{\partial p}{\partial x}=\frac{dp}{dx}$.

In the wave frame the non-dimensional boundary conditions are

$$\left.\begin{array}{ll}\psi=0,\quad \frac{\partial^2\psi}{\partial y^2}=0, & \text{at } y=0\\ \frac{\partial\psi}{\partial y}=0,\quad \psi=q, & \text{at } y=h\end{array}\right\} \quad (2.33)$$

where q is the non-dimensional mean flow and non-dimensional surface of the peristaltic wave h becomes

$$h(x)=1+\phi\sin x \quad (2.34)$$

where $\phi\left(=\frac{b}{a}\right)$ is the amplitude ratio and $0\le\phi<1$.

3. SOLUTION

The Equations (2.28) and (2.33) are non-linear and it is difficult to get a closed form solution. However for vanishing Deborah number Γ, the boundary value problem is amenable to an easy analytical solution. In this case the equations become linear and can be solved. Accordingly the stream function, volume flow rate and pressure are assumed in the following form

$$\left.\begin{array}{l}\psi=\psi_0+\Gamma\psi_1+\Gamma^2\psi_2+\ldots\ldots,\\ q=q_0+\Gamma q_1+\Gamma^2q_2+\ldots\ldots,\\ p=p_0+\Gamma p_1+\Gamma^2p_2+\ldots\ldots,\end{array}\right\} \quad (3.1)$$

Upon making use of Equation (3.1) into Equations (2.28), (2.29), (2.33) and equating the coefficients of like powers of Γ and then solving the resulting systems upto O(Γ^2), we have the following expressions upto second order of the Deborah number

$$
\begin{aligned}
\psi = {} & \frac{q_0\alpha+\tanh\alpha h}{h\alpha-\tanh\alpha h}\left[y-\frac{\sinh\alpha y}{\alpha\cosh\alpha h}\right]-\frac{\sinh\alpha y}{\alpha\cosh\alpha h} \\
& +\Gamma\ [y(\alpha\cosh\alpha h\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right)+\left(\varepsilon\frac{dp_0}{dx}\right)^3\{\frac{3\cosh\alpha h}{16\alpha^4\cosh^3\alpha h} \\
& -\frac{3}{4\alpha^4\cosh^2\alpha h}-\frac{3h\sinh\alpha h}{4\alpha^3\cosh^3\alpha h}\})-\sinh\alpha y\left\{\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right\} \\
& -\left(\frac{\varepsilon dp_0}{dx}\right)^3\left\{\frac{\sinh 3\alpha y}{16\alpha^4\cosh^3\alpha h}-\frac{3y\cosh\alpha y}{4\alpha^3\cosh^3\alpha h}\right\}] \\
& +\Gamma^2\{[\frac{q_1\alpha\cosh\alpha h}{\alpha h\cosh\alpha h-\sinh\alpha h}-\alpha\cosh\alpha h\left(\varepsilon\frac{dp_0}{dx}\right)^2\frac{q_1 H_1}{\alpha h\cosh\alpha h-\sinh\alpha h} \\
& -\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0 H_1-\alpha\cosh\alpha h\left(\varepsilon\frac{dp_0}{dx}\right)^5 H_2] \\
& -[\left(\varepsilon\frac{dp_0}{dx}\right)^2\left\{\frac{9\cosh 3\alpha h}{16\alpha\cosh^2\alpha h}-\frac{9\cosh\alpha h}{4\alpha\cosh^2\alpha h}-\frac{9h\sinh\alpha h}{4\cosh^2\alpha h}\right\} \\
& \times\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right)] \\
& -\left(\varepsilon\frac{dp_0}{dx}\right)^5[\frac{45\cosh 5\alpha h}{256\alpha^6\cosh^5\alpha h}-\frac{54\cosh 3\alpha h}{128\alpha^6\cosh^5\alpha h}+\frac{225\cosh\alpha h}{64\alpha^6\cosh^5\alpha h}+\frac{261h\sinh\alpha h}{64\alpha^5\cosh^5\alpha h} \\
& -\frac{27h\sinh 3\alpha h}{64\alpha^5\cosh^5\alpha h}+\frac{9h^2\cosh\alpha h}{32\alpha^4\cosh^5\alpha h}] \\
& -[\frac{q_2\sinh\alpha y}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^2\left(\frac{q_1\sinh\alpha y}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right)H_1 \\
& -\sinh\alpha y\left(\varepsilon\frac{dp_0}{dx}\right)^5 H_2] \\
& +\left(\varepsilon\frac{dp_0}{dx}\right)^2\left\{\frac{3\sinh 3\alpha y}{16\alpha^2\cosh^2\alpha h}-\frac{9y\cosh\alpha y}{4\alpha\cosh^2\alpha h}\right\}\times\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right) \\
& +\left(\varepsilon\frac{dp_0}{dx}\right)^5[\frac{9\sinh 5\alpha y}{256\alpha^7\cosh^5\alpha h}-\frac{12\sinh 3\alpha y}{128\alpha^7\cosh^5\alpha h}+\frac{225\cosh\alpha y}{64\alpha^6\cosh^5\alpha h}-\frac{9y\cosh 3\alpha y}{64\alpha^6\cosh^5\alpha h}+\frac{9y^2\sinh\alpha y}{32\alpha^5\cosh^5\alpha h}]\}
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
u = {} & \frac{q_0\alpha+\tanh\alpha h}{h\alpha-\tanh\alpha h}\left[1-\frac{\cosh\alpha y}{\cosh\alpha h}\right]-\frac{\cosh\alpha y}{\cosh\alpha h} \\
& +\Gamma\ [\alpha\cosh\alpha h\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right)+\left(\varepsilon\frac{dp_0}{dx}\right)^3\{\frac{3\cosh\alpha h}{16\alpha^4\cosh^3\alpha h} \\
& -\frac{3}{4\alpha^4\cosh^2\alpha h}-\frac{3h\sinh\alpha h}{4\alpha^3\cosh^3\alpha h}\}-\alpha\cosh\alpha y\left\{\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3 H_0\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\varepsilon dp_0}{dx}\right)^3\left\{\frac{3\cosh 3\alpha y}{16\alpha^4\cosh^3\alpha h}-\frac{3(\cosh\alpha y+\alpha y\sinh\alpha y)}{4\alpha^3\cosh^3\alpha h}\right\}]\\
&+\Gamma^2\ \{[\frac{q_2\alpha\cosh\alpha h}{\alpha h\cosh\alpha h-\sinh\alpha h}-\alpha\cosh\alpha h\left(\varepsilon\frac{dp_0}{dx}\right)^2(\frac{q_1H_1}{\alpha h\cosh\alpha h-\sinh\alpha h}\\
&-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0H_1)-\alpha\cosh\alpha h\left(\varepsilon\frac{dp_0}{dx}\right)^5H_2]\\
&-\left(\varepsilon\frac{dp_0}{dx}\right)^2\left\{\frac{9\cosh 3\alpha h}{16\alpha\cosh^2\alpha h}-\frac{9\cosh\alpha h}{4\alpha\cosh^2\alpha h}-\frac{9h\sinh\alpha h}{4\cosh^2\alpha h}\right\}\times\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0\right)\\
&-\left(\varepsilon\frac{dp_0}{dx}\right)^5[\frac{45\cosh 5\alpha h}{256\alpha^6\cosh^5\alpha h}-\frac{54\cosh 3\alpha h}{128\alpha^6\cosh^5\alpha h}+\frac{225\cosh\alpha h}{64\alpha^6\cosh^5\alpha h}\\
&+\frac{261h\sinh\alpha h}{64\alpha^5\cosh^5\alpha h}-\frac{27h\sinh 3\alpha h}{64\alpha^5\cosh^5\alpha h}+\frac{9h^2\cosh\alpha h}{32\alpha^4\cosh^5\alpha h}]\\
&-[\frac{q_2\alpha\cosh\alpha y}{\alpha h\cosh\alpha h-\sinh\alpha h}-\alpha\cosh\alpha y\left(\varepsilon\frac{dp_0}{dx}\right)^2(\frac{q_1H_1}{\alpha h\cosh\alpha h-\sinh\alpha h}\\
&-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0H_1)-\alpha\cosh\alpha y\left(\varepsilon\frac{dp_0}{dx}\right)^5H_2]\\
&+\left(\varepsilon\frac{dp_0}{dx}\right)^2\left\{\frac{9\cosh 3\alpha y}{16\alpha\cosh^2\alpha h}-\frac{9\cosh\alpha y}{4\alpha\cosh^2\alpha h}-\frac{9y\sinh\alpha y}{4\cosh^2\alpha h}\right\}\times\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0\right)\\
&+\left(\varepsilon\frac{dp_0}{dx}\right)^5[\frac{45\cosh 5\alpha y}{256\alpha^6\cosh^5\alpha h}-\frac{54\cosh 3\alpha y}{128\alpha^6\cosh^5\alpha h}+\frac{225\cosh\alpha y}{64\alpha^6\cosh^5\alpha h}\\
&+\frac{261y\sinh 3\alpha y}{64\alpha^5\cosh^5\alpha h}-\frac{27y\sinh 3\alpha y}{64\alpha^5\cosh^5\alpha h}+\frac{9y^2\sinh\alpha y}{32\alpha^4\cosh^5\alpha h}]
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\frac{dp}{dx}=&-\frac{\alpha^2}{\varepsilon}\left(\frac{q_0\alpha+\tanh\alpha h}{h\alpha-\tanh\alpha h}+1\right)+\frac{\Gamma}{\varepsilon}[-\alpha^3\cosh\alpha h\left\{\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0\right\}\\
&+\left(\varepsilon\frac{dp_0}{dx}\right)^3\left\{\frac{3}{4\alpha^2\cosh^2\alpha h}+\frac{3h\sin\alpha h}{4\alpha\cosh^3\alpha h}-\frac{3\cos 3\alpha h}{16\alpha^2\cosh^3\alpha h}\right\}]\\
&+\frac{\Gamma^2}{\varepsilon}[-\{\frac{q_2\alpha^3\cosh\alpha h}{\alpha h\cosh\alpha h-\sinh\alpha h}-\alpha^3\cosh\alpha h\left(\varepsilon\frac{dp_0}{dx}\right)^2(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}\\
&-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0)H_1-\alpha^3\cosh\alpha h\left(\varepsilon\frac{dp_0}{dx}\right)^5H_2\}\\
&+\alpha^2\left(\varepsilon\frac{dp_0}{dx}\right)^2\left\{\frac{9\cosh 3\alpha h}{16\alpha\cosh^2\alpha h}-\frac{9\cosh\alpha h}{4\alpha\cosh^2\alpha h}-\frac{9h\sinh\alpha h}{4\cosh^2\alpha h}\right\}\\
&\times\left(\frac{q_1}{\alpha h\cosh\alpha h-\sinh\alpha h}-\left(\varepsilon\frac{dp_0}{dx}\right)^3H_0\right)\\
&+\left(\varepsilon\frac{dp_0}{dx}\right)^5(\frac{45\cosh 5\alpha h}{256\alpha^4\cosh^5\alpha h}-\frac{54\cosh 3\alpha h}{128\alpha^4\cosh^5\alpha h}+\frac{225\cosh\alpha h}{64\alpha^4\cosh^5\alpha h}
\end{aligned}
$$

$$+\frac{261h\sinh 3\alpha h}{64\alpha^3\cosh^5\alpha h}-\frac{27h\sinh 3\alpha h}{64\alpha^3\cosh^5\alpha h}+\frac{9h^2\sinh\alpha h}{32\alpha^2\cosh^5\alpha h})] \tag{3.4}$$

where

$$H_0=\frac{1}{\alpha h\cosh\alpha h-\sinh\alpha h}\left\{\frac{3h\cosh 3\alpha h}{16\alpha^4\cosh^3\alpha h}-\frac{\sinh 3\alpha h}{16\alpha^5\cosh^3\alpha h}-\frac{3h^2\sinh\alpha h}{4\alpha^3\cosh^3\alpha h}\right\} \tag{3.5}$$

$$H_1=\frac{1}{\alpha h\cosh\alpha h-\sinh\alpha h}\left\{\frac{3h\sinh 3\alpha h}{16\alpha^4\cosh^3\alpha h}-\frac{9h\cosh 3\alpha h}{16\alpha\cosh^2\alpha h}-\frac{9h^2\sinh\alpha h}{4\cosh^2\alpha h}\right\} \tag{3.6}$$

$$H_2=\frac{1}{\alpha h\cosh\alpha h-\sinh\alpha h}[\frac{9\sinh 5\alpha h}{256\alpha^7\cosh^5\alpha h}-\frac{12\sinh 3\alpha h}{128\alpha^7\cosh^5\alpha h}-\frac{45h\cosh 5\alpha h}{256\alpha^6\cosh^5\alpha h}$$
$$+\frac{36h\cosh 3\alpha h}{128\alpha^6\cosh^5\alpha h}-\frac{243h2\sinh\alpha h}{64\alpha^5\cosh^5\alpha h}+\frac{27h^2\sinh 3\alpha h}{64\alpha^5\cosh^5\alpha h}-\frac{9h^3\cosh\alpha h}{32\alpha^4\cosh^5\alpha h}] \tag{3.7}$$

The result of our analysis can be expressed to second order by defining

$$q=q_0+\Gamma q_1+\Gamma^2 q_2 \tag{3.8}$$

Using Equation (3.8) into Equation (3.4) and neglecting the terms greater than O(Γ^2), we find that

$$\frac{dp}{dx}=\frac{\alpha^3(q+h)\cosh\alpha h}{\varepsilon\left(\alpha h\cos\alpha h-\sinh\alpha h\right)}+\frac{\Gamma}{\varepsilon}[\frac{\alpha^8(q+h)^3}{(\alpha h\cosh\alpha h-\sinh\alpha h)^3}\left\{-\frac{3\cosh\alpha y}{4\alpha}-\frac{3h\sinh\alpha y}{4}+\frac{3\cosh\alpha y}{16\alpha}\right\}$$
$$-\frac{\alpha^{12}(q+h)^3\cosh^4\alpha h}{(\alpha h\cosh\alpha h-\sinh\alpha h)^4}H_0]$$
$$+\frac{\Gamma^2}{\varepsilon}\{[\frac{(q+h)^5\alpha^{18}\cosh^6\alpha h}{(\alpha h\cosh\alpha h-\sinh\alpha h)^6}H_0H_1-\frac{(q+h)^5\alpha^{18}\cosh^6\alpha h}{(\alpha h\cosh\alpha h-\sinh\alpha h)^6}H_2]$$
$$+\frac{(q+h)^5\alpha^{17}H_0\cosh^3\alpha h}{(\alpha h\cosh\alpha h-\sinh\alpha h)^5}\left\{\frac{9\cosh 3\alpha h}{16\alpha}-\frac{9h\sinh\alpha h}{4\alpha}+\frac{225\cosh\alpha h}{64\alpha^2}\right\}$$
$$-\frac{\alpha^{13}(q+h)^5}{(\alpha h\cosh\alpha h-\sinh\alpha h)^5}(\frac{45\cosh 5\alpha h}{256\alpha^2}-\frac{54h\cosh 3\alpha h}{128\alpha^2}+\frac{225\cosh\alpha h}{64\alpha^2}$$
$$-\frac{27h\sinh 3\alpha h}{64\alpha}+\frac{9h^2\cosh\alpha h}{32}+\frac{261h\sinh\alpha h}{64\alpha})\} \tag{3.9}$$

The volume flow rate q in the wave frame is given by $q=\int_0^h u\,dy$

The dimensionless time averaged flux $\overline{Q}$ over one period in the fixed frame of reference is given by

$$\overline{Q}=\frac{1}{T}\int_0^T\int_0^h U\,dY\,dt=\frac{1}{2\pi}\int_0^{2\pi}\int_0^h (u+1)\,dy\,dx=q+1\,. \tag{3.10}$$

The pressure rise Δp and friction force F (on the wall) per one wavelength of the wave in their non dimensional forms are given by

$$\Delta p=\int_0^{2\pi}\frac{dp}{dz}dz \tag{3.11}$$

$$F=\int_0^{2\pi}h\left(-\frac{dp}{dz}\right)dz \tag{3.12}$$

4. RESULTS AND DISCUSSION

In order to get a the physical insight of the problem, axial pressure gradient, pumping and friction force are computed numerically for different values of the various parameters, viz., Darcy number Da, porosity ε, Deborah number Γ and amplitude ratio ϕ and are presented in figures 2-12.

Fig. 2 shows the profiles of axial pressure gradient dp/dx for different values of Darcy number Da with $\phi = 0.3, \varepsilon = 0.7$ and $\Gamma = 0.01$. It is observed that dp/dx decreases with an increase in the Darcy number.

The variation of dp/dx with x for various values of ε with $\phi = 0.3, Da = 0.1$ and $\Gamma = 0.01$ is shown in Fig. 3. It is found that the dp/dx first increases and then decreases with an increase in ε.

In order to see the effect of Deborah number Γ on dp/dx we have plotted Fig 4. It is observed that the dp/dx increase with an increases in Γ.

The variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of the Darcy number Da with $\phi = 0.3, \varepsilon = 0.7$ and $\Gamma = 0.01$ is depicted in Fig. 5. It is observed that as the Darcy number Da increases, the $\overline{Q}$ decreases in the pumping region $(\Delta p > 0)$, and it decreases in both free-pumping $(\Delta p = 0)$ and co- pumping $(\Delta p < 0)$ regions.

Fig. 6 shows the variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of porosity ε with $\phi = 0.3, Da = 0.1$ and $\Gamma = 0.01$. It is found that, the time averaged flux $\overline{Q}$ decreases with an increase in ε, in both the pumping and free-pumping regions. Further, it is observed that the $\overline{Q}$ increases with an increase in ε for appropriately chosen $\Delta p(< 0)$.

The variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of Deborah number Γ with $\phi = 0.3, \varepsilon = 0.7$ and $Da = 0.1$ is represented in Fig. 7. It is observed that the $\overline{Q}$ increases with an increase in Γ, in both the pumping and free-pumping regions. Further, the $\overline{Q}$ decreases with an increase in Γ for appropriately chosen $\Delta p(< 0)$.

In order to see the effect of amplitude of the peristaltic wave on the pumping characteristics we have plotted Fig. 8, by fixing $Da = 0.1,\ \varepsilon = 0.7$ and $\Gamma = 0.01$. It is found that, as the amplitude ratio ϕ increases, the $\overline{Q}$ increases in the pumping, free-pumping and co-pumping regions.

The variation of friction force with time averaged flux $\overline{Q}$ for different values of Darcy number Da with $\phi = 0.3, \varepsilon = 0.7$ and $\Gamma = 0.01$ is represented in Fig. 9. It is observed that the friction force F decreases with an increase in Da.

Fig. 10 represents the variation of friction force F with $\overline{Q}$ for different values of porosity ε with $\phi = 0.3, Da = 0.1$ and $\Gamma = 0.01$. It is observed that, as the Da increases friction force F first increases and then decreases.

In order to see the effect of Deborah number Γ on friction force F with $\overline{Q}$, we have plotted Fig. 11. It is found that the friction force F initially decreases and then increases with an increase in Deborah number Γ.

Fig. 12 is plotted to see the effect of amplitude ratio on friction force F. It is observed that the friction force decreases with an increase in ϕ.

5. CONCLUSIONS

The peristaltic pumping of third order fluid through a porous medium in a symmetric channel under the assumptions of long wavelength and low Reynolds number is studied. The effects of various emerging parameters on the pressure gradient, pressure rise and frictional force over one wavelength are discussed. It is observed that, the pressure gradient increases as the Darcy number or Deborah number increases. It is found that, as the Deborah number increases the pumping and magnitude of the friction force both increase. Both the pumping and friction force decrease with an increase in the Darcy number. Further, the pumping is less for Newtonian fluid $(\Gamma \to 0)$ than that of third order fluid $(0 < \Gamma < 1)$.

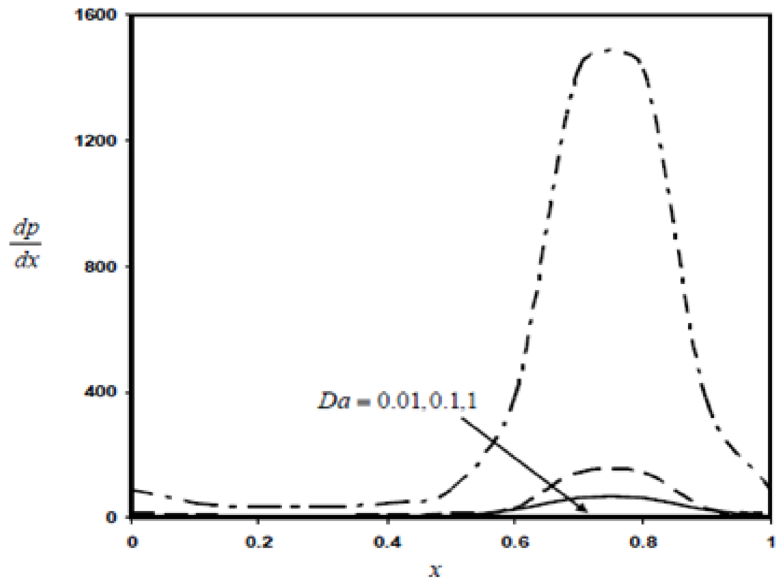

Fig. 2: Profiles of axial pressure gradient dp/dx for different values of Darcy number Da with $\phi = 0.3, \varepsilon = 0.7$ and $\Gamma = 0.01$.

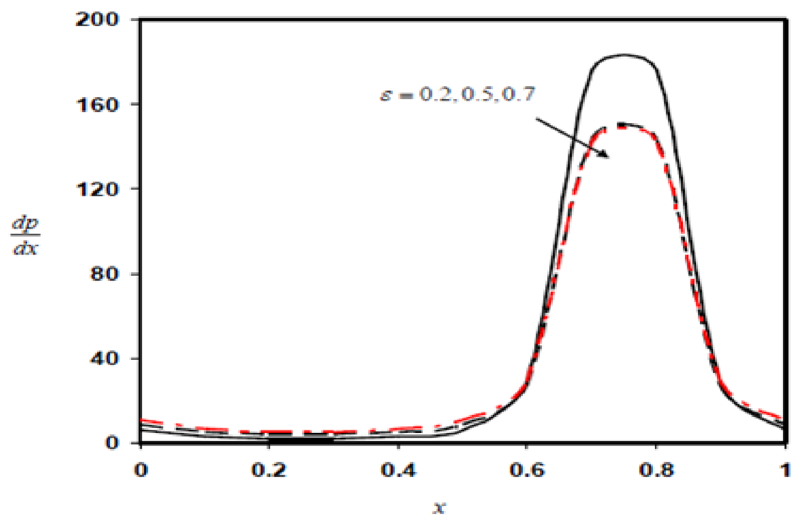

Fig. 3: Profiles of axial pressure gradient dp/dx for different values of porosity ε with $\phi = 0.3, Da = 0.1$ and $\Gamma = 0.01$.

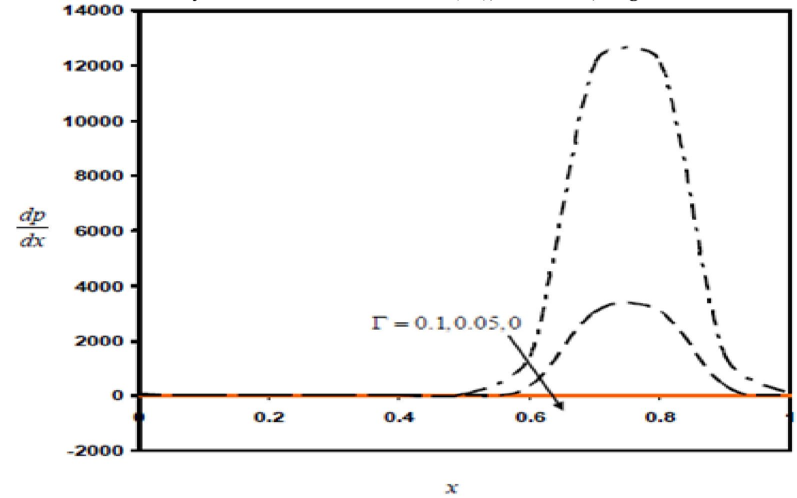

Fig. 4: Profiles of axial pressure gradient dp/dx for different values of Deborah number Γ with $\phi = 0.3, \varepsilon = 0.7$ and $Da = 0.1$.

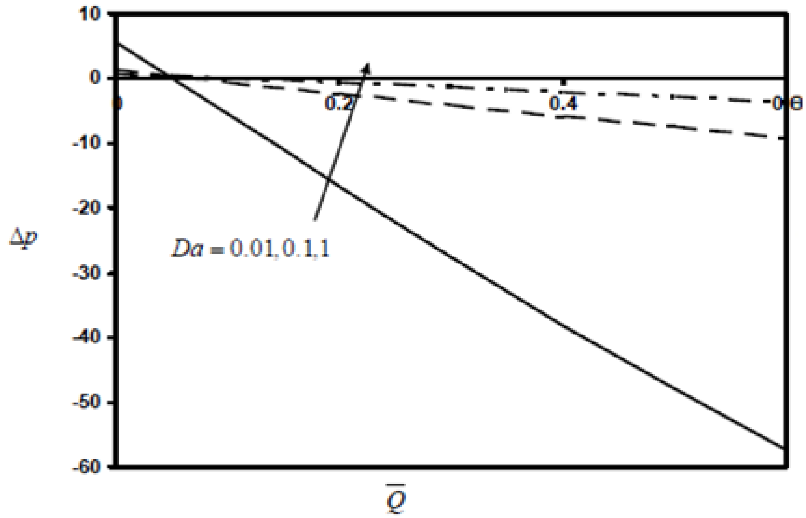

Fig. 5: The variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of Darcy number Da with $\phi = 0.3, \varepsilon = 0.7$ and $\Gamma = 0.01$.

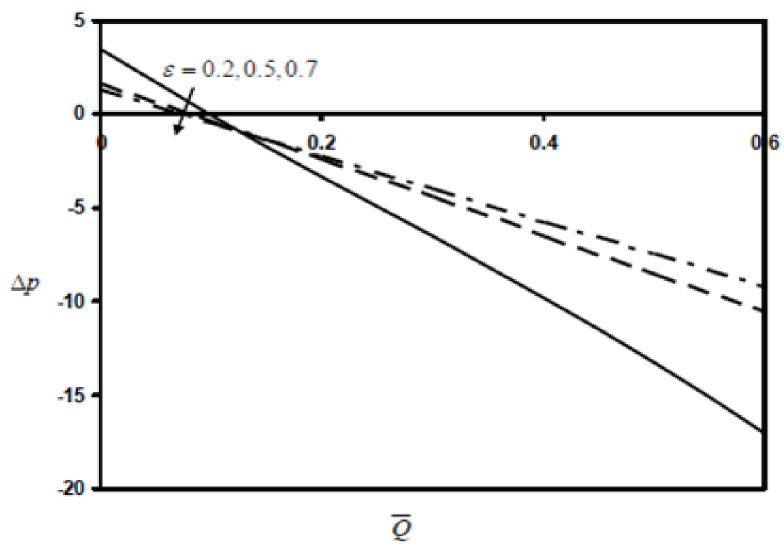

Fig. 6: The variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of porosity ε with $\phi = 0.3,\ Da = 0.1$ and $\Gamma = 0.01$.

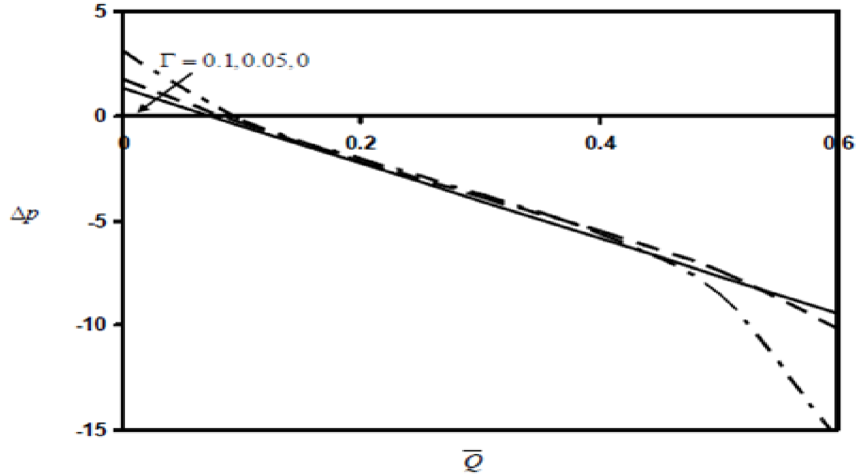

Fig. 7: The variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of Deborah number Γ with $\phi = 0.3, \varepsilon = 0.7$ and $Da = 0.1$.

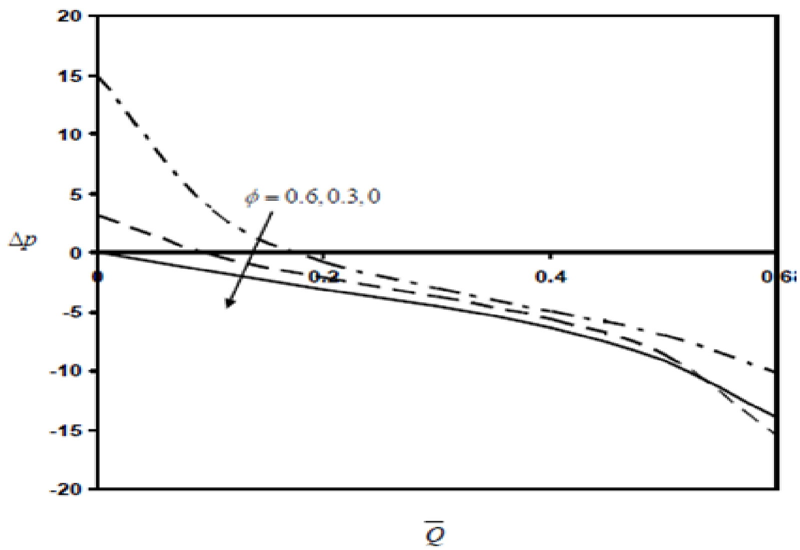

Fig. 8: The variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of ϕ with $Da = 0.1$, $\varepsilon = 0.7$ and $\Gamma = 0.01$.

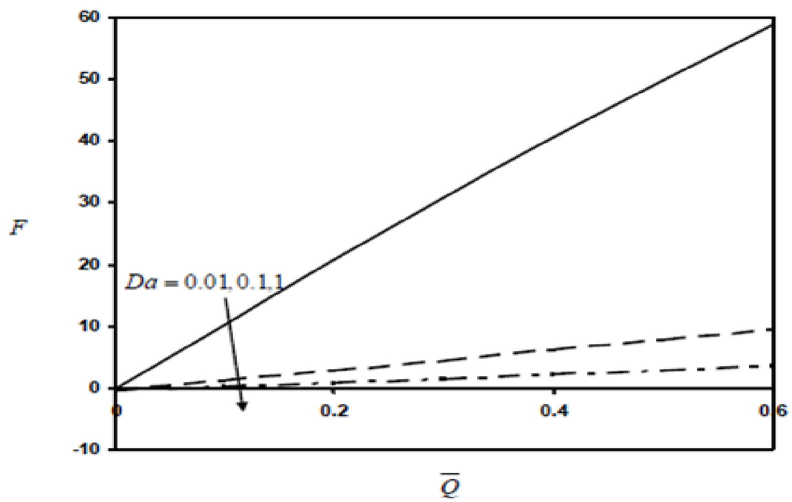

Fig. 9: The variation of friction force F with time averaged flux $\overline{Q}$ for different values of Darcy number Da with $\phi = 0.3, \varepsilon = 0.7$ and $\Gamma = 0.01$.

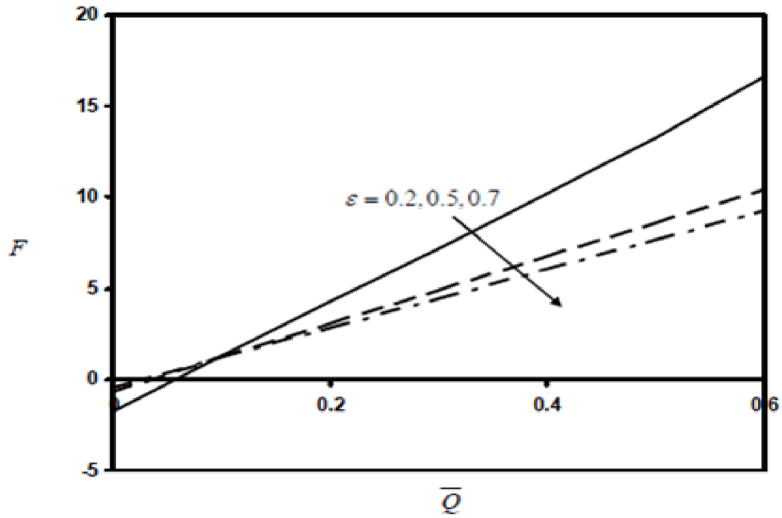

Fig. 10: The variation of friction force F with time averaged flux $\overline{Q}$ for different values of porosity ε with $\phi = 0.3,\ Da = 0.1$ and $\Gamma = 0.01$.

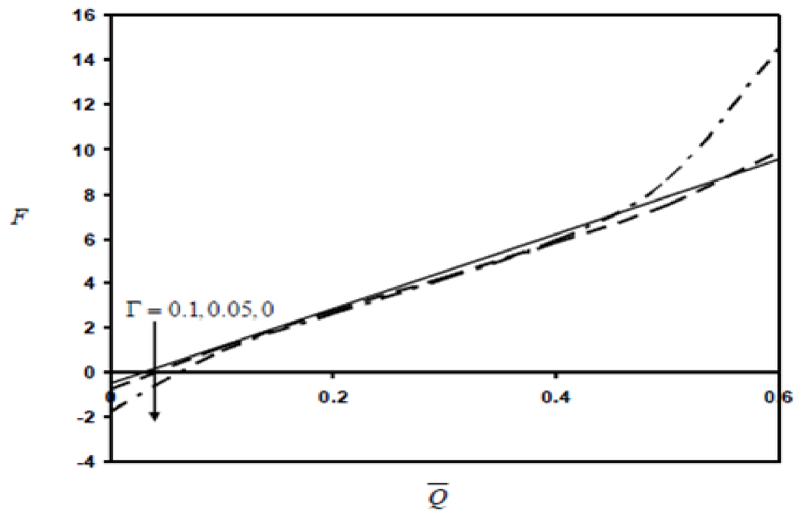

Fig. 11: The variation of friction force F with time averaged flux $\overline{Q}$ for different values of Deborah number Γ with $\phi = 0.3, \varepsilon = 0.7$ and $Da = 0.1$.

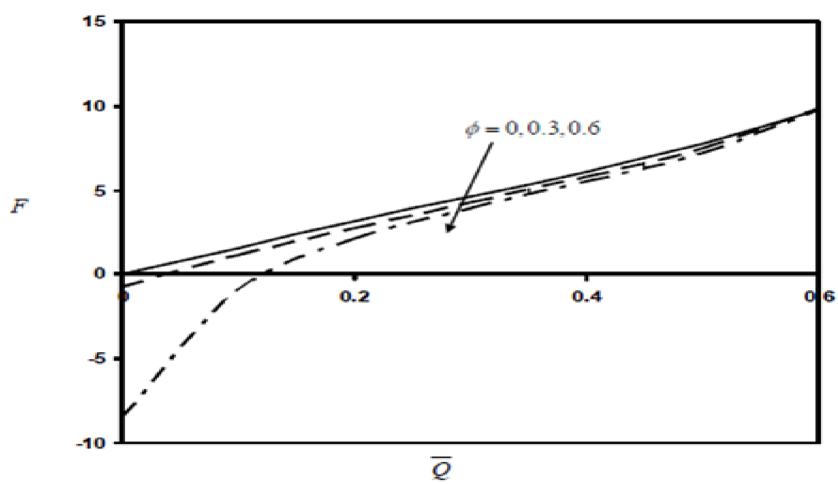

Fig. 12: The variation of friction force F with time averaged flux $\overline{Q}$ for different values of ϕ with $Da = 0.1, \varepsilon = 0.7$ and $\Gamma = 0.01$.

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