LEFT AND RIGHT 2-ENGLE ELEMENTS OF DERIVATIVE OF GROUPS

1H. Khosravi*, 2H. Golmakani and 3H. M. Mohammadinezhd

1,2Department Of Mathematics, Faculty Of Science, Mashhad Branch, Islamic Azad University, Mashhad, 91735-413, Iran

Email: Khosravi@mshdiau.ac.ir, H.golmakani@mshdiau.ac.ir, Hmohmmadin@math.birjand.ac.ir

(Received on: 20-09-11; Accepted on: 05-10-11)

ABSTRACT

In this paper we study right and left 2-Engle elements in derivative of groups. In particular, we prove that $R_2(G)$ is a characteristic subgroup of $G$. (let $G$ be a group and $G'$ be a derivative of $G$). As a consequence, in a special cases $[x, y]^2$ is a abelian subgroup of $G$ and the inverse of a right 2-Engle element of $G'$ is a left 3-Engle element of $G$.

1. INTRODUCTION:

Let $G$ be group and $G'$ be a derivative of $G$. Consider the set $R_2(G) = \{g \in G \mid [g, x, x] = 1 \text{ for all } x \in G\}$ of right 2-Engle elements of $G$. It is also known; see [1], that the inverse of a right 2-Engle element is a left 3-Engle element. For any two elements $a$ and $b$ of $G$ we define inductively $[a, b]^n$ the $n$-Engle commutator of the pair $(a, b)$, as follows:

$$[a, b] = a, [a, b] = a^{-1}b^{-1}ab \quad \text{and} \quad [a, b]^n = [[a, b], b] \quad \text{for all } n > 0.$$ 

$R_2(G)$ is a characteristic subgroup of $G$. It is known also that $R_2(G) \subseteq L_2(G)$ [3].

2. THE RESULTS:

Definition 1: Let $[x, y]$ and $[z, t]$ be two elements of $G$. We define the commentator of the pair $([x, y], [z, t])$, as follows:

$$[[x, y], [z, t]] = [x, y]^{-1}[z, t]^{-1}[x, y][z, t].$$

And define inductively

$$([x, y], a [z, t]) \text{ the n-Engle commentator of the pair } ([x, y], [z, t]), \text{ as follows:}$$

$$[[x, y], a [z, t]] = [[[x, y], (a-1) [z, t]], [z, t]] \quad \text{for all } n > 0.$$ 

Definition 2: An element $[x, y]$ of $G$ is called a left n-Engle element if $[[z, t], [x, y]] = 1$ for all $[z, t] \in G$. We denote by $L_n(G)$, the set of all left n-Engle elements of $G$ and an element $[x, y]$ of $G$ is called a right n-Engle element if $[[x, y], a [z, t]] = 1$ for all $[z, t] \in G$.

The set of all right n-Engle elements of $G$, denote by $R_n(G)$.

Theorem 1: In any derivative of group $G$ the inverse of right 2-Engle element is a left 2-Engle element. Thus $R_2(G') \subseteq L_2(G)$

Proof: Let $[x, y] \in R_2(G')$, using the definition of $R_2(G')$ we have

$$[[x, y], [x, y][z, t], [x, y][z, t]] = 1 \quad ; [z, t] \in G$$

$$1 = [[x, y], [x, y][z, t], [x, y][z, t]] = [[x, y], [z, t], [x, y][z, t]]$$

$$= [[x, y], [z, t]]^{-1}[y, y][z, t]^{-1}[x, y][y][z, t]$$

$$= [[x, y], [z, t]]^{-1}[y, y][x, y][y][z, t]$$

*Corresponding author: 1H. Khosravi*, E-mail: Khosravi@mshdiau.ac.ir
\[= [z, t]^{-1}[[x, y], [z, t]]^{-1}[[x, y], [z, t]][[x, y], [z, t]]\]
\[= [z, t]^{-1}[[x, y], [z, t]][[x, y], [z, t]]\]
\[= [[x, y], [z, t], [x, y]][x, y][z, t]\]

Since \([[x, y], [z, t], [x, y]] = 1\]
So \([[x, y], [z, t], [x, y]] = 1\]

By using three subgroup lemma \([3]\) we have \([[x, y], [z, t], [x, y]] = 1\).
Thus \([x, y] \in L_2(G^\circ)\).

**Theorem 2:** In any derivative of group G the inverse of a right Engle element is a left Engle element and the inverse of a right n-Engle element is a left \((n+1)\)-Engle element.

Thus \(R(G^\circ) \subseteq L(G^\circ)\) and \(R_{n+1}(G^\circ) \subseteq L_{n+1}(G^\circ)\)

**Proof:** Let \([x, y]\) and \([z, t]\) be elements of \(G^\circ\). Using the fundamental commutator identities \([2, 3]\) we obtain
\[=[[x, y], [[z, t], [z, t]]] = 1\]

Thus \([x, y] \in L_2(G)\).

**Theorem 3:** Let \(G\) be a group and \(G'\) be a derivative of \(G\). The set of elements of \(R_2(G')\) is a characteristic subgroup of \(G'\).

**Proof:** Let \(\alpha \in Aut(G)\) be an arbitrary automorphism and \([x, y] \in R_2(G')\). By definition of right 2-Engle element of \(G'\), for any \([z, t] \in G'\), we have
\[=[[x, y], [z, t], [z, t]] = 1\]
So \(\alpha([x, y], [z, t]) = 1\)

Therefore \([x, y] \in R_2(G')\).
Theorem 4: Let $G$ be a group and $G'$ be a derivative of $G$ and $[x, y] \in R_2(G')$. Then $[x, y]^G$ is abelian.

Proof: Let $[x, y], [z, t], [m, n] \in G'$, so

$$
[x, y]^{[z, t]}[x, y]^{[m, n]} = [[x, y]^{[z, t]}[m, n]^{-1}, [x, y]]^{[m, n]}
$$

$$
= [[x, y][z, t][m, n]^{-1}, [x, y]]^{[m, n]}
$$

$$
= [[x, y], [x, y]]^{[z, t][m, n]^{-1}}^{[m, n]} [[x, y], [z, t][m, n], [x, y]]^{[m, n]}
$$

$$
= [[[z, t][m, n]^{-1}, [x, y]]^{[m, n]}^{-1}, [x, y]]^{[m, n]}
$$

$$
= ( [[[z, t][m, n]^{-1}, [x, y]]^{-1}[m, n]] [[z, t][m, n]^{-1}, [x, y]]^{-1} )^{[m, n]} [[z, t][m, n]^{-1}, [x, y]]^{-1}
$$

$$
= 1.
$$

The last identities holds since $R_2(G') \subseteq L_2(G')$ and, therefore $[x, y]^G$ is abelian.

Definition 3: Let $G$ be a group and $G'$ be a derivative of $G$. Then the verbal subgroup, $E_1(G')$, define as follows:

$$
E_1(G') = \{ [x, y] \in G' : [[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]] \text{ for all } [z, t], [m, n] \in G' \}
$$

Remark: Now it is shown that there is a close connection between the elements of $R_2(G')$ and verbal subgroups of $G'$.

Theorem 5: Let $G$ be a group and $G'$ be a derivative of $G$. Then $E_1(G') = R_2(G')$.

Proof: By definition we have

$$
E_1(G') = \{ [x, y] \in G' : [[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]] \text{ for all } [z, t], [m, n] \in G' \}
$$

$$
R_2(G') = \{ [x, y] \in G' : [[x, y], [z, t], [z, t]] = 1 \text{ for all } [z, t] \in G' \}
$$

Now if $[x, y] \in E_1(G')$, then for all $[z, t], [m, n] \in G'$ we have

$$
[[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]].
$$

For $z = t = 1$ the identities hold. So for all $[m, n] \in G'$

$$
[[x, y], [m, n], [m, n]] = 1.
$$

Therefore $[x, y] \in R_2(G')$ and $E_1(G') \subseteq R_2(G')$ (I)

Conversely, if $[x, y] \in R_2(G')$, then for all $[z, t] \in G'$

$$
[[x, y], [z, t], [z, t]] = 1
$$

Now we have

$$
[[x, y][z, t], [m, n], [m, n]] = [[[x, y], [m, n]]^{[z, t]}[[z, t], [m, n]], [m, n]]
$$

$$
= [[[x, y], [m, n]]^{[z, t]}, [m, n]]^{[[z, t][m, n]]} [[z, t], [m, n]], [m, n]]
$$

$$
= [[[x, y], [m, n]][[x, y], [z, t][m, n]], [m, n]]^{[[z, t][m, n]]} [[z, t], [m, n]], [m, n]]
$$

$$
= [[[x, y], [m, n]][[x, y], [m, n]], [m, n]]^{[[z, t][m, n]]} [[z, t], [m, n]], [m, n]]
$$

$$
[[x, y], [m, n], [z, t], [m, n]]^{[[z, t][m, n]]} [[z, t], [m, n]], [m, n]].
$$

By using of assumption of last theorem, equal to $[[z, t], [m, n]], [m, n]]$. (II)

So $[x, y] \in E_1(G')$ and therefore $R_2(G') \subseteq E_1(G')$.
Now with (I) and (II) we have the result.

**Theorem 6:** Let $G, G'$ be group and derivative of group, respectively. And $[z, t] \in G', [x, y] \in R_2(G')$. Then

(i) $[x, y]^n$ is abelian.

(ii) For all $r, s \in \mathbb{N}$, $[x, y], [z, t]^n = [x, y]^i, [z, t]^j$.

**Proof:**

(i) It’s shown in Theorem 4.

(ii) We proceed by induction in two times.

As a first time, we proved by induction

$$[[x, y], [z, t]^n] = [[x, y]^n, [z, t]].$$

For $n=1$, being obvious.

Assume the result holds for $n-1$, i.e,

$$[[x, y], [z, t]]^{n-1} = [[x, y]^{n-1}, [z, t]].$$

Then we have

$$[[x, y]^n, [z, t]] = [[x, y]^{n-1}, [x, y], [z, t]]$$

$$= [[x, y]^{n-1}, [z, t][x, y], [z, t]]$$

$$= [[x, y]^{n-1}, [z, t][[x, y]^{n-1}, [z, t], [x, y]][[x, y], [z, t]].$$

By assumption, we have

$$[[[x, y]^{n-1}, [z, t], [x, y]] = [[[x, y], [z, t]]^{n-1}, [x, y]]$$

$$= [[[z, t], [x, y]]^{n-1}, [x, y]]^{n-1}, [z, t]].$$

We know that $R_2(G') \subseteq L_2(G')$, so if $[x, y] \in R_2(G')$, then $[x, y] \in L_2(G')$. Therefore, for each $[z, t] \in G'$ we have $[[z, t], [x, y], [x, y]] = 1$.

So $[[z, t], [x, y]]$ commutes with $[x, y]$ and $[x, y]^{-1}$.

Therefore

$$[[[x, y]^{n-1}, [z, t], [x, y]] = [[x, y]^{-1}, [z, t], [x, y]][[x, y], [z, t]]^{n-1}, [x, y] = 1.$$  

So we have

$$[[x, y]^n, [z, t]] = [[x, y]^{n-1}, [z, t][[x, y], [z, t]]$$

$$= [[x, y], [z, t]]^{n-1}, [x, y], [z, t]]$$

$$= [[x, y], [z, t]]^{n-1}.$$  

As a second time, we proved by induction

$$[[x, y], [z, t]]^{n-1} = [[x, y], [z, t]]^{n-1}.$$  

No we have

$$[[x, y], [z, t]^n] = [[x, y], [z, t]^{n-1}[z, t]]$$

$$= [[x, y], [z, t]][[x, y], [z, t]^{n-1}[z, t]]$$

$$= [[x, y], [z, t]][[x, y], [z, t]^{n-1}][[x, y], [z, t]^{n-1}].$$  

As a similar to the last case, proved.
\[ [[x, y], [z, t]]^{n-1}, [z, t] = 1 \]

So

\[ [[x, y], [z, t]] = [[x, y], [z, t]] [[x, y], [z, t]]^{n-1} = [[x, y], [z, t]]^n \]

ACKNOWLEDGMENTS:

The authors thank the research council of Mashhad Branch, (Islamic Azad University) for support.

REFERENCES:


********************