



Almost ν -open mappings

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ABSTRACT

The aim of the paper is to study basic properties of ν -open mappings and interrelations with other mappings.

Keywords: ν -open sets, ν -continuity, ν -irresolute, ν -open mappings and almost ν -open mappings.

Ams: 54C10, 54C08, 54C05.

1. INTRODUCTION:

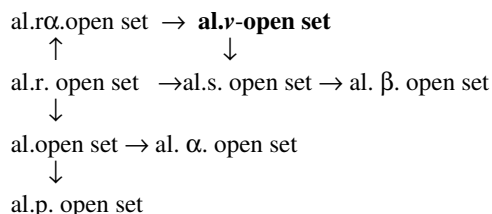
Mappings plays an important role in the study of modern mathematics, especially in Topology and Functional Analysis. Open mappings are one such mapping which are studied for different types of open sets by various mathematicians for the past many years. In this paper we tried to study a new variety of open maps called almost ν -open maps. Throughout the paper X, Y means a topological spaces (X, τ) and (Y, σ) unless otherwise mentioned without any separation axioms defined in it.

2. Preliminaries:

Definition 2.1: $A \subseteq X$ is called

- (i) pre-open if $A \subseteq (\text{cl } A)^\circ$ and pre-closed if $\text{cl}\{(A^\circ)\} \subseteq A$;
- (ii) semi-open if $A \subseteq \text{cl}\{(A^\circ)\}$ and semi-closed if $(\text{cl } A)^\circ \subseteq A$;
- (iii) semipre-open [β -open] if $A \subseteq \text{cl}\{(\text{cl } A)^\circ\}$ and semipre-closed [β -closed] if $(\text{cl}\{(A^\circ)\})^\circ \subseteq A$;
- (iv) α -open if $A \subseteq (\text{cl}\{(A^\circ)\})^\circ$ and α -closed if $\text{cl}\{(\text{cl } A)^\circ\} \subseteq A$;
- (v) regular open if $A = (\text{cl } A)^\circ$ and regular closed if $A = \text{cl}\{(A^\circ)\}$;
- (vi) ν -open if there exists a regular open set U such that $U \subseteq A \subseteq \text{cl } U$.
- (vii) $\nu\alpha$ -closed if there exists a regular closed set U such that $\alpha(U)^\circ \subseteq A \subseteq U$.
- (viii) g -closed [resp: rg -closed] if $\text{cl } A \subseteq U$ whenever $A \subseteq U$ and U is open [resp: regular open].

Note 1: From the above definition we have the following implication diagram.



Definition 2.2: A function $f: X \rightarrow Y$ is said to be

- (1) continuous [resp: pre-continuous; semi-continuous; β -continuous; α -continuous; nearly-continuous; ν -continuous; $\nu\alpha$ -continuous] if the inverse image of every open set is open [resp: pre-open; semi-open; β -open; α -open; regular-open; ν -open; $\nu\alpha$ -open]

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(2) irresolute [resp:pre-irresolute; β -irresolute; α -irresolute; nearly-irresolute; ν -irresolute; $r\alpha$ -irresolute] if the inverse image of every semi-open[resp:pre-open; β -open; α -open; regular-open; ν -open; $r\alpha$ -open] set is semi-open[resp:pre-open; β -open; α -open; regular-open; ν -open; $r\alpha$ -open]

(3) open [resp:pre-open; semi-open; β -open; α -open; nearly-open; $r\alpha$ -open] if the image of every open set is open[resp:pre-open; semi-open; β -open; α -open; regular-open; $r\alpha$ -open]

(4) almost open[resp:almost pre-open; almost semi-open; almost β -open; almost α -open; almost nearly-open; almost $r\alpha$ -open] if the image of every regular open set is open[resp:pre-open; semi-open; β -open; α -open; regular-open; $r\alpha$ -open]

(5) g-continuous [resp:rg-continuous] if the inverse image of every open set is g-open[resp:rg-open]

Definition 2.3: X is said to be $T_{1/2}[r-T_{1/2}]$ if every [regular-] generalized closed set is [regular-]closed

3. Almost ν -open mappings:

Definition 3.1: A function $f: X \rightarrow Y$ is said to be almost ν -open if image of every regular open set in X is ν -open in Y

Theorem 3.1:

- (i) Every almost-r-open map is almost ν -open but not conversely.
- (ii) Every almost-r-open map is almost $r\alpha$ -open but not conversely.
- (iii) Every almost $r\alpha$ -open map is almost ν -open but not conversely.
- (iv) Every almost ν -open map is almost semi-open but not conversely.
- (v) Every almost ν -open map is almost β -open but not conversely.
- (vi) Every almost-r-open map is almost open but not conversely.
- (vii) Every almost-r-open map is almost semi-open but not conversely.

Proof: (i) f is almost-r-open \rightarrow image of every regular open set is r-open \rightarrow image of every regular open set is ν -open [since every r-open set is ν -open] $\rightarrow f$ is almost ν -open.

Similarly we can prove the remaining parts using definition 2.1 and Note 1.

Example 1: Let $X = Y = \{a, b, c\}$; $\tau = \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f: X \rightarrow Y$ is identity map. Then f is almost ν -open.

Example 2: Let $X = Y = \{a, b, c\}$; $\tau = \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f: X \rightarrow Y$ be the map defined as $f(a) = b$; $f(b) = c$ and $f(c) = a$ is not almost ν -open.

Example 3: Let $X = Y = \{a, b, c\}$; $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and let $f: X \rightarrow Y$ is identity map. Then f is almost ν -open, almost semi-open, almost β -open and almost $r\alpha$ -open but not almost-open, almost-r-open, almost-pre-open and almost α -open.

Example 4: Let $X = Y = \{a, b, c\}$; $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, X\}$ and let $f: X \rightarrow Y$ is identity map. Then f is almost β -open but not almost ν -open, almost semi-open, almost $r\alpha$ -open, almost-open, almost-r-open, almost-pre-open and almost α -open.

Example 5: Let $X = Y = \{a, b, c\}$; $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, X\}$ and let $f: X \rightarrow Y$ be the map defined as $f(a) = c$; $f(b) = b$ and $f(c) = a$. Then f is almost β -open and almost-pre-open but not almost ν -open, almost semi-open, almost $r\alpha$ -open, almost-open, almost-r-open and almost α -open.

Example 6: Let $X = Y = \{a, b, c\}$; $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ and let $f: X \rightarrow Y$ be the map defined as $f(a) = b$; $f(b) = b$ and $f(c) = c$. Then f is almost α -open and almost-open but not almost ν -open.

Theorem 3.2:

- (i) If $R\alpha O(Y) = RO(Y)$, then f is almost $r\alpha$ -open iff f is almost-r-open.
- (ii) If $R\alpha O(Y) = \nu O(Y)$, then f is almost $r\alpha$ -open iff f is almost ν -open.
- (iii) If $\nu O(Y) = RO(Y)$, then f is almost-r-open iff f is almost ν -open.
- (iv) If $\nu O(Y) = \alpha O(Y)$, then f is almost α -open iff f is almost ν -open.

Corollary 3.1:

- (i) Every almost $r\alpha$ -open map is almost semi-open and hence almost β -open but not conversely

(ii) Every almost-r-open map is almost β -open but not conversely

Note 2:

- (i) almost open map and almost ν -open map are independent to each other
- (ii) almost α -open map and almost ν -open map are independent to each other
- (iii) almost pre-open map and almost ν -open map are independent to each other

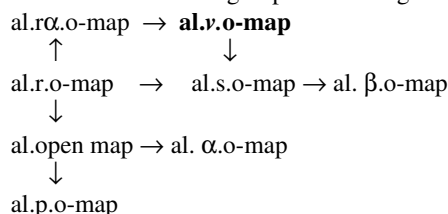
Example 6: In Example 2 above, f is almost open; almost pre-open and almost α -open but not almost ν -open.

Example 7: f as in Example 3 is almost ν -open but not almost open; almost pre-open and almost α -open.

Example 8: f as in Example 6 is almost open; almost pre-open and almost α -open but not almost ν -open.

Example 9: f as in Example 5 is almost β -open but not almost ν -open

Note 4: We have the following implication diagram among the open maps.



Theorem 3.3:

- (i) If f is almost open and g is ν -open then $g \circ f$ is almost ν -open
- (ii) If f is almost open and g is r -open then $g \circ f$ is almost ν -open
- (iii) If f and g are almost-r-open then $g \circ f$ is almost ν -open
- (iv) If f is almost-r-open and g is almost ν -open then $g \circ f$ is almost ν -open

Proof: (i) Let A be regular open set in $X \Rightarrow f(A)$ is open in $Y \Rightarrow g(f(A))$ is ν -open in $Z \Rightarrow g \circ f(A)$ is ν -open in $Z \Rightarrow g \circ f$ is almost ν -open.

Similarly we can prove the remaining parts and so omitted.

Corollary 3.2:

- (i) If f is almost open and g is almost ν -open[almost-r-open] then $g \circ f$ is almost semi-open and hence almost β -open
- (ii) If f and g are almost-r-open then $g \circ f$ is almost semi-open and hence almost β -open
- (iii) If f is almost-r-open and g is almost ν -open then $g \circ f$ is almost semi-open and hence almost β -open

Theorem 3.4: If $f: X \rightarrow Y$ is almost ν -open, then $f(A^\circ) \subset \nu(f(A))^\circ$

Proof: Let $A \subset X$ be regular open and $f: X \rightarrow Y$ is almost ν -open gives $f(A^\circ)$ is ν -open in Y and $f(A^\circ) \subset f(A)$ which in turn gives $\nu(f(A^\circ))^\circ \subset \nu(f(A))^\circ$ (1)

Since $f(A^\circ)$ is ν -open in Y , $\nu(f(A^\circ))^\circ = f(A^\circ)$ (2)

combining (1) and (2) we have $f(A^\circ) \subset \nu(f(A))^\circ$ for every subset A of X .

Remark 1: converse is not true in general.

Corollary 3.3: If $f: X \rightarrow Y$ is almost-r-open, then $f(A^\circ) \subset \nu(f(A))^\circ$

Theorem 3.5: If $f: X \rightarrow Y$ is almost ν -open and $A \subset X$ is regular open, then $f(A)$ is τ_ν -open in Y .

Proof: Let $A \subset X$ be regular open and $f: X \rightarrow Y$ is almost ν -open implies $f(A^\circ) \subset \nu(f(A))^\circ$ which in turn implies $f(A) \subset \nu(f(A))^\circ$, since $f(A) = f(A^\circ)$. But $\nu(f(A))^\circ \subset f(A)$. Combining we get $f(A) = \nu(f(A))^\circ$. Therefore $f(A)$ is τ_ν -open in Y .

Corollary 3.4: If $f: X \rightarrow Y$ is almost-r-open, then $f(A)$ is τ_ν -open in Y if A is r -open set in X .

Theorem 3.6: If $\nu(A)^{\circ} = r(A)^{\circ}$ for every $A \subset Y$, then the following are equivalent:

- (i) $f: X \rightarrow Y$ is almost ν -open map
- (ii) $f(A^{\circ}) \subset \nu(f(A))^{\circ}$

Proof: (i) \Rightarrow (ii) follows from theorem 3.4

(ii) \Rightarrow (i) Let A be any regular open set in X , then $f(A) = f(A^{\circ}) \subset \nu(f(A))^{\circ}$ by hypothesis. We have $f(A) \subset \nu(f(A))^{\circ}$. Combining we get $f(A) = \nu(f(A))^{\circ} = r(f(A))^{\circ}$ [by given condition] which implies $f(A)$ is r -open and hence ν -open. Thus f is almost ν -open.

Theorem 3.7: $f: X \rightarrow Y$ is almost ν -open iff for each subset S of Y and each regular open set U containing $f^{-1}(S)$, there is a ν -open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof: Assume f is almost ν -open, $S \subset Y$ and $U \in RO(X, f^{-1}(S))$, then $f(X - U) \in \nu O(Y)$ and $V = Y - f(X - U)$ is ν -open in Y . $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - U) = U$.

Conversely let F be a regular open in X , then $f^{-1}(f(F^{\circ})) \subset F^{\circ}$. By hypothesis, exists $V \in \nu O(Y)$ such that $f(F^{\circ}) \subset V$ and $f^{-1}(V) \subset F^{\circ}$ and so $f \subset (f^{-1}(V))^{\circ}$. Hence $V \subset f(F) \subset f[(f^{-1}(V))^{\circ}] \subset V^{\circ}$ implies $f(F) \subset V^{\circ}$, which implies $f(F) = V^{\circ}$. Thus $f(F)$ is ν -open in Y and therefore f is ν -open.

Remark 2: composition of two almost ν -open maps is not almost ν -open in general

Theorem 3.8: Let X, Y, Z be topological spaces and every ν -open set is r -open in Y , then the composition of two almost ν -open maps is almost ν -open.

Proof: Let A be regular open in $X \Rightarrow f(A)$ is ν -open in $Y \Rightarrow f(A)$ is r -open in Y [by assumption] $\Rightarrow g(f(A))$ is ν -open in $Z \Rightarrow g \circ f(A)$ is ν -open in $Z \Rightarrow g \circ f$ is almost ν -open.

Theorem 3.9: If $f: X \rightarrow Y$ is almost g -open; $g: Y \rightarrow Z$ is ν -open [r -open] and Y is $T_{1/2}$ [$r-T_{1/2}$], then $g \circ f$ is almost ν -open.

Proof: (i) Let A be regular open in $X \Rightarrow f(A)$ is g -open in $Y \Rightarrow f(A)$ is open in Y [since Y is $T_{1/2}$] $\Rightarrow g(f(A))$ is ν -open in $Z \Rightarrow g \circ f(A)$ is ν -open in $Z \Rightarrow g \circ f$ is almost ν -open.

(ii) Since every g -open set is rg -open, this part follows from the above case.

Corollary 3.5: If $f: X \rightarrow Y$ is almost g -open; $g: Y \rightarrow Z$ is ν -open [r -open] and Y is $T_{1/2}$ [$r-T_{1/2}$], then $g \circ f$ is almost semi-open and hence almost β -open.

Corollary 3.6: If $f: X \rightarrow Y$ is almost g -open; $g: Y \rightarrow Z$ is almost ν -open [almost r -open] and Y is $r-T_{1/2}$, then $g \circ f$ is almost semi-open and hence almost β -open.

Proof: Since every g -open set is rg -open, part follows from the above theorem.

Theorem 3.10: If $f: X \rightarrow Y$ is almost rg -open; $g: Y \rightarrow Z$ is ν -open [r -open] and Y is $r-T_{1/2}$, then $g \circ f$ is almost ν -open.

Proof: Let A be regular open in $X \Rightarrow f(A)$ is rg -open in $Y \Rightarrow f(A)$ is r -open in Y [since Y is $r-T_{1/2}$] $\Rightarrow f(A)$ is open in Y [since every r -open set is open] $\Rightarrow g(f(A))$ is ν -open in $Z \Rightarrow g \circ f(A)$ is ν -open in $Z \Rightarrow g \circ f$ is almost ν -open.

Theorem 3.11: If $f: X \rightarrow Y$ is almost rg -open; $g: Y \rightarrow Z$ is almost ν -open [almost r -open] and Y is $r-T_{1/2}$, then $g \circ f$ is almost ν -open.

Corollary 3.7: If $f: X \rightarrow Y$ is almost rg -open; $g: Y \rightarrow Z$ is ν -open [r -open] and Y is $r-T_{1/2}$, then $g \circ f$ is almost semi-open and hence almost β -open.

Corollary 3.8: If $f: X \rightarrow Y$ is almost rg -open; $g: Y \rightarrow Z$ is almost ν -open [almost r -open] and Y is $r-T_{1/2}$, then $g \circ f$ is almost semi-open and hence almost β -open.

Theorem 3.12: If $f: X \rightarrow Y$; $g: Y \rightarrow Z$ be two mappings such that $g \circ f$ is ν -open [r -open]. Then the following are true

- (i) If f is continuous [r -continuous] and surjective, then g is ν -open
- (ii) If f is g -continuous, surjective and X is $T_{1/2}$, then g is ν -open
- (iii) If f is g -continuous [rg -continuous], surjective and X is $r-T_{1/2}$, then g is ν -open

Proof: (i) Let A be regular open in $Y \Rightarrow f^{-1}(A)$ is open in $X \Rightarrow g \circ f(f^{-1}(A))$ is ν -open in $Z \Rightarrow g(A)$ is ν -open in $Z \Rightarrow g$ is almost ν -open.

Similarly we can prove the remaining parts and so omitted.

Corollary 3.9: If $f: X \rightarrow Y; g: Y \rightarrow Z$ be two mappings such that $g \circ f$ is ν -open[r-open]. Then the following are true

- (i) If f is continuous[r-continuous] and surjective, then g is semi-open and hence β -open
- (ii) If f is g -continuous, surjective and X is $T_{1/2}$, then g is semi-open and hence β -open
- (iii) If f is g -continuous[rg-continuous], surjective and X is $r-T_{1/2}$, then g is semi-open and hence β -open

Theorem 3.13: If X is ν -regular, $f: X \rightarrow Y$ is r -closed, nearly-continuous, ν -open surjection and $A^\circ = A$ for every ν -open set in Y , then Y is ν -regular.

Proof: Let $p \in U \in \nu O(Y)$, there exists a point $x \in X$ such that $f(x) = p$ by surjection. Since X is ν -regular and f is nearly-continuous exists $V \in RC(X)$ such that

$$x \in V^0 \subset V \subset f^{-1}(U) \text{ which implies } p \in f(V^0) \subset f(V) \subset U \quad (1)$$

$$\text{for } f \text{ is } \nu\text{-open, } f(V^0) \subset U \text{ is } \nu\text{-open. By hypothesis } f(V^0)^\circ = f(V^0) \text{ and } f(V^0)^\circ = \{f(V)\}^\circ \quad (2)$$

combining (1) and (2) $p \in f(V)^\circ \subset f(V) \subset U$ and $f(V)$ is r -closed. Hence Y is ν -regular.

Corollary 3.10: If X is ν -regular, $f: X \rightarrow Y$ is r -open, nearly-continuous, ν -open surjection and $A^\circ = A$ for every r -open set in Y , then Y is ν -regular.

Theorem 3.14: If $f: X \rightarrow Y$ is almost ν -open[almost- r -open] and

- (i) A is regular open set of X , then $f_A: (X, \tau_A) \rightarrow (Y, \sigma)$ is ν -open.
- (ii) X is $T_{1/2}$ and A is g -open set of X , then $f_A: (X, \tau_A) \rightarrow (Y, \sigma)$ is almost ν -open.

Proof: Let F be r -open set in A . Then $f = A \cap E$ for some r -open set E of X and so F is r -open in X which implies $f(A)$ is ν -open in Y . But $f(F) = f_A(F)$ and therefore f_A is ν -open.

Corollary 3.11: If $f: X \rightarrow Y$ is almost ν -open[almost- r -open] and

- (i) A is regular open set of X , then $f_A: (X, \tau_A) \rightarrow (Y, \sigma)$ is semi-open and hence β -open.
- (ii) X is $T_{1/2}$ and A is g -open set of X , then $f_A: (X, \tau_A) \rightarrow (Y, \sigma)$ is almost semi-open and hence almost β -open.

Theorem 3.15: If $f_i: X_i \rightarrow Y_i$ be almost ν -open[almost- r -open] for $i = 1, 2$. Let $f: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is almost ν -open.

Proof: Let $U_1 \times U_2 \subset X_1 \times X_2$ where U_i is regular open in X_i for $i = 1, 2$. Then $f(U_1 \times U_2) = f_1(U_1) \times f_2(U_2)$ a ν -open set in $Y_1 \times Y_2$. Thus $f(U_1 \times U_2)$ is ν -open and hence f is almost ν -open.

Corollary 3.12: If $f_i: X_i \rightarrow Y_i$ be ν -open[r-open] for $i = 1, 2$. Let $f: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be defined as follows:

$f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is almost semi-open and hence almost β -open.

Theorem 3.16: Let $h: X \rightarrow X_1 \times X_2$ be almost ν -open[almost- r -open]. Let $f_i: X \rightarrow X_i$ be defined as $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$. Then $f_i: X \rightarrow X_i$ is almost ν -open for $i = 1, 2$.

Corollary 3.13: Let $h: X \rightarrow X_1 \times X_2$ be almost ν -open[almost- r -open]. Let $f_i: X \rightarrow X_i$ be defined as $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$. Then $f_i: X \rightarrow X_i$ is almost semi-open and hence almost β -open for $i = 1, 2$.

CONCLUSION: We studied some properties and interrelations of almost ν -open mappings.

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