International Journal of Mathematical Archive-2(10), 2011, Page: 1980-1984
Available online through www.ijma.info ISSN 2229 – 5046

LEFT AND RIGHT 2-ENGLE ELEMENTS OF DRIVATIVE OF GROUPS

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(Received on: 20-09-11; Accepted on: 05-10-11)

ABSTRACT

In this paper we study right and left 2-Engle elements in derivative of groups. In particular, we prove that $R_2(G')$is a characteristic subgroup of G'.(let G be a group and G' be a derivative of G). As a consequence, in a special cases $[x, y]^{G'}$ is a abelian subgroup of $G'$and the inverse of a right 2-Engle element of G' is a left 3-Engle element of G'.

1. INTRODUCTION:

Let G be group and G' be a derivative of G. Consider the set $R_2(G) =\{g \in G \mid [g, x, x] = 1$ for all $x \in G\}$of right 2-Engle elements of G. It is also known; see [1], that the inverse of a right 2-Engle element is a left 3-Engle element. For any two elements a and b of G we define inductivily $[a, {}_nb]$ the n-Engle commutator of the pair (a, b), as follows:

$[a,{}_0b] = a$, $[a, b] = a^{-1}b^{-1}ab$ and $[a, {}_nb] = [[a,{}_{n-1}b],b]$ for all $n>0$.

$R_2 (G)$ is a characteristic subgroup of G. It is known also that $R_2 (G)\subseteq L_2(G)$[3].

2. THE RESULTS:

Definition 1: Let [x, y] and [z, t] be two elements of G' .We define the commentator of the pair ([x, y], [z, t]), as follows:

$[[x, y], [z, t]]=[x, y]^{-1}[z, t]^{-1}[x, y] [z, t]$.

And define inductively

$([x, y],{}_n [z, t])$ the n-Engle commentator of the pair ([x, y], [z, t]), as follows:

$[[x, y],{}_n [z, t]]= [[[x, y],{}_{(n-1)} [z, t]], [z, t]]$ for all $n > 0$.

Definition 2: An element [x, y] of G' is called a left n-Engle element if $[[z, t],{}_n[x, y]] = 1$ for all $[z, t]\in G'$. we denote by $L_n(G')$, the set of all left n-Engle elements of G' and an element [x, y] of G' is called a right n-Engle element if $[[x, y],{}_n[z, t]] = 1$ For all $[z, t]\in G'$.

The set of all right n-Engle elements of G', denote by $R_n(G')$.

Theorem 1: In any derivative of group G the inverse of right 2-Engle element is a left 2-Engle element. Thus
$R_2 (G') \subseteq L_2 (G')$

Proof: Let $[x, y]\in R_2(G')$, using the definition of $R_2(G')$ we have

$[[x, y], [x, y][z, t], [x, y][z, t]] = 1 \quad ; [z, t]\in G'$

$1= [[x, y], [x, y][z ,t], [x, y][z, t]] = [[x, y], [z, t], [x, y][z ,t]]$

$=[[x, y], [z, t]]^{-1}([x, y][z, t])^{-1}[[x, y][z, t]]([x, y][z, t])$

$= [[x, y], [z, t]]^{-1}[z, t]^{-1}[x, y]^{-1}[[x, y],[z, t]][x, y][z, t]$

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$=[z, t]^{-1}[[x, y], [z, t]]^{-1}[x, y]^{-1}[[x, y],[z, t]][x, y][z, t]$

$=[z, t]^{-1}[[x, y], [z, t],[x, y]][z, t]$

$=[[x, y], [z, t], [x, y]]^{[z ,t]}$

Since $[[x, y], [z, t], [x, y]^{[z, t]} = 1$

So $[[x, y], [z, t], [x, y]] =1$

By using three subgroup lemma [3] we have $[[x, y], [z, t], [x, y]] = 1$.

Thus $[x, y]\in L_2(G\acute{})$.

Theorem 2: In any derivarive of group G the inverse of a right Engle Element is a left Engle element and the inverse of a right n-Engle element is a left (n+1)-Engle element.

Thus $R(G\acute{}) \subseteq L(G\acute{})$ and $R_{n+1}(G\acute{}) \subseteq L_{n+1}(G\acute{})$

Proof: Let [x, y] and [z, t] be elements of Ǵ . Usiug the fundamental commutator identities [2, 3] we obtain

$$
\begin{aligned}
[[x, y], {}_{n+1}[z, t]] &= [[[x, y],[z, t]], {}_{n}[z, t]] \\
&= [[[z, t]^{-1}, [x, y]]^{[z,t],} {}_{n}[z, t]] \\
&= [[[z, t]^{-1},[x, y]], {}_{n}[z, t]]^{[z,t]} \\
&= [[z, t][z, t]^{-[x,y]} , {}_{n}[z, t]]^{[z,t]} \\
&= [[z, t]^{-[x,y]} , {}_{n}[z, t]]^{[z,t]} \\
&=[[z, t]^{-[x,y]} , ({}_{n}[z, t]^{[x,y]})^{-[x,y]}]^{[z,t]} \\
&= [[z, t], {}_{n}[z, t]^{[x,y]}]^{-[x,y][z,t]}
\end{aligned}
$$

since $[z, t] \in R_n(G\acute{})$

so $[[z, t],{}_{n}[z, t]^{[x,y]}]^{-[x,y][z,t]} =1$

therefore $[[x, y], {}_{n+1}[z, t]] =1$.

Theorem 3: Let G be a group and G' be a derivative of G. The set of elements of R_2 (G') is a characteristic subgroup of G'.

Proof: Let $\alpha\in Aut(G)$ be an arbitarary automorphism and $[x, y]\in R_2(G')$. By definition of right 2-Engle element of G', for any $[z, t]\in G'$,we have

$[[x, y], [z, t], [z, t]] = 1$

So
$\alpha[[x, y], [z, t], [z, t]] =1$

$\rightarrow [\alpha[x, y], \alpha[z, t],\alpha[z,t]] =1$

$\rightarrow [[\alpha(x),\alpha(y)],[\alpha(z),\alpha(t)],[\alpha(z),\alpha(t)]]=1$

Since $\alpha \in Aut(G)$,for any $z, t\in G$, there exists $z', t'\in G$ such that $\alpha(z) = z'$ and $\alpha(t) = t'$,

Therefore $[\alpha[x, y], [z',t'], [z',t']] = 1$.

Thus $\alpha[x, y]\in R_2(G')$.

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Theorem 4:.Let G be a group and G' be a derivative of G and $[x, y]\in R_2(G')$. Then $[x, y]^{G'}$ is abelian.

Proof: Let [x, y], [z, t], [m, n] be elements of G', so

$$[[x,y]^{[z,t]},[x,y]^{[m,n]}] = [[x,y]^{[z,t][m,n]^{-1}}, [x, y]]^{[m,n]}$$

$$= [[x, y][[x, y],[z, t][m, n]^{-1}],[x, y]]^{[m,n]}$$

$$= [[x,y], [x,y]]^{[[x,y],[z,t][m,n]^{-1}]^{[m,n]}}[[x, y], [z, t][m, n],[x,y]]^{[m,n]}$$

$$= [[x, y], [z, t][m, n]^{-1},[x, y]]^{[m,n]}$$

$$= [[[z, t][m, n]^{-1},[x, y]]^{-1},[x, y]]^{[m,n]}$$

$$=([[[z,t][m,n]^{-1}, [x,y]], [x,y]]^{-1})^{[m,n][[z,t][m,n]^{-1},[x,y]]^{-1}}$$

$$=1.$$

The last identities holds since $R_2(G') \subseteq L_2(G')$ and, therefore $[x, y]^{G'}$ is abelian.

Definition 3: Let G be a group and G' be a derivative of G. Then the verbal subgroup, $E_1(G')$, define as follows:

$E_1(G')=\{[x, y]\in G' : [[x, y][z, t],[m, n], [m, n]] = [[z, t], [m, n], [m, n]]$ For all $[z, t], [m, n]\in G'\}$

Remark: Now it is shown that there is a close connection between the elements of $R_2(G')$ and verbal subgroups of G'.

Theorem 5: Let G be a group and G' be a derivative of G. Then $E_1(G') = R_2(G')$.

Proof: By definition we have

$E_1(G') =\{ [x, y]\in G' : [[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]]$ For all $[z, t],[m, n]\in G'\}$

$R_2(G') = \{[x, y]\in G' : [[x, y], [z, t], [z, t]] =1$ For all $[z, t] \in G'\}$.

Now if $[x, y] \in E_1(G')$,then for all $[z, t],[m, n]\in G'$ we have

$[[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]]$.

For z = t = 1 the identities hold. So for all $[m, n]\in G'$

$[[x, y], [m, n], [m, n]] = 1$.

Therefore $[x, y]\in R_2(G')$ and $E_1(G') \subseteq R_2(G')$ (I)

Conversely .If $[x, y]\in R_2(G')$, then for all $[z, t]\in G'$

$[[x, y], [z, t], [z, t]] = 1$

Now we have

$$[[x, y][z, t], [m, n], [m, n]] = [[[x, y], [m, n]]^{[z,t]} [[z, t], [m, n]], [m, n]]$$

$$= [[[x, y], [m, n]]^{[z,t]}, [m, n]]^{[[z,t],[m,n]]} [[z, t], [m, n], [m, n]]$$

$$= [[[x, y], [n, m]][[x, y], [m, n], [z, t]],[m, n]]^{[[z,t],[m,n]]} [[z, t], [m, n], [m, n]]$$

$$=[[x, y], [m, n], [m, n]]^{[[z,t],[m,n]][[x,y],[m,n],[z,t]]}$$

$$[[x, y], [m, n], [z, t], [m, n]]^{[[z,t],[m,n]]} [[z, t], [m, n] [m, n]].$$

By using of assumption of last theorem, equal to [[z, t], [m, n], [m, n]].

So $[x, y]\in E_1(G')$ and therefore $R_2(G') \subseteq E_1(G')$ (II)

Now with (I) and (II) we have the result.

Theorem 6: Let G, G' be group and derivative of group, respectively. And $[z, t]\in G'$, $[x, y]\in R_2(G')$. Then
(i) $[x, y]^{G'}$ is abelian.

(ii) For all $r, s\in N$ $[[x, y], [z, t]]^{rs} = [[x, y]^r, [z, t]^s]$.

Proof: (i) Its shown in Theorem 4.

(ii) We proceed by induction in two times.

As a first time, we proved by induction

$[[x, y], [z, t]^n] = [[x, y]^n, [z, t]]$.

For n=1, being obvious.

Assume the result holds for n-1, i.e,

$[[x, y], [z, t]]^{n-1} = [[x, y]^{n-1}, [z, t]]$.

Then we have
$[[x, y]^n, [z, t]] = [[x, y]^{n-1} [x, y], [z, t]]$

$= [[x, y]^{n-1}, [z, t]]^{[x,y]} [[x, y],[z, t]]$

$= [[x, y]^{n-1}, [z, t]][[x, y]^{n-1}, [z, t], [x, y]][[x, y], [z, t]]$.

By assumption, we have

$[[[x, y]^{n-1}, [z, t], [x, y]] = [[[x, y], [z, t]]^{n-1}, [x, y]]$

$= [[z, t], [x, y]]^{n-1} [x, y]^{-1} [[x, y], [z, t]]^{n-1} [x, y]$.

We know that $R_2(G') \subseteq L_2(G')$, so if $[x, y]\in R_2(G')$, then$[x, y]\in L_2(G')$. Therefore, for each $[z, t]\in G'$ we have
$[[z, t], [x, y], [x, y]] = 1$.

So $[[z, t], [x, y]]$ commutes with $[x, y]$ and $[x, y]^{-1}$.

Therefore
$[[[x,y]^{n-1},[z,t],[x,y]]=[x,y]^{-1}[[z,t],[x,y]]^{n-1}[[x,y],[z,t]]^{n-1}[x,y]=1$

So we have
$[[x, y]^n, [z, t]] = [[x, y]^{n-1}, [z, t]][[x, y],[z, t]]$

$= [[x, y], [z, t]]^{n-1}[[x, y], [z, t]]$

$= [[x, y], [z, t]]^n$

As a second time, we proved by induction

$[[x, y], [z, t]^{n-1}] = [[x, y], [z, t]]^{n-1}$

No we have

$[[x, y], [z, t]^n] = [[x, y], [z, t]^{n-1}[z, t]]$

$= [[x, y], [z, t]][[x, y], [z, t]^{n-1}]^{[z,t]}$

$= [[x, y], [z, t]][[x, y], [z, t]^{n-1}][[x, y], [z, t]^{n-1}, [z, t]]$

As a similar to the last case, proved

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$[[x, y], [z, t]^{n-1}, [z, t]] = 1$

So

$[[x, y], [z, t]^{n}] = [[x, y], [z, t]][[x, y], [z, t]]^{n-1} = [[x, y],[z, t]]^{n}$

ACKNOWLEDGMENTS:

The authors thank the research council of Mashhad Branch, (Islamic Azad University) for support.

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