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# *Double Full Subsets By m Of Z*

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***ABSTRACT***

*Let A be a subset of Z such that $A= A^+ \cup A^-$, where $A^+ =\{a_1,\dots, a_k\}$, $A^- =\{-a_k,\dots, -a_1\}=\{b_k,\dots, b_1\}$ and $a_i \geq 0$, $a_1<\dots <a_k$. We say that A is double full by m if $\sum A^+= [m]$ and $\sum A^- =[-m]$ for a positive integer m, where $\sum A^+$ is the set of all positive integers and $\sum A^-$ is the set of all negative integers. We show that a set $A^+$ is full if and only if $a_1=1$ and $a_i \leq + \dots + a_{i-1} +1$ for each i, $2 \leq i \leq k$ and $A^-$ is full if and only if $b_1= -1$ and $b_i \geq b_1 + \dots + b_{i-1}-1$ for each i, $2 \leq i \leq k$.*

*We also prove that for each integer $m \notin \{\pm 2, \pm 4, \pm 5, \pm 8, \pm 9\}$ there is an double full by m set. We also give formula for F(m), the number of m full sets of $Z^+$ and F(-m), the number of -m full sets of $Z^-$.*

***Keywords:*** *Double Full, Double Full By m, Partition Of Integer*

## 1. INTRODUCTION:

Let n be a positive integer and denote by D (n) and σ(n) the set of its positive divisors and the sum of its positive divisors, respectively.

Let A be a subset of Z. Define the sum set of A, dented by $\sum A^{+,-}$ :

$$\sum A^+ = \{ a_{i_1} + \dots + a_{i_r} : a_{i_1} < \dots < a_{i_r} , 1 \leq r \leq$$
$$\sum A^- = \{ b_{i_1} + \dots + b_{i_r} : b_{i_1} > \dots > b_{i_r} , 1 \leq r \leq k\}$$

For what positive integer m does there exist a set $A= A^+ \cup A^-$ with $\sum A^+=[m]$ and $\sum A^-= [-m]$, where $[m] =\{1, \dots , m\}$ and $[-m] = \{-1\dots-m\}$?

We show that each integer $m \notin \{\pm 2, \pm4, \pm5, \pm8, \pm 9\}$ has this property and determine the numbers:

$\alpha(m) = \min\{|A| : \sum A^+ =[m]\}$.

$\beta(m) = \max\{|A| : \sum A^+ =[m]\}$.

$L(m) = \min \{\max A^+ : \sum A^+ = [m]\}$.

$U(m) =\max \{\max A^+ : \sum A^+ = [m]\}$.

We define α (-m), β (-m), L (-m) and U (-m) similar as above.

**Example 1:** If m=1, then A= {1} is 1 full subset of $Z^+$.

**Example 2:** If m=3, then A= {1, 2} is 3 full subset of $Z^+$, because (i) $a_1=1$ and $\sum A= [3]$, (ii) $a_2=2 \leq a_1+1=2$

**Example 3:** If m=-6 then A= {-1,-2,-3} is -6 full subset of $Z^-$, because (i) $a_1=-1$ and $\sum A= [-6]$, (ii) $a_2=-2\geq a_1-1=-1-1$ and $a_3=-3 \geq a_1+a_2-1=-4$.

**Example 4:** If A= {±1, ±2}, then A is double full subset of Z, because $A^-= \{-2,-1\}$ is -3 full subset of $Z^-$ and $A^+= \{1, 2\}$ is 3 full subset of $Z^+$, this means that A is double full by 3.

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**2. THE RESULTS:**

**Definition 1:** Let m be a positive integer. A subset $A=A^+ \cup A^-$ of Z is called double full by m if $\sum A^+ = [m]$ and $\sum A^- = [-m]$. A is called double full if it is double full by m for some positive integer m.

**Theorem 1:**.A subset $A=A^+ \cup A^-$ of Z where $A^+= \{a_1,\dots,a_k\}$, $A^-=\{-a_k,..,a_1\}=\{b_k,\dots,b_1\}$ with $a_1<\dots<a_k$.

**(i)** $A^+$ is full if and only if $a_1=1$ and $a_i \le a_1+\dots+ a_{i-1} + 1$ for each i, $2\le i\le k$ and

**(ii)** $A^-$ is full if and only if $b_1=-1$ and $b_i \ge b_1+\dots+ b_{i-1}- 1$ for each i, $2\le i\le k$.

**Proof:** Let $A=A^+ \cup A^-$ be double full and $\sum A^+ = [m]$, $\sum A^- = [-m]$ for a positive integer m.

**(i)** Its shown in [3].

**(ii)** Clearly $b_1=1$. If $b_j > b_1+\dots+ b_{j-1}- 1$ for some j, $2\le j\le k$, then $b_1 +\dots+ b_{j-1}- 1$ is not of a sum of distinct elements of $A^-$.

But $-m = b_1+\dots+b_k \le b_1+\dots+ b_{j-1}- 1\le -1$. This contradicts to the fact that $\sum A^- = [-m]$.

Conversely, suppose that $b_1=-1$ and $b_i \ge b_1+\dots+ b_{i-1}- 1$ for each i, $2\le i\le k$. We claim that $\sum A^- = [b_1+\dots+b_k]$.

We prove this by induction on k. For k=1 the result is obvious. Suppose that the result is true for k-1.
Then $\sum A^- \setminus \{b_k\} = [b_1+\dots+b_{k-1}]$.

Now suppose that $b_1+\dots+b_k\le L\le b_1+\dots+b_{k-1} -1$ and write $L=b_k+b$. If $b = 0$, then $L=b_k \in \sum A^-$ and if $b \neq 0$, then $b \in [b_1+\dots+b_{k-1}] = \sum A^- \setminus \{b_k\}$. Thus $L \in \sum A^-$.

**Proposition 1(i):** Let $n = p_1^{\alpha_1}\dots p_r^{\alpha_r}$ , with $p_1<\dots<p_r$ primes, be a positive integer. Then $D(n) = \{d \in Z^+: d|n\}$ is full if and only if $p_1=2$ and $p_i\le \sigma(p_1^{\alpha_1}\dots p_r^{\alpha_r}) +1$ for each i, $2\le i\le r$, and

**(ii)** Let $n=-p_1\dots p_r$, with $p_1<\dots<p_r$ primes, be a positive integer. Then

$D(n) = \{d \in Z^- : d|n\}$ is full if and only if $p_1=2$ and $p_i\le \sigma(p_1^{\alpha_1}\dots p_r^{\alpha_r}) +1$ for each i, $2\le i\le r$.

**Proof (i):** If D (n) is m full, then $m=\sigma(n)$. since $p_1^{\alpha_1}\dots p_r^{\alpha_r} | n$ and $p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}} \neq n$.

We have $\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}}) < \sigma(n)$.

Hence $\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}}) +1$ is a member of $[\sigma(n)]$. Thus if

$p_i>\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ for some i , then the member $\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ is a member of $[\sigma(n)]$ which is not a sum of distinct elements of D(n). On the other hand , if the condition $p_i< \sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ for each i , $2\le i\le r$, is satisfied, then using an argument similar to the one used in theorem 1 , we can inductively prove that each element of $[\sigma(n)]$ can be written as a sum of distinct elements of D(n).

**(ii)** This part proved just like as part (i).

**Definition 2:** Define F (m) is number of m full sets of $Z^+$ and F (-m) is the number of -m full sets of $Z^-$.

**Theorem 2:** Let m be a positive integer. There is a set A such that $\sum A^+ = [m]$ and $\sum A^- = [-m]$ if and only if
$m \neq \{2,4,5,8,9\}$.

**Proof:** Its shown in [3].

**Example 5 D:** $(6) = \{1, 2, 3, 6\}$ is 12 full subset of $Z^+$, because (i) $P_1=2$ and (ii) $3\le \sigma(2) +1$.

**Example 6** D: $(-6) = \{-6,-3,-2,-1\}$ is -12 full subset of $Z^-$ , because (i) $p_1=2$ and (ii) $3\le \sigma(2)+1$ or $-3\ge -\sigma(2)-1$.

**Example 7:** For m=12 and m=-12 note that $A^+=D(6)=\{1,2,3,6\}$ and $A^- = D(-6) = \{-6,-3,-2,-1\}$ are 12 full and -12 full subsets of Z, respectively . Therefore, $A=A^+ \cup A^-$ is double full by 12 set of Z.

 

**Theorem 3 (i):** If $\alpha(m) = \min\{|A| : \sum A^+ = [m]\}$, $\beta(m) = \max\{|A| : \sum A^+ = [m]\}$. Then

$\alpha(m) = [\log_2(m+1)]$ ,

$\beta(m) = \max\{l : \frac{l(l+1)}{2} \leq m\}$.

**(ii)** If $m \neq \{2,4,5,8,9,14\}$and $L(m) = \min\{\max A : \sum A^+ = [m]\}$.and $m = \frac{n(n+1)}{2} + r$ ,where r = 0,1,...,n, then

$$L(m) = \begin{cases} n & r = 0 \\ n+1 & 1 \leq r \leq n-2 \\ n+2 & 1 \leq r = n-1 \text{ or } n \end{cases}$$

**(iii)** If $n \geq 20$ and $U(m) = \max\{\max A$: then $\sum A^+ = [m]\}$, then $U(m) = [\frac{m}{2}]$.

**Proof:** Its shown in [2].

**Theorem 4:**.Let m be a positive integer and F(m, i) denote the number of m full sets A with max A=i, where $L(m) \leq i \leq U(m)$ , then

$F(m, i) = \sum_{j=L(m-i)}^{\min\{U(m-i), i-1\}} F(m-i, j)$

**Proof:** Its shown in [3].

**Example 8:**.By definition for L (m) and U (m), in [1]. We have

| m | 1 | 3 | 6 | 7 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L(m) | 1 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 7 | 5 | 6 | 6 | 6 | 7 | 7 |
| U(m) | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 7 | 8 | 6 | 7 | 8 | 9 | 10 |

**Theorem 5 (i):** Let me be a positive integer and denote the number of m full sets A by F (m). Then

$F(m, i) = \sum_{j=L(m)}^{U(m)} F(m, i)$

**(ii)** F (m) =F (-m)

**Proof (i):** Its shown in [3].

**(ii)** Its obvious by definition.

**Corollary 1:** By theorem 5, the number of double full by m sets of Z are $2\sum_{j=L(m)}^{U(m)} F(m, i)$.

**Example 9:** By definition of F (m), the first few values of F (m) are

| m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| F(m) | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 3 | 4 | 5 |

**Example 10:** For evaluate F(21) by using theorem 4 and theorem 5 , we have

$$F(21) = \sum_{j=L(21)}^{U(21)} F(21, i) = \sum_{i=6}^{12} F(21, i) = F(21,6) + F(21,7) + \cdots + F(21,12)$$
$$= F(15,5)+F(13,6)+F(13,7)+F(12,5)+F(12,6)+F(11,5)+F(10,4)=7.$$

This means that, there are the seven 21 full sets are

{1,2,3,4,5,6} , {1,2,4,6,8} , {1,2,3,7,8} , {1,2,4,5,9} , {1,2,3,6,9}, {1,2,3,5,10} , {1,2,3,4,11}

**Example 11:** For evaluate F (-6), by using theorem 4 and theorem 5 , we have

F (-6) =F (6), so by theorem 5, we have

 

$$F(6) = \sum_{j=L(6)}^{U(6)} F(6, i) = \sum_{i=3}^{3} F(6, i)$$

$$= \sum_{j=L(6-3)}^{\min\{U(6-3),3-1\}} F(6-3, j) = \sum_{j=2}^{2} F(3, j) = F(3,2) = \sum_{j=L(3-2)}^{\min\{U(1),1\}} F(1, j) = \sum_{1}^{1} F(1,1) = 1$$

This means that, there is a -6 full set such that define as follows; A= {-1,-2,-3}

**Example 12:** The number of double full by 6 sets of Z is one and define as follows; A= {±1, ±2, ±3}.

## 3. ACKNOWLEDGMENTS:

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