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HOMOMORPHISM OR ANTI- HOMOMORPHISM OF LEFT (α, 1) - DERIVATIONS IN PRIME RINGS

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ABSTRACT

Let R be a 2- torsion free ring and let U be a Lie ideal of R. Suppose that α, 1 are automorphisms of R. An additive mapping d: R→ R is said to be a left (α, 1)-derivation (resp. Jordan left (α, 1)-derivation) of R if $d(xy) = \alpha(x)d(y) + yd(x)$ (resp. $d(x^2) = \alpha(x)d(x) + xd(x)$) holds for all x, y∈ R. In this paper it is established that if R admits a nonzero left (α, 1)-derivation which acts as a homomorphism or as an anti-homomorphism on I of R, then d = 0 on R. Also we prove that if G: R→ R is an additive mapping satisfying $G(xy) = \alpha(x)G(y) + yd(x)$ for all x, y∈ R and a left (α, 1)-derivation d of R such that G also acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then either R is commutative or d = 0 on R.

Keywords: *Prime ring, Lie ideal, (α, 1)-derivation, Jordan (α, 1)-derivation, Left (α, 1)-derivation, Jordan left (α, 1)-derivation, Generalized left (α, 1)-derivation, Generalized Jordan left (α, 1)-derivation, Homomorphism, Anti-homomorphism.*

1. INTRODUCTION

The study of left derivation was initiated by Bresar and Vukman in [6] and it was shown that if a prime ring R of characteristic different from 2 and 3 admits a nonzero Jordan left derivation then R must be commutative. Ashraf and Ali [9] introduced the concepts of generalized left derivation and generalized Jordan left derivation. Bell and Kappe [5] proved that if K is a non-zero right ideal of a prime ring R and d: R→ R a derivation of R such that d acts as a homomorphism on K, then d = 0 on R. Ashraf *et al.* [7, 8] and Jaya Subba Reddy *et al.* [10] studied Lie ideals, generalized left derivation, left (α, β)-derivation and generalized left (α, β)-derivation on prime rings R which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then d = 0 on R. Several authors have proved commutativity theorems and some results of lie ideals with left derivation in prime rings (See. [1, 2, 3, 4]). In general every left derivation is a Jordan left derivation but the converse need not be true. Ashraf [8] proved that the converse is true in the case when the underlying ring is 2-torsion free prime ring. The purpose of this paper to study some commutative properties of left (α, 1)-derivation in prime rings. Also we prove that if d or G acts as a homomorphism and as an anti-homomorphism on R, then either commutative or d = 0 on R.

2. PRELIMINARIES

Throughout the present paper, R will denote an associative ring with center Z(R). Recall that R is prime if xRy = {0} implies that either x = 0 or y = 0. For any x, y∈ R, the symbol [x, y] stands for the commutator xy– yx. Let S be a non-empty subset of R and d: R→ R a derivation of R. If d(xy) = d(x)d(y) (resp. d(xy) = d(y)d(x)) holds for all x, y∈ S, then d is said to act as a homomorphism (resp. anti-homomorphism) on S. An additive subgroup U of R is said to be a Lie ideal of R if [u, r]∈ U for all u∈ U, r∈ R. Suppose that α, 1 are endomorphisms of R. An additive mapping d: R→ R is called a (α, 1)-derivation (resp. Jordan (α, 1)-derivation) if $d(xy) = d(x)y + \alpha(x)d(y)$ (resp. $d(x^2) = d(x)x + \alpha(x)d(x)$) holds for all x, y∈ R. An additive mapping d: R→ R is called a left (α, 1)-derivation (resp. Jordan left (α, 1)-derivation)

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if $d(xy) = \alpha(x)d(y) + yd(x)$ (resp. $d(x^2) = \alpha(x)d(x) + xd(x)$) holds for all $x, y\in R$. An additive mapping $G: R\to R$ is said to be a generalized left (α, 1)-derivation (resp. generalized Jordan left (α, 1)-derivation) if there exists a Jordan left (α, 1)-derivation $d: R\to R$ such that $G(xy) = \alpha(x)G(y) + yd(x)$ (resp. $G(x^2) = \alpha(x)G(x) + xd(x)$) holds for all $x, y\in R$. The definition of generalized right(α, 1)-derivation (resp. generalized Jordan right (α, 1)-derivation) is self-explanatory. We shall make use of the following results, all but one of which is known.

Lemma 1 ([3, Lemma 2]): If $U\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b\in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.

Lemma 2 ([6, Lemma 4]): Let G and H are additive groups and let R be a 2-torsion free ring. Let $f: G\times G\to H$ and $g: G\times G\to R$ be biadditive mappings. Suppose that for each pair $a, b \in G$ either $f(a, b) = 0$ or $g(a, b)^2 = 0$. In this case either $f = 0$ or $g(a, b)^2 = 0$ for all $a, b\in G$ respectively.

Lemma 3 ([4, Theorem 4]): Let R be a 2-torsion free prime ring and U a Lie ideal of R. If R admits a derivation d such that $d(u)^n = 0$ for all $u\in U$, where $n \geq 1$ is a fixed integer, then $d(u) = 0$ for all $u\in U$.

Lemma 4 ([2, Lemma 1.3]): Let R be a 2-torsion free semiprime ring. If U is a commutative Lie ideal of R, then $U\subseteq Z(R)$.

3. MAIN RESULTS

Theorem 1: Let R be a prime ring and I a nonzero ideal of R, and let α, 1 be automorphisms of R. Suppose $d: R\to R$ is a (α, 1)-derivation of R.

(i) If d acts as a homomorphism on I, then $d = 0$ on R.

(ii) If d acts as an anti-homomorphism on I, then $d = 0$ on R.

Proof: (i) Let d acts as a homomorphism on I. By our hypothesis, we have

$$d(vu) = d(v)u + \alpha(v)d(u), \text{ for all } u, v\in I . \tag{1}$$

Replacing v by vr, for any $r\in R$ in the equation (1), we get

$$d(v)\alpha(r)d(u) = \alpha(v)\alpha(r)d(u),$$

and so, $(d(v)- \alpha(v))\alpha(r)d(u) = 0$, for all $v\in I$ and $r\in R$. (2)

Hence, $\alpha^{-1}(d(v)- \alpha(v))R\alpha^{-1}(d(u)) = \{0\}$, for all $v\in I$ and $r\in R$.

By the primeness of R, we conclude that either $\alpha^{-1}(d(v)- \alpha(v)) = 0$ or $\alpha^{-1}(d(u)) = 0$. Since α is an automorphism, we find that $d(v)- \alpha(v) = 0$ or $d(v) = 0$, for all $v\in I$.

For this yields that $d(v) = \alpha(v)$, for all $v\in I$. (3)

Replacing v by sv, for any $s\in R$ in the equation (3), we get

$$d(s)v + \alpha(s)d(v) = \alpha(s)\alpha(v), \text{ for all } v\in I \text{ and } s\in R.$$

Using equation (3) in the above relation, we get $d(s)v = 0$, for all $v\in I$ and $s\in R$. (4)

Hence, $d(s)I = \{0\}$, for all $s\in R$. Since the primeness of R, we have either $d(s) = 0$ or $I = 0$.

But $I \neq \{0\}$, this implies that I is central and hence R is commutative.

The last relation yields that $d(s) = 0$, i.e., $d = 0$ on R.

(ii) Let d acts as an anti-homomorphism on I. By our hypothesis, we have

$$d(uv) = d(u)v + \alpha(u)d(v), \text{ for all } u, v\in I . \tag{5}$$

Replacing u by uv in the equation (5), we get $d(uvv)= d(v)d(uv)= d(uv)v+ \alpha(uv)d(v)$

$$d(v)\alpha(u)d(v) = \alpha(u)\alpha(v)d(v), \text{ for all } u, v\in I. \tag{6}$$

Replacing u by uw in the equation (6) and using (6), we obtain

$$[\alpha(w), d(v)]\alpha(u)d(v) = 0, \text{ for all } u, v, w\in I.$$

Hence, $\alpha^{-1}([\alpha(w), d(v)])U\alpha^{-1}(d(v)) = \{0\}$, for all $v, w\in I$. Since I is a nonzero ideal of prime ring R, we obtain either $[w, \alpha^{-1}(d(v))] = 0$ or $\alpha^{-1}(d(v)) = 0$, for all $v, w\in I$.

Since α is an automorphism, we find that either $[\alpha(w), d(v)] = 0$ or $d(v) = 0$, for all $v, w\in I$.

Define for fixed $w\in R$, $I_1 = \{v\in I/ d(v) = 0\}$ and $I_2 = \{v\in I/ [\alpha(w), d(v)] = 0\}$. It is clearly shows that I_1 and I_2 are subgroups of I whose union is I. Hence either $I_1 = I$ or $I_2 = I$.

If $I_1 = I$. From the case (i), we have $d(v) = 0$. It shows $d = 0$ on R.

If $I_2 = I$, so we have $[\alpha(w), d(v)] = 0$, for all $v, w\in I$. (7)

Replacing w by rw, for any $r\in R$ in the equation (7), we get $[\alpha(r), d(v)]\alpha(w) = 0$, for all $v, w\in I$ and $r\in R$. So, we have $[R, \alpha^{-1}(d(v))]I = 0$, for all $v\in I$.

Since I is a nonzero ideal of prime ring R, we get $d(v)\subset Z(R)$ and it follows $d = 0$ from (i).

Hence the theorem proof is completed.

Theorem 2: Let R be a prime ring and I a nonzero ideal of R, and let α, 1 be automorphisms of R. Suppose $d: R \rightarrow R$ is a left (α, 1)- derivation of R.

(i) If d acts as a homomorphism on I, then $d = 0$ on R.
(ii) If d acts as an anti-homomorphism on I, then $d = 0$ on R.

Proof: (i) Let d acts as a homomorphism on I. By our hypothesis, we have

$d(u)d(v) = d(uv) = \alpha(u)d(v) + vd(u)$, for all $u, v\in I$. (8)

Replacing u by uv in the equation (8), we get $d(uv)d(v) = \alpha(u)\alpha(v)d(v) + vd(uv)$, for all $u, v\in I$.

The application of (8) yields that $\alpha(u)d(v)d(v) = \alpha(u)\alpha(v)d(v)$, for all $u, v\in I$.

This implies that $\alpha(u)(d(v) - \alpha(v))d(v) = 0$, for all $u, v\in I$. (9)

Replacing u by ur, for any $r\in R$ in the equation (9), we get $\alpha(u)\alpha(r)(d(v) - \alpha(v))d(v) = 0$, for all $u, v\in I$, and hence, $uR\alpha^{-1}((d(v) - \alpha(v))d(v)) = \{0\}$ for all $u, v\in I$.

Since by the primeness of R, we have either $u = 0$ or $\alpha^{-1}((d(v)- \alpha(v))d(v)) = 0$. Since I is a nonzero ideal of R, we have $\alpha^{-1}((d(v) - \alpha(v))d(v)) = 0$, this yields that $(d(v)- \alpha(v))d(v) = 0$.

That is $d(v^2) = \alpha(v)d(v)$. Since d is a left (α, 1)- derivation, we find that $\alpha(v)d(v) = 0$.

Linearizing the latter relation, we have $vd(u) + ud(v) = 0$, for all $u, v\in I$.

Replacing u by vu in the above relation, we get $vud(v) = 0$ for all $u, v\in I$. (10)

Replacing u by su in the equation (10), we get $vsud(v) = 0$, for all $u, v\in I$ and $s\in R$, and hence $vRud(v) = \{0\}$. Since by the primeness of R, we have either $v = 0$ or $ud(v) = 0$, for all $u, v\in I$. Since I is a nonzero ideal of R, we have $ud(v)= 0$, for all $u, v\in I$. (11)

Replacing u by ut in the equation (11), we get $utd(v) = 0$, for all $u, v\in I$ and $t\in R$, and hence $IRd(v) = \{0\}$. By the primeness of R, we have either $I = 0$ or $d(v) = 0$, for all $u\in I$. But $I\neq \{0\}$, so we have $d(v) = 0$. From in the case of theorem 1(i), we conclude that $d = 0$ on R.

(ii) Let d acts as an anti-homomorphism on I. By our hypothesis, we have

$d(uv) = \alpha(u)d(v) + vd(u)$, for all $u, v\in I$. (12)

Replacing v by uv in the equation (12), we get

$d(uv)d(u) = d(u(uv)) = \alpha(u)d(uv) + uvd(u)$, for all $u, v\in I$. (13)

Now right multiplying (12) by d(u) and using the fact that d is an anti- homomorphism on I, we get

$d(uv)d(u) = \alpha(u)d(uv) + vd(u)d(u)$, for all $u, v\in I$. (14)

Comparing (13) and (14), we get $uvd(u) = vd(u)d(u)$. (15)

Replacing v by rv in the equation (15), we get

$$urvd(u) = rvd(u)d(u), \text{ for all } u, v\in I \text{ and } r\in R. \quad (16)$$

Left multiplying (15) by r and comparing with (16), we obtain

$$[r, u]vd(u) = 0, \text{ for all } u, v\in I \text{ and } r\in R. \quad (17)$$

Replacing v by sv in the equation (17), we get $[r, u]svd(u) = 0$, for all $u, v\in I$ and $r, s\in R$. Hence, $[r, u]Rvd(u) = \{0\}$, for all $u, v\in I$ and $r\in R$.

Let $I_a= \{u\in I/ vd(u) = 0, \text{ for all } v\in I\}$ and $I_b= \{u\in I/ [r, u] = 0, \text{ for all } r\in R\}$. It is clearly shows that I_a and I_b are subgroups of I whose union is I. Since by the primeness of R, we have either $vd(u) = 0$ or $[r, u] = 0$, for all $u, v\in I$ and $r\in R$.

If $[r, u] = 0$, replacing u by su, we get $[r, s]u = 0$, for all $u\in I$ and $r, s\in R$.

This implies that $[r, s]I = \{0\}$. Since I is a nonzero ideal of prime ring R, we have either $u = 0$ or $[r, s] = 0$. But $I \neq \{0\}$, so we have $[r, s] = 0$, for all $r, s\in R$. i.e., R is commutative. i.e., d is a (α, 1)-derivation which acts as an anti-homomorphism on I.

Hence by Theorem 1(ii), we have $d = 0$ on R.

If $vd(u) = 0$, for all $u, v\in I$. (18)

In the above case we get the result (i) from (11). In the same technique by using we get the required result from (18).

Hence the theorem proof is completed.

Theorem 3: Let R be a prime ring and let I be a nonzero ideal of R. Suppose that α, 1 are automorphisms of R and $G: R\rightarrow R$ is a generalized left (α, 1)-derivation of R with associated left (α, 1)- derivation d.

(i) If G acts as a homomorphism on I, then either R is commutative or d= 0 on R.
(ii) If G acts as an anti-homomorphism on I, then either R is commutative or d= 0 on R.

Proof: (i) Let G acts as a homomorphism on I. By our hypothesis, we have

$$G(uv) = G(u)G(v) = \alpha(u)G(v) + vd(u), \text{ for all } u, v\in I. \quad (19)$$

Replacing v by vw in the equation (19), we get

$$G(uvw) = G(u(vw)) = \alpha(u)G(vw) + vwd(u), \text{ for all } u, v, w\in I. \quad (20)$$

On the other hand, we have

$$\begin{aligned} G(uvw) = G((uv)w)) &= G(uv)G(w) \\ &= \alpha(u)G(v)G(w) + vd(u)G(w). \end{aligned} \quad (21)$$

Comparing (21) and (20) and using (19), we get

$$vwd(u) = vd(u)G(w), \text{ for all } u, v, w\in I. \quad (22)$$

This implies that $v\{wd(u) - d(u)G(w)\} = 0$, for all $u, v, w\in I$.

Replacing v by vr, for any $r\in R$, we find that $vr\{wd(u) - d(u)G(w)\} = \{0\}$, for all $u, v, w\in I$.

Hence $IR\{wd(u) - d(u)G(w)\} = \{0\}$ for all $u, v, w\in I$. Since I is a nonzero ideal of prime ring R, this yields that

$$wd(u) = d(u)G(w), \text{ for all } u, w\in I. \quad (23)$$

Replacing u by uv in the equation (23), we get

$$w\alpha(u)d(v) + wvd(u) = \alpha(u)d(v)G(w) + vd(u)G(w), \text{ for all } u, v, w\in I. \quad (24)$$

Using (23) in (24), we have $[w, v]d(u) + [w, \alpha(u)]d(v) = 0$, for all $u, v, w\in I$. (25)

Hence in particular, we find that $[v, \alpha(u)]d(v) = 0$, for all $u, v\in I$. (26)

Replacing u by ru in the equation (26), for any $r\in R$ and using (26), we get

$$[v, \alpha(r)]\alpha(u)d(v) = 0, \text{ for all } u, v\in I.$$

The above relation implies that $\alpha^{-1}([v, \alpha(r)])u\alpha^{-1}(d(v)) = 0$ for all $u, v\in I$ and $r\in R$.

This can be rewritten as $\alpha^{-1}([v, \alpha(r)])IR\alpha^{-1}(d(v)) = \{0\}$, for all $v\in I$ and $r\in R$. Since by the primeness of R, we have either $\alpha^{-1}([v, \alpha(r)])I = \{0\}$ or $\alpha^{-1}(d(v)) = 0$.

Now define $I_\alpha = \{v\in I/ \alpha^{-1}([v, \alpha(r)])I = \{0\}$, for all $r\in R\}$ and $I_\beta = \{v\in I/ \alpha^{-1}(d(v)) = 0\}$. It is clearly shows that I_α and I_β are subgroups of I whose union is I. Since by the primeness of R, we have either $I_\alpha = I$ or $I_\beta = I$.

If $I_\alpha = I$, then we have $\alpha^{-1}([v, r'])I = \{0\}$, for all $v\in I$ and $\alpha(r) = r'\in R$. Since I is a nonzero ideal of prime ring R and α is an automorphism, this yields that $[v, r'] = 0$. This implies that I is central and hence R is commutative.

If $I_\beta = I$, then $\alpha^{-1}(d(v)) = 0$, for all $v\in I$. Since α is an automorphism and from the case of theorem 1(i), it follows that d = 0 on R.

(ii) Let G acts as an anti-homomorphism on I. By our hypothesis, we have

$$G(uv) = G(v)G(u) = \alpha(u)G(v) + vd(u), \text{ for all } u, v \in I. \tag{27}$$

Replacing v by uv in the equation (27), we get

$$G(u^2v) = G(uv)G(u) = \alpha(u)G(uv) + uvd(u), \text{ for all } u, v\in I. \tag{28}$$

Now using (27) in (28), we get

$$\alpha(u)G(v)G(u) + vd(u)G(u) = \alpha(u)G(uv) + uvd(u). \tag{29}$$

Again using (27) in (29), we get $uvd(u) = vd(u)G(u)$, for all $u, v\in I$. (30)

Replacing v by rv, for any $r\in R$ in the equation (30), we get

$$urvd(u) = rvd(u)G(u), \text{ for all } u, v\in I. \tag{31}$$

Left multiplying (30) by r, we get $ruvd(u) = rvd(u)G(u)$. (32)

Comparing (31) and (32), we get $[u, r]vd(u) = 0$, for all $u, v\in I$ and $r\in R$.
This relation can be written as $[u, r]I(d(u)) = \{0\}$. i.e., $[u, r]IRd(u) = \{0\}$, for all $u\in I$ and $r\in R$. This implies that for each fixed $u\in I$ either $[u, r]I = \{0\}$ or $d(u) = 0$.

Now using similar techniques as above, we get the required result.

Hence the theorem proof is completed.

5. CONCLUSION

In this study, we have introduced the (α, 1)-derivation (resp. Jordan (α, 1)-derivation), Left (α, 1)-derivation (resp. Jordan left (α, 1)-derivation), Generalized left (α, 1)-derivation (resp. Generalized Jordan left (α, 1)-derivation) and also established that if R admits a nonzero left (α, 1)-derivation acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then either R is commutative or d = 0 on R.

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