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CLASS OF ANALYTIC FUNCTIONS CONSTRUCTED USING q - DERIVATIVE OPERATOR

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ABSTRACT

In this paper, we are using a q - derivative operator of quantum calculus. By using this operator, we define Fekete – Szegö Inequality for a new class of analytic functions.

Keywords *– analytic function, Fekete – Szegö Inequality, q - derivative operator, concept of subordination, starlike functions.*

2010 Mathematics Subject Classification*: 30C45, 30C50.*

INTRODUCTION

Here, we are dealing with q – calculus and as we know, it has many applications on various branches of mathematics, so till now many researchers have worked on it. First of all, the concept of q – derivative and q – integral was developed , which was given by Jackson [9,10]. After that, Aral and Gupta proved an operator based on q- analogue [2, 3]. Recently, the concept of q- derivative operator was defined by Abdullah Alsoboh and Maslina Darus [1] which was based on q – operator and the concept of q – operator was studied by Mohammed and Darus[16].
$\mathcal{A}$ be the family of analytic functions of the type

$$f(z) = z + \sum_{j=2}^{\infty} a_j \; z^j$$

with the normalization $f(0) = 0$, $f'(0) = 1$.

$\mathcal{S}$ be the family of functions of the type

$$f(z) = z + \sum_{j=2}^{\infty} a_j \; z^j$$

with the normalization $f(0) = 0$, $f'(0) = 1$ and univalent in the open disk $E = \{ z \in C: |z| < 1\}$.

$S^*(\phi)$ be the class of functions in $f \in S$, for which

$$\frac{z\,f'(z)}{f(z)} \prec \phi\,(z)$$

which was introduced by Ma and Minda [15].

In this equation, "$\prec$" denotes subordination [which states that if $f(z)$ and $g(z)$ are two analytic functions , then there exists a Schwarzian function $w(z)$ (which is analytic in E) in such a way that $|w(z)| < 1$, $w(0) = 0$ and $f(z) = g(w(z))$; $z \in E$, then the function $f(z)$ is subordinate to $g(z)$ and we write it as $f(z) \prec g(z)$].

The concept of subordination was given by Lindelof [13]. Here, $\phi(z)$ is an analytic function with positive real part on E; which maps the unit disk E onto a region starlike with respect to 1 as well as symmetric with respect to real axis, satisfying conditions $\phi\,(0) = 0$ and $\phi'(0) > 0$.

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Miller *et.al* [14] proved the conditions

$$|c_1| \le 1,\ |c_2| \le 1-|c_1|^2$$

for Schwarzian function, which is a function of the type $w(z)=\sum_{n=1}^{\infty} c_n z^n$, with the conditions $w(0) = 0$ and $|w(z)| < 1$.
For $j \in N$,

the quantum number, $|j|_q = \frac{q^j-1}{q-1}$; $0 < q < 1$, $z \ne 0$

and the quantum derivative, $D_q f(z) = \frac{f(qz)-f(z)}{(q-1)z}$; $q \ne 0$, $z \ne 0$.

Here $|j|_q \to j$ and $D_q f(z) \to f'(z)$, as $q \to 1^-$.

Now, a new class of q- starlike of order β for $0 \le \beta < 1$, $0 < q < 1$ and $n \in N$, is defined by

$$S_{q,n}^*(\beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{z D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) > \beta \ ; z \in E \right\}$$

And a new q – differential operator denoted by $M_q^n f(z)$, defined as

$$M_q^0 f(z) = f(z)\ ,\ M_q^1 f(z) = zD_q f(z) = z + \sum_{j=2}^{\infty} |j|_q\, a_j\, z^j$$
$$M_q^n f(z) = zD_q\{M_q^{n-1} f(z)\} = z + \sum_{j=2}^{\infty} {|j|_q}^n\, a_j\, z^j$$

The terms $S_{q,n}^*(\beta)$ and $M_q^n f(z)$ were defined by Abdullah Alsoboh and Maslina Darus [1].
It is to be noticed here that

$$S_{q,0}^*(\beta) \equiv S^*(\beta);\ q \to 1$$

And this concept was introduced by Seoudy and Aouf [21].

Now by using all the above defined terms, we are proving this inequality for the class $TS_{q,n}^*(\alpha,\beta)$ defined below

$$(1-\alpha)\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) \prec \phi(z).$$

As $\frac{z D_q\{M_q^n f(z)\}}{M_q^n f(z)} \prec \phi(z)$; proved by Abdullah Alsoboh and Maslina Darus [1].

MAIN RESULTS

Theorem 1: Let $f(z) \in TS_{q,n}^*(\alpha,\beta)$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$; $w(z)$ is a Schwarzian function, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}; & \mu \le \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; & \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \mu \le \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} - \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}; & \mu \ge \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS_{q,n}^*(\alpha,\beta)$,

$$(1-\alpha)\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) = \frac{1+w(z)}{1-w(z)} \tag{1.1}$$

By putting all the values in (1.1), we get

$$1 + (1-2\alpha+\alpha|2|_q)|2|_q^n a_2 z + \left[(1-2\alpha+\alpha|3|_q)|3|_q^n a_3 - \alpha|2|_q^{2n}(|2|_q-1)a_2^2\right] z^2 + \dots$$
$$= 1 + 2c_1 z + 2(c_2 + c_1^2) z^2 + \dots$$

By comparing the coefficients, we get

$$a_2 = \frac{2c_1}{(1-2\alpha+\alpha|2|_q)|2|_q^n} \text{ and } a_3 = \frac{2c_2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} + \frac{[2|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}|3|_q^n}.$$

So, we get

$$a_3 - \mu a_2^2 = \frac{2c_2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} + \frac{[2|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}|3|_q^n} - \mu\frac{4c_1^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}.$$

Applying mode on both sides and using $|c_2| \le 1-|c_1|^2$, we get

$$|a_3 - \mu a_2^2| \le \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$$
$$+ \left\{\left|\frac{[2|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}]}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\right| - \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\right\}|c_1|^2.$$

Case-1: When $\mu \le \frac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{2(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n}$.

Then, $|a_3-\mu a_2^2| \le \frac{2}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n} + \left\{\frac{4\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}}\right\}|c_1|^2$.

Subcase-1 (a): If $\mu \le \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \le 1$, we get

$$|a_3-\mu a_2^2| \le \frac{2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} \tag{1.2}$$

Subcase-1 (b): If $\mu \ge \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $$|a_3-\mu a_2^2| \le \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \tag{1.3}$$

Case-2: When $\mu \ge \frac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{2(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n}$

Then, $|a_3-\mu a_2^2| \le \frac{2}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n} + \left\{\frac{4\mu}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} - \frac{4|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n}\right\}|c_1|^2$.

Subcase-2 (a): If $\mu \ge \frac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \le 1$, we get

$$|a_3-\mu a_2^2| \le \frac{4\mu}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} - \frac{2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} \tag{1.4}$$

Subcase-2 (b): If $\mu \le \frac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $$|a_3-\mu a_2^2| \le \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \tag{1.5}$$

Combining (1.2), (1.3), (1.4) and (1.5) we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z\,[1+az]^n$$

where $a = \frac{2(1-2\alpha+\alpha|3|_q)|3|_q^n - 2|2|_q^{2n}\{\alpha|2|_q(\alpha|2|_q-4\alpha+4)+(1+4\alpha^2-6\alpha)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)|3|_q^n|2|_q^n}$

and $n = \frac{(1-2\alpha+\alpha|3|_q)|3|_q^n}{(1-2\alpha+\alpha|3|_q)|3|_q^n - |2|_q^{2n}\{\alpha|2|_q(\alpha|2|_q-4\alpha+4)+(1+4\alpha^2-6\alpha)\}}$.

For second equation, extremal is

$$f(z) = z\,[1+2z^2]^{\frac{1}{(1-2\alpha+\alpha|3|_q)|3|_q^n}} .$$

Corollary 2: $\mathrm{T}S_{q,n}^*(1,\beta) = \mathrm{T}S_{q,n}^*(\beta)$; as by substituting $\alpha = 1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} - \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}};\ \mu \le \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)}; \\ \frac{2}{|3|_q^n(|3|_q-1)};\ \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \le \mu \le \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}; \\ \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} - \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)};\ \mu \ge \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}. \end{cases}$$

which is the required result given by Abdullah Alsoboh and Maslina Darus [1].

Theorem 3: Let $f(z) \in \mathrm{T}S_{q,n}^*(\alpha,\beta,\delta)$ and $\phi(z) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}$; $w(z)$ is a Schwarzian function, then

$| a_3-\mu a_2^2| \le$

$$\begin{cases} \frac{2\delta^2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}};\ \mu \le \frac{|2|_q^{2n}\{(\delta^2-\delta)(1-2\alpha+\alpha\,|2|_q)^2+2\alpha\delta^2(|2|_q-1)\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{2\delta}{(1-2\alpha+\alpha|3|_q)|3|_q^n};\ \frac{|2|_q^{2n}\{(\delta^2-\delta)(1-2\alpha+\alpha\,|2|_q)^2+2\alpha\delta^2(|2|_q-1)\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \mu \le \frac{|2|_q^{2n}\{(\delta^2+\delta)(1-2\alpha+\alpha\,|2|_q)^2+2\alpha\delta^2(|2|_q-1)\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} - \frac{2\delta^2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n};\ \mu \ge \frac{|2|_q^{2n}\{(\delta^2+\delta)(1-2\alpha+\alpha\,|2|_q)^2+2\alpha\delta^2(|2|_q-1)\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $\mathrm{T}S^{*}_{q,n}(\alpha,\beta,\delta)$,

$$(1\text{-}\alpha)\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z\, D_q\,\{M_q^n\, f\,(z)\}}{M_q^n f(z)}\right) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta} \tag{3.1}$$

By putting all the values in (3.1), we get

$$1 + (1\text{-}2\alpha + \alpha\,|2|_q)\,|2|_q^n\; a_2 z + \left[\left(1-2\alpha+\alpha|3|_q\right)|3|_q^n\; a_3 - \alpha\,|2|_q^{2n}\,\left(|2|_q\; - 1\right)a_2^2\;\right] z^2 + \ldots$$
$$= 1 + 2\delta\, c_1 z + 2\,(\delta c_2 + \delta^2 c_1^2)\, z^2 + \ldots$$

By comparing the coefficients, we get

$$a_2 = \frac{2\delta c_1}{(1-2\alpha+\alpha\,|2|_q)\,|2|_q^n} \text{ and } a_3 = \frac{2\delta\, c_2}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n} + \frac{[2\delta^2|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n}.$$

So, we get

$$a_3 - \mu\, a_2^2 = \frac{2\delta\, c_2}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n} + \frac{[2\delta^2|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n} - \mu\,\frac{4\delta^2 c_1^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}}\,.$$

Applying mode on both sides and using $|c_2| \le 1 - |c_1|^2$, we get

$$|\,a_3 - \mu a_2^2\,| \le \frac{2\delta}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n} + \left\{\left|\frac{[2\delta^2|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}]}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}}\right| - \frac{2\delta}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n}\right\}|c_1|^2.$$

Case-1: When $\mu \le \dfrac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{2(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n}$.

Then, $|a_3 - \mu a_2^2\,| \le \dfrac{2\delta}{(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n} + \left\{\dfrac{2(\delta^2-\delta)|2|_q^{2n}(1-2\alpha+\alpha\,|2|_q)^2+4\alpha\delta^2|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n} - \dfrac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}}\right\}|c_1|^2$.

Subcase-1 (a): If $\mu \le \dfrac{|2|_q^{2n}\left\{(\delta^2-\delta)(1-2\alpha+\alpha\,|2|_q)^2 + 2\alpha\delta^2(|2|_q-1)\right\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \le 1$, we get

$$|a_3 - \mu a_2^2\,| \le \frac{2\delta^2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} \tag{3.2}$$

Subcase-1 (b): If $\mu \ge \dfrac{|2|_q^{2n}\left\{(\delta^2-\delta)(1-2\alpha+\alpha\,|2|_q)^2 + 2\alpha\delta^2(|2|_q-1)\right\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $$|a_3 - \mu a_2^2\,| \le \frac{2\delta}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \tag{3.3}$$

Case-2: When $\mu \ge \dfrac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{2(1-2\alpha+\alpha\,|3|_q)\,|3|_q^n}$

Then, $|a_3 - \mu a_2^2\,| \le \dfrac{2\delta}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$+\left\{\frac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} - \frac{2(\delta^2+\delta)|2|_q^{2n}(1-2\alpha+\alpha\,|2|_q)^2+4\alpha\delta^2|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|2|_q^{2n}|3|_q^n}\right\}|c_1|^2\,.$$

Subcase-2 (a): If $\mu \ge \dfrac{|2|_q^{2n}\{(\delta^2+\delta)(1-2\alpha+\alpha\,|2|_q)^2+2\alpha\delta^2(|2|_q-1)\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \le 1$, we get

$$|a_3 - \mu a_2^2\,| \le \frac{4\mu\delta^2}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} - \frac{2\delta^2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} \tag{3.4}$$

Subcase-2 (b): If $\mu \le \dfrac{|2|_q^{2n}\{(\delta^2+\delta)(1-2\alpha+\alpha\,|2|_q)^2+2\alpha\delta^2(|2|_q-1)\}}{2\delta^2(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $$|a_3 - \mu a_2^2\,| \le \frac{2\delta}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \tag{3.5}$$

Combining (3.2), (3.3), (3.4) and (3.5), we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z\,[1+az]^n$$

where $a = \dfrac{2\delta(1-2\alpha+\alpha|3|_q)|3|_q^n - 2\delta|2|_q^{2n}\{\alpha|2|_q(\alpha|2|_q-4\alpha+4)+(1+4\alpha^2-6\alpha)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)|3|_q^n|2|_q^n}$

and $n = \dfrac{(1-2\alpha+\alpha|3|_q)|3|_q^n}{(1-2\alpha+\alpha|3|_q)|3|_q^n - |2|_q^{2n}\{\alpha|2|_q(\alpha|2|_q-4\alpha+4)+(1+4\alpha^2-6\alpha)\}}$.

For second equation, extremal is

$$f(z) = z\,[1+2\delta z^2]^{\frac{1}{(1-2\alpha+\alpha|3|_q)|3|_q^n}}\ .$$

Corollary 4: $TS^*_{q,n}(\alpha,\beta,1) = TS^*_{q,n}(\alpha,\beta)$, as by putting $\delta = 1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\ ; & \mu \le \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n};\\ \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\ ; & \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \mu \le \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n};\\ \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} - \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}\ ; & \mu \ge \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

which is same as $TS^*_{q,n}(\alpha,\beta)$.

Corollary 5: $TS^*_{q,n}(1,\beta,1) = TS^*_{q,n}(\beta)$, as by putting α=1 and $\delta = 1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} - \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}}\ ; & \mu \le \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)};\\ \frac{2}{|3|_q^n(|3|_q-1)}\ ; & \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \le \mu \le \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}\ ;\\ \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} - \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)}; & \mu \ge \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}\ . \end{cases}$$

which is same as that of $TS^*_{q,n}(\beta)$ given by Abdullah Alsoboh and Maslina Darus (1).

Theorem 6: Let $f(z) \in TS^*_{q,n}(\alpha,\beta,A,B)$ and $\phi(z) = \frac{1+Aw(z)}{1+Bw(z)}$; *w(z)* is a Schwarzian function, then

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{\{-B(A-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\ ; & \mu \le \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n};\\ \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\ ; & \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \mu \le \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n};\\ \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} + \frac{\{B(A-B)(1-2\alpha+\alpha|2|_q)^2-\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}\ ; & \mu \ge \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS^*_{q,n}(\alpha,\beta,A,B)$,

$$(1-\alpha)\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z\,D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) = \frac{1+Aw(z)}{1+Bw(z)} \tag{6.1}$$

By putting all the values in (6.1), we get

$$1 + (1-2\alpha+\alpha|2|_q)|2|_q^n\, a_2 z + \left[(1-2\alpha+\alpha|3|_q)|3|_q^n\, a_3 - \alpha|2|_q^{2n}(|2|_q-1)a_2^2\right] z^2 + \dots$$
$$= 1 + (A-B)c_1 z + [B(B-A)c_1^2 - (B-A)c_2]z^2 + \ \dots$$

By comparing the coefficients, we get

$$a_2 = \frac{(A-B)c_1}{(1-2\alpha+\alpha|2|_q)|2|_q^n} \text{ and } a_3 = \frac{(A-B)c_2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} + \frac{[(A-B)\{\alpha(A-B)(|2|_q-1)-B(1-2\alpha+\alpha|2|_q)^2\}]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}\ .$$

So, we get

$$a_3 - \mu a_2^2 = \frac{(A-B)c_2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} + \frac{[(A-B)\{\alpha(A-B)(|2|_q-1)-B(1-2\alpha+\alpha|2|_q)^2\}]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \mu\frac{(A-B)^2c_1^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\ .$$

Applying mode on both sides and using $|c_2| \le 1-|c_1|^2$, we get

$$|a_3-\mu a_2^2| \le \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$$
$$+\left\{\left|\frac{[(A-B)\{\alpha(A-B)(|2|_q-1)-B(1-2\alpha+\alpha|2|_q)^2\}]}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\right| - \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\right\}|c_1|^2.$$

Case-1: When $\mu \le \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-B(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$.

Then, $|a_3-\mu a_2^2| \le \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} + \left\{\frac{\alpha(A-B)^2(|2|_q-1)-(A-B)(B+1)(1-2\alpha+\alpha|2|_q)^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\right\}|c_1|^2$.

Subcase-1 (a): If $\mu \leq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3-\mu a_2^2| \leq \frac{\{-B(A-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} \tag{6.2}$$

Subcase-1 (b): If $\mu \geq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $|a_3-\mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$ (6.3)

Case-2: When $\mu \geq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-B(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $|a_3-\mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$+\left\{\frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} - \frac{\alpha(A-B)^2(|2|_q-1)+(A-B)(1-B)(1-2\alpha+\alpha|2|_q)^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}\right\}|c_1|^2 .$$

Subcase-2 (a): If $\mu \geq \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3-\mu a_2^2| \leq \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} + \frac{\{B(A-B)(1-2\alpha+\alpha|2|_q)^2-\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} \tag{6.4}$$

Subcase-2 (b): If $\mu \leq \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $|a_3-\mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$ (6.5)

Combining (6.2), (6.3), (6.4) and (6.5) we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z\,[1+az]^n$$

where $a = \frac{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n - 2|2|_q^{2n}\{-B\alpha|2|_q(\alpha|2|_q-4\alpha+3)-B(1+4\alpha^2-5\alpha)+\alpha A(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)|3|_q^n|2|_q^n}$

and $n = \frac{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n - 2|2|_q^{2n}\{-B\alpha|2|_q(\alpha|2|_q-4\alpha+3)-B(1+4\alpha^2-5\alpha)+\alpha A(|2|_q-1)\}}$.

For second equation, extremal is

$$f(z) = z\,[1+(A-B)z^2]^{\frac{1}{(1-2\alpha+\alpha|3|_q)|3|_q^n}} .$$

Corollary 7: $\mathrm{T}S_{q,n}^{*}(\alpha,\beta,1,-1) = \mathrm{T}S_{q,n}^{*}(\alpha,\beta)$, as by putting A=1 and B = -1, the result becomes

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}};\ \mu \leq \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n};\ \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \leq \mu \leq \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} - \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n};\ \mu \geq \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

which is same as $\mathrm{T}S_{q,n}^{*}(\alpha,\beta)$.

Corollary 8: $\mathrm{T}S_{q,n}^{*}(1,\beta,1,-1) = \mathrm{T}S_{q,n}^{*}(\beta)$, as by putting α = 1,A = 1 and B = -1, the result becomes

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} - \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}};\ \mu \leq \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)}; \\ \frac{2}{|3|_q^n(|3|_q-1)};\ \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \leq \mu \leq \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}; \\ \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} - \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)};\ \mu \geq \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}. \end{cases}$$

which is same as given by Abdullah Alsoboh and Maslina Darus [1].

Theorem 9: Let $f(z) \in TS^*_{q,n}(\alpha,\beta,A,B,\delta)$ and $\phi(z) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta}$; $w(z)$ is a Schwarzian function, then

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)\right\}\left(1-2\alpha+\alpha|2|_q\right)^2+\alpha\delta^2(A-B)^2(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)\left(1-2\alpha+\alpha|2|_q\right)^2|3|_q^n} - \frac{\mu\delta^2(A-B)^2}{\left(1-2\alpha+\alpha|2|_q\right)^2|2|_q^{2n}}; \\ \quad \mu \le \frac{|2|_q^{2n}\left[\alpha\delta(A-B)(|2|_q-1)+\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)-1\right\}(1-2\alpha+\alpha|2|_q)^2\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \quad \frac{|2|_q^{2n}\left[\alpha\delta(A-B)(|2|_q-1)+\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)-1\right\}\left(1-2\alpha+\alpha|2|_q\right)^2\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \\ \quad \mu \le \frac{|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)+1\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta(A-B)(|2|_q-1)\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{\mu\delta^2(A-B)^2}{\left(1-2\alpha+\alpha|2|_q\right)^2|2|_q^{2n}} - \frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)\right\}\left(1-2\alpha+\alpha|2|_q\right)^2+\alpha\delta^2(A-B)^2(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)\left(1-2\alpha+\alpha|2|_q\right)^2|3|_q^n}; \\ \quad \mu \ge \frac{|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)+1\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta(A-B)(|2|_q-1)\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS^*_{q,n}(\alpha,\beta,A,B,\delta)$

$$(1-\alpha)\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z\,D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \tag{9.1}$$

By putting all the values in (9.1), we get

$$1+(1-2\alpha+\alpha|2|_q)|2|_q^n\,a_2 z+\left[\left(1-2\alpha+\alpha|3|_q\right)|3|_q^n\,a_3-\alpha|2|_q^{2n}\left(|2|_q-1\right)a_2^2\right]z^2+\ldots$$
$$=1+\delta(A\text{-}B)c_1 z+\left[\delta(A\text{-}B)c_2+\frac{\delta}{2}\{\delta(A-B)^2-(A^2-B^2)\}c_1^2\right]z^2+\ldots$$

By comparing the coefficients, we get

$$a_2=\frac{\delta(A-B)c_1}{(1-2\alpha+\alpha|2|_q)|2|_q^n} \quad \text{and}$$

$$a_3=\frac{\delta(A-B)c_2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}+\frac{\left[\left\{\alpha\delta^2(A-B)^2(|2|_q-1)+\frac{\delta}{2}\{\delta(A-B)^2-(A^2-B^2)\}(1-2\alpha+\alpha|2|_q)^2\right\}\right]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}.$$

So, we get

$$a_3-\mu a_2^2=\frac{\delta(A-B)c_2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}+$$
$$\frac{\left[\frac{\delta}{2}\{\delta(A-B)^2-(A^2-B^2)\}\left(1-2\alpha+\alpha|2|_q\right)^2+\alpha\delta^2(A-B)^2(|2|_q-1)\right]c_1^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}-\mu\frac{\delta^2(A-B)^2c_1^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}.$$

Applying mode on both sides and using $|c_2| \le 1-|c_1|^2$, we get

$$|a_3-\mu a_2^2| \le \frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$$
$$+\left\{\left|\frac{\left[\frac{\delta}{2}\{\delta(A-B)^2-(A^2-B^2)\}\left(1-2\alpha+\alpha|2|_q\right)^2+\alpha\delta^2(A-B)^2(|2|_q-1)\right]}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}-\frac{\mu\delta^2(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\right|-\frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\right\}|c_1|^2.$$

Case-1: When $\mu \le \frac{|2|_q^{2n}\left\{\alpha\delta^2(A-B)^2(|2|_q-1)+\frac{\delta}{2}\{\delta(A-B)^2-(A^2-B^2)\}(1-2\alpha+\alpha|2|_q)^2\right\}}{\delta^2(A-B)^2(1-2\alpha+\alpha|3|_q)|3|_q^n}$.

Then, $|a_3-\mu a_2^2| \le \frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$+\left\{\frac{\alpha\delta^2(A-B)^2(|2|_q-1)+\delta(A-B)\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)-1\right\}(1-2\alpha+\alpha|2|_q)^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}-\frac{\mu\delta^2(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\right\}|c_1|^2.$$

Subcase-1 (a): If $\mu \le \frac{|2|_q^{2n}\left[\alpha\delta(A-B)(|2|_q-1)+\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)-1\right\}(1-2\alpha+\alpha|2|_q)^2\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \le 1$, we get

$$|a_3-\mu a_2^2| \le \frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta^2(A-B)^2(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}-\frac{\mu\delta^2(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} \tag{9.2}$$

Subcase-1 (b): If $\mu \ge \frac{|2|_q^{2n}\left[\alpha\delta(A-B)(|2|_q-1)+\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)-1\right\}(1-2\alpha+\alpha|2|_q)^2\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $$|a_3-\mu a_2^2| \le \frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \tag{9.3}$$

Case-2: When $\mu \geq \frac{|2|_q^{2n}\left\{\alpha\delta^2(A-B)^2(|2|_q-1)+\frac{\delta}{2}\{\delta(A-B)^2-(A^2-B^2)\}(1-2\alpha+\alpha|2|_q)^2\right\}}{\delta^2(A-B)^2(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $|a_3-\mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$+\left\{\frac{\mu\delta^2(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}-\frac{\alpha\delta^2(A-B)^2(|2|_q-1)+\delta(A-B)\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)+1\right\}(1-2\alpha+\alpha|2|_q)^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}\right\}|c_1|^2 .$$

Subcase-2 (a): If $\mu \geq \frac{|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)+1\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta(A-B)(|2|_q-1)\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3-\mu a_2^2| \leq \frac{\mu\delta^2(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}-\frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta^2(A-B)^2(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} \tag{9.4}$$

Subcase-2 (b): If $\mu \leq \frac{|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)+1\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta(A-B)(|2|_q-1)\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

Then, $$|a_3-\mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \tag{9.5}$$

Combining (9.2), (9.3), (9.4) and (9.5) we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z\,[1+az]^n$$

where $a = \frac{\delta^2(A-B)^2(1-2\alpha+\alpha|3|_q)|3|_q^n-2|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta(|2|_q-1)(A-B)\right]}{\delta(A-B)(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)|3|_q^n|2|_q^n}$

and $n = \frac{\delta^2(A-B)^2(1-2\alpha+\alpha|3|_q)|3|_q^n}{\delta^2(A-B)^2(1-2\alpha+\alpha|3|_q)|3|_q^n-2|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B)-\frac{1}{2}(A+B)\right\}(1-2\alpha+\alpha|2|_q)^2+\alpha\delta(|2|_q-1)(A-B)\right]}$.

For second equation, extremal is

$$f(z) = z\,[1+\delta(A-B)z^2]^{\frac{1}{(1-2\alpha+\alpha|3|_q)|3|_q^n}}.$$

Corollary 10: $\mathrm{T}S_{q,n}^*(\alpha,\beta,A,B,1) = \mathrm{T}S_{q,n}^*(\alpha,\beta,A,B)$, as by putting $\delta=1$, the result becomes

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{\{-B(A-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}-\frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}\ ;\ \mu \leq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\ ;\frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n} \leq \mu \leq \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}+\frac{\{B(A-B)(1-2\alpha+\alpha|2|_q)^2-\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}\ ;\mu \geq \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

same as $\mathrm{T}S_{q,n}^*(\alpha,\beta,A,B)$.

Corollary 11: $\mathrm{T}S_{q,n}^*(\alpha,\beta,1,-1,1) = \mathrm{T}S_{q,n}^*(\alpha,\beta)$, as by putting A=1, B= -1 and $\delta=1$, the result becomes

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}-\frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}};\ \mu \leq \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}\ ;\frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \leq \mu \leq \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}-\frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n};\mu \geq \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

Same as $\mathrm{T}S_{q,n}^*(\alpha,\beta)$.

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