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# PARAMETER ESTIMATION OF FRECHET DISTRIBUTION

ARUN KUMAR RAO[1] & HIMANSHU PANDEY[2*]

[1,2]Department of Mathematics & Statistics,
DDU Gorakhpur University, Gorakhpur, INDIA.

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*ABSTRACT*

*Frechet distribution is considered. Bayesian method of estimation is employed in order to estimate the scale parameter of inverted exponential distribution by using quasi and gamma priors. In this paper, the Bayes estimators of the scale parameter have been obtained under squared error, precautionary and weighted loss functions.*

***Keywords:*** *Frechet distribution, Bayesian method, quasi and gamma priors, squared error, precautionary and weighted loss functions.*

## 1. INTRODUCTION

Frechet distribution is also known as extreme value distribution of type-II. Kotz and Nadarajah [1] describe this distribution and discussed its applicability in accelerated life testing. Ramos *et al.* [2] consider the problem of estimating the parameters of the Frechet distribution from both frequentist and Bayesian points of view. The cumulative distribution function of Frechet distribution is given by

$$F\left(x;\theta\right)=e^{-\theta x^{-a}} \quad ; x\geq 0,\ \theta>0. \tag{1}$$

Therefore, the probability density function of Frechet distribution is given by

$$f\left(x;\theta\right)=a\theta x^{-(a+1)}e^{-\theta x^{-a}} \quad ; x\geq 0,\ \theta>0. \tag{2}$$

The joint density function or likelihood function of (2) is given by

$$f\left(\underline{x};\theta\right)=\left(a\theta\right)^{n}\left(\prod_{i=1}^{n}x_i^{-(a+1)}\right)e^{-\theta\sum_{i=1}^{n}\left(x_i\right)^{-a}}. \tag{3}$$

The log likelihood function is given by

$$log\ f\left(\underline{x};\theta\right)=n\,log\ a+n\,log\ \theta+log\left(\prod_{i=1}^{n}x_i^{-(a+1)}\right)-\theta\sum_{i=1}^{n}\left(x_i\right)^{-a} \tag{4}$$

Differentiating (4) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given as

$$\hat{\theta}=\frac{n}{\sum_{i=1}^{n}\left(x_i\right)^{-a}}. \tag{5}$$

## 2. BAYESIAN METHOD OF ESTIMATION

In Bayesian analysis the fundamental problem are that of the choice of prior distribution g (θ) and a loss function $L\left(\hat{\theta},\theta\right)$. The squared error loss function for the scale parameter θ are defined as

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^{2}. \tag{6}$$

*Corresponding Author: Himanshu Pandey[2*]*
*[1,2]Department of Mathematics & Statistics, DDU Gorakhpur University, Gorakhpur, INDIA.*

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean,

*i.e,* $$\hat{\theta}_S = E(\theta). \tag{7}$$

This loss function is often used because it does not lead to extensive numerical computations but several authors (Zellner [3], Basu and Ebrahimi [4]) have recognized that the inappropriateness of using symmetric loss function. J.G.Norstrom [5] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}. \tag{8}$$

Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_P$ and is obtained by solving the following equation.

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2}. \tag{9}$$

Weighted loss function (Ahamad *et al*. [6]) is given as

$$L\left(\hat{\theta},\theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\theta}. \tag{10}$$

The Bayes estimator under weighted loss function is denoted by $\hat{\theta}_W$ and is obtained as

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{11}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.
(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \ ; \theta > 0,\ d \geq 0, \tag{12}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.
(ii) Gamma prior: The most widely used prior distribution of θ is the gamma distribution with parameters $\alpha$ and $\beta(>0)$ with probability density function given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \ ; \theta > 0. \tag{13}$$

**3. BAYES ESTIMATORS UNDER $g_1(\theta)$**

The posterior density of θ under $g_1(\theta)$, on using (3), is given by

$$f(\theta/\underline{x}) = \frac{(a\theta)^n\left(\prod_{i=1}^n x_i^{-(a+1)}\right)e^{-\theta\sum_{i=1}^n (x_i)^{-a}}\theta^{-d}}{\int_0^\infty (a\theta)^n\left(\prod_{i=1}^n x_i^{-(a+1)}\right)e^{-\theta\sum_{i=1}^n (x_i)^{-a}}\theta^{-d}d\theta}$$

$$= \frac{\theta^{n-d}\ e^{-\theta\sum_{i=1}^n (x_i)^{-a}}}{\int_0^\infty \theta^{n-d}\ e^{-\theta\sum_{i=1}^n (x_i)^{-a}}d\theta}$$

$$= \frac{\left(\sum_{i=1}^n (x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)}\theta^{n-d}\ e^{-\theta\sum_{i=1}^n (x_i)^{-a}} \tag{14}$$

 

**Theorem 1:** Assuming the squared error loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_S = \frac{n-d+1}{\sum_{i=1}^{n}(x_i)^{-a}} . \tag{15}$$

**Proof:** From equation (7), on using (14),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{n-d+1}\, e^{-\theta\sum_{i=1}^{n}(x_i)^{-a}}\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+2)}{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+2}}$$

or, $\hat{\theta}_S = \dfrac{(n-d+1)}{\sum_{i=1}^{n}(x_i)^{-a}}$ .

**Theorem 2:** Assuming the precautionary loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_P = \frac{\left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}}}{\sum_{i=1}^{n}(x_i)^{-a}} \tag{16}$$

**Proof:** From equation (9), on using (14),

$$\left(\hat{\theta}_P\right)^2 = E(\theta^2) = \int \theta^2 f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{n-d+2}\, e^{-\theta\sum_{i=1}^{n}(x_i)^{-a}}\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+3)}{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+3}}$$

$$= \frac{(n-d+2)(n-d+1)}{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^2}$$

$$\Rightarrow \qquad \hat{\theta}_P = \frac{\left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}}}{\sum_{i=1}^{n}(x_i)^{-a}} .$$

 

**Theorem 3:** Assuming the weighted loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_W = \frac{n-d}{\sum_{i=1}^{n}(x_i)^{-a}} \ . \tag{17}$$

**Proof:** From equation (11), on using (14),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f\left(\theta/\underline{x}\right) d\theta\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{n-d-1} e^{-\theta\sum_{i=1}^{n}(x_i)^{-a}} d\theta\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d)}{\left(\sum_{i=1}^{n}(x_i)^{-a}\right)^{n-d}}\right]^{-1}$$

$$= \left[\frac{\sum_{i=1}^{n}(x_i)^{-a}}{n-d}\right]^{-1}$$

or, $$\hat{\theta}_W = \frac{n-d}{\sum_{i=1}^{n}(x_i)^{-a}} \ .$$

## 4. BAYES ESTIMATORS UNDER $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (3), is obtained as

$$f\left(\theta/\underline{x}\right) = \frac{(a\theta)^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-\theta\sum_{i=1}^{n}(x_i)^{-a}} \dfrac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}}{\int_0^\infty (a\theta)^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-\theta\sum_{i=1}^{n}(x_i)^{-a}} \dfrac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1} e^{-\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)\theta}}{\int_0^\infty \theta^{n+\alpha-1} e^{-\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1} e^{-\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)\theta}}{\Gamma(n+\alpha) \Big/ \left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)^{n+\alpha}}$$

$$= \frac{\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)\theta} \tag{18}$$

 

**Theorem 4:** Assuming the squared error loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_S = \frac{n+\alpha}{\beta + \sum_{i=1}^{n} (x_i)^{-a}} \; . \tag{19}$$

**Proof:** From equation (7), on using (18),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{n+\alpha}\, e^{-\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)\theta} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha+1)}{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^{n+\alpha+1}}$$

or, $$\hat{\theta}_S = \frac{n+\alpha}{\beta + \sum_{i=1}^{n} (x_i)^{-a}} .$$

**Theorem 5:** Assuming the precautionary loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_P = \frac{\left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}}}{\beta + \sum_{i=1}^{n} (x_i)^{-a}} \; . \tag{20}$$

**Proof:** From equation (9), on using (18),

$$\left(\hat{\theta}_P\right)^2 = E(\theta^2) = \int \theta^2 f(\theta/\underline{x}) d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{n+\alpha+1}\, e^{-\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)\theta} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha+2)}{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^{n+\alpha+2}}$$

$$= \frac{(n+\alpha+1)(n+\alpha)}{\left(\beta + \sum_{i=1}^{n} (x_i)^{-a}\right)^2}$$

or, $$\hat{\theta}_P = \frac{\left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}}}{\beta + \sum_{i=1}^{n} (x_i)^{-a}} \; .$$

 

**Theorem 6:** Assuming the weighted loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_W = \frac{n+\alpha-1}{\beta+\sum_{i=1}^{n}(x_i)^{-a}} \tag{21}$$

**Proof:** From equation (11), on using (18),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f\left(\theta/\underline{x}\right) d\theta\right]^{-1}$$

$$= \left[\frac{\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{n+\alpha-2}\, e^{-\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)\theta} d\theta\right]^{-1}$$

$$= \left[\frac{\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha-1)}{\left(\beta+\sum_{i=1}^{n}(x_i)^{-a}\right)^{n+\alpha-1}}\right]^{-1}$$

$$= \left[\frac{\beta+\sum_{i=1}^{n}(x_i)^{-a}}{n+\alpha-1}\right]^{-1}$$

or, $$\hat{\theta}_W = \frac{n+\alpha-1}{\beta+\sum_{i=1}^{n}(x_i)^{-a}} .$$

## CONCLUSION

In this paper, we have obtained a number of estimators of parameter of Frechet distribution. In equation (15), (16) and (17) we have obtained the Bayes estimators under squared error, precautionary and weighted loss functions using quasi prior. In equation (19), (20) and (21) we have obtained the Bayes estimators under squared error, precautionary and weighted loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

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