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A NOTE ON INVARIANT SUBSPACES OF SOME OPERATORS IN HILBERT SPACE

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ABSTRACT

In this paper, we show that if M is a nontrivial invariant for both T and S, then M is ST −invariant or $TS -$ invariant. An example is provided to illustrate that if M is TS – invariant, then it is not necessarily invariant for either T and S. However if TS, S and T have same structure and M is invariant for TS, then it is also invariant for T and S.

Keywords: *Invariant subspaces, Nilpotent operators.*

1. INTRODUCTION

The invariantsubspaces of an operator plays a central role in operator they as they give information on the structure of the operator. They are a direct analogue of the eigen-vectors of a linear operator. The motivation behind the study of invariant subspaces come from the interest of structure of operators and from approximation theory. Let H be a Hilbert space and $B(H)$ denotes all bounded linear operators on H. A subspace M of H is a invariant under operator Tif $T\,(M) \subseteq M$, that is,$x \in M$ for every$Tx \in M$ or $TM \subset M.$ If T is any subset of $B(H)$, we denote by $\{T\}$' the commutant of T, that is $\{T\}' = \{T \in B\,(H): ST = TS\}$.

A subspace $M \subset H$ is said to be nontrivial hyper-invariant subspace (n.h.s) for a fixed operator in $T \in B(H)$ if $0 \neq M \neq H$ and $SM \subset M$ for each $S \in \{T\}'$.An operator $T \in B(H)$is nilpotentif $T^n = 0$.

Theorem 1.1: If $T \in B(H)$,then the following subspaces are invariant under T:

(i) $\{0\}$.
(ii) H.
(iii) $Ker\,(T)$
(iv) $Ran\,(T)$

Proof:

(i) If $x \in \{0\}$, then$x = 0$ hence $Tx = 0 \in \{0\}$. Thus $\{0\}$ is invariant under T.
(ii) If $x \in H$,then $Tx \in H$ since$T$on Hilbert space H is bounded, then it is bounded below and hence its range is closed. Thus H is invariant under$T$.
(iii) If $x \in Ker(T), then\ Tx = 0$ and hence$Tx \in Ker\,(T)$. Thus $Ker\,(T)$ is invariant under T.
(iv) Note that, since the operators T on a Hilbert space H is bounded below and hence its range is closed subspace of H. Thus $T(Ran(T))$ is contained in $Ran(T)$. Let $x \in Ran(T)$, then $Tx \in Ran(T)$. Thus$Ran(T)$is invariant under T.

Lemma 1.2: Let $U_1, U_2 \subset H$be invariant subspaces. Then $U_1 \cap U_2$and $U_1 + U_2$are invariant subspaces.

Proof: SupposeU_1 and U_2 are both under T , and $u \in U_1 \cap U_2$. Since U_1 is invariant under T, $t$hen $T\,(u) \in U_1$.

Similarly, sinceU_2 is invariant under T, then $T\,(u) \in U_2$and so

$$T(u) \in U_1 \cap U_2. \text{ Thus } U_1 \cap U_2 \text{ is invariant under T.}$$

Suppose$u \in U_1 + U_2$. Then$u = u_1 + u_2$ where $u_i \in U_i$ for $i = 1, 2$. Applying the linear operator on both sides of the equation we have

$$T(u) = T(u_1 + u_2) = T(u_1) + T(u_2).$$

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Because U_1, U_2 are all invariant subspace under T, and since $u_i \in U_i$ we have T$(u_i) \in U_i$

For $i = 1,2$. Hence T (u) is contained in $U_1 + U_2$ and therefore $U_1 + U_2$ is invariant under T.

Proposition 1.3: Let T and L be nonzero on a Hilbert space H. If $LT = 0$, then $Ker(L)$ and $Ran\,(T)$ are nontrivial invariant subspaces for both $T$and L.

Proof: If LT=0, then Ran (T) ⊆ Ker (L). Hence $T(\,Ker(L))$ ⊆T (H)= $Ran\ (T)\ \subseteq Ker\ (L)$.
Since $T \neq 0, Ran\ (T) \neq 0, so\ that\ Ker(L) \neq 0. Since\ L \neq 0\ \ Ker(L) \neq H$. Therefore $Ker\ (L)$ is nontrivial invariant subspace for T. Dually since T*L*=0, L*≠ 0 it follows that $Ker\ (T^*)^{\perp}$ is nontrivial invariant subspace for L^*, and hence $Ran(T) = Ker\ (T^*)^{\perp}$ is a nontrivial invariant subspace for L.

Remark 1.1: $Ker(L)$ and $\overline{Ran(T)}$ are invariant subspaces for L and T.

Proposition 1.3 leads to the following result.

Corollary 1.1: Every nilpotent operator has a nontrivial invariant subspace.

Proof: Recall that, an operator is nilpotent if $T^n = 0$. Thus $T^n = T(T^{n-1})$ which can be written as a product of two operators and by Proposition 1.3 Ker(T) and $\overline{Ran(T^{n-1})}$are nontrivial invariant subspaces.

Proposition 1.4: Let$T \in B(H)$and M be subspace of a Hilbert space H. If M is $T-$invariant, Then $(T|_M)^* = PT^*|_M$ where $P$is the orthogonal projection of H onto M.

Proof: Let M be an invariant subspace for T so that $T\ (M) \subseteq\ M$, and let P be the orthogonal projection onto M.

Since $P\ v\ = v$ for every $v \in M\ and$ using the fact that $P$is self-adjoint,
we have $\langle (T|_M)^* u, v\rangle = \langle u, T|_M v\rangle = \langle u, Tv\rangle = \langle u, TPv\rangle = \langle PT^* u, v\rangle = \langle PT^*|_M u, v\rangle$ forevery $u, v \in M$,
hence $(T|_M)^* = PT^*|_M$.

Proposition 1.5: Let $T,\ S \in B(H)$and M be a nontrivial invariant subspace for both $T$and S. Then M is $TS -$ invariant.

Proof: If M is invariant for both T and S then we have $T(M)\ \subseteq\ M$ and $S(M) \subseteq M$.
Thus we have $TSM = T(SM) \subseteq T(\,M) \subseteq M$. Therefore M is TS – invariant.

Proposition 1.6: Let $T, S\ \in B(H)$ and Mbe a nontrivial invariant subspace for both T and S. Then M is ST –invariant

Proof: If M is invariant for both $T$and S, then we have $T\ (M)\ \subseteq\ M$ and $S\ (M)\ \subseteq M$.
Thus we have $STM =\ S(TM)\ \subseteq S(M)\ \subseteq\ M$. Therefore M is $ST -$ invariant.

Question: If M is TS $-$invariant, is it true that M is T $-$invariant or $S -$ invariant?

Answer: We answer this question with the following example.
Let $TS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We observe that $Lat\ (TS)\ =\ \left\{\{0\}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, R^2\right\}$. However TS can be written, not uniquely, as a product of $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$and $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We notice that $M =\ span\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is invariant for TS but it is not invariant for T and S.
This leads to the following remarks:

Remark 1.2: Let M be subspace of a Hilbert space H and $T, S \in B(H)$. If M is $TS -$ invariant, then M is not necessarily $T -$ or S $-$invariant.

However if $TS, T$and S have the same structure, then if$M$is $TS -$ invariant the M is also invariant for both T and S.

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