International Journal of Mathematical Archive-13(4), 2022, 1-6
IJMA Available online through www.ijma.info ISSN 2229 – 5046

SOME CHARACTERISTICS OF MONADIC AND SIMPLE MONADIC ALGEBRAS

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(Received On: 02-09-21; Revised & Accepted On: 20-04-22)

ABSTRACT

In this paper, we investigate and extension of monadic and their properties were given by Halmos, we study the ideals, filters, homomorphism, constant mapping, and simple monadic algebra. We also derive some results which associate ideals and filters under mapping homomorphism.

***Keywords**: Monadic Algebras, monadic ideal, monadic filter, monadic homomorphism, Simple Monadic Algebras.*

1. INTRODUCTION

The father of algebraic logic is George Bool who introduced Boolean algebras in the 1850's to express statement logic in algebraic form [2]. The subsequent steps, Tarski and Thompson [3] were to present algebras capable of expressing monadic and polyadic logics when introducing cylindrical algebras. The other approaches were developing monadic and polyadic algebras from Boolean algebras with the inclusion of a certain operator to represent existential and universal quantifiers. This is due to Halmos [7, 10], This approach is preferred, because it is a direct generalization of Boolean algebra. In [1], we investigated, explained, and enlarged existential and universal quantifier operators on Boolean algebras with their properties. The aim of this paper is the extension of monadic algebras and their property were given in [7]. In this section we introduce the background of the next section.

Definition 1.1: [1,3,10] A Boolean Algebra is an algebraic structure $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ consists of a set B, two binary operations $\vee$ (join) and $\wedge$ (meet), one unary operation '(complementation) and two nullary operations 0 and 1(fixed elements) which satisfies the following axioms:
BA_1. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a, \forall\, a, b \in B$(commutative axiom).
BA_2. $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c), \forall\, a, b, c \in B$((associative axiom).
BA_3. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall\, a, b, c \in B$(distributive axiom).
BA_4. $a \vee a = a$ and $a \wedge a = a$(idempotent axiom).
BA_5 . $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a, \forall\, a, b \in B$(absorption axiom).
BA_6. $a \wedge 1 = a$ and $a \vee 0 = a, \forall\, a \in B$(existence of zero and unit elements axioms)and
BA_7. $\forall\, a \in B \Longrightarrow \exists\, a' \in B \ni a \vee a' = 1$ and $a \wedge a' = 0$(existence of complment axiom).

The following theorems gives us the main properties of elements of Boolean Algebras.

Theorem 1.2: [1, 7] Let B be a Boolean algebra and $S \neq \emptyset \subset B$. Then S is called sub-algebra of B, if for any a and $b \in S$, then $a \vee b \in S, a \wedge b \in S$ and $a' \in S$.

Definition 1.3: [1, 7] Let$X$ be a nonempty set, B a Boolean Algebra. The set of all functions from set X into set of B, given by $B^X = \{p | p: X \longrightarrow B \; is\ a\ function\}$ for any $p, q \in B^X$. Define ,$p \vee q, p \wedge q, p'$, 0 and 1 respectively in B^X as follows:

1.$(p \vee q)(x) = p(x) \vee q(x), \forall\, x \in X$;
2.$(p \wedge q)(x) = p(x) \wedge q(x), \forall\, x \in X$;
3.$p'(x) = (p(x))', \forall\, x \in X$;
4.$0(x) = 0, \forall\, x \in X$ and
5.$1(x) = 1, \forall\, x \in X$.

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Theorem 1.4: [1] Consider p, q and $r \in B^X$, Then: $(B^X, \vee, \wedge, ', 0, 1)$ is a Boolean algebra.

Definition 1.5: [3, 10] Let B be a Boolean algebra. A non-empty set subset I of B is called an ideal of B, if it is satisfying the following conditions: 1. If $a \in I$ and $b \in I$,then $a \vee b \in I$ and 2. If $a \in I$ and $b \in B$, then $a \wedge b \in I$. $\{0\}$ and B are two ideals.

Definition 1.6: [3, 10] Let B be a Boolean algebra. A non-empty set subset F of B is called a filter of B, if it is satisfying the following conditions: 1. If $a \in F$ and $b \in F$,then $a \wedge b \in F$ and 2. If $a \in F$ and $b \in B$, then $a \vee b \in F$. $\{1\}$ and B are two filters.

The following theorems give us the main properties of ideals and filters with a corroding to Halmos, see [9].

Theorem 1.7: [10] Let B be a Boolean algebra and I is an ideal of B, then 1. $0 \in I$, and, 2. If $a \in I$ and $b \in B$ such that $b \leq a$, then $b \in I$.

Theorem 1.8: [10] Let B be a Boolean algebra and F is a filter of B, then 1. $1 \in F$, and, 2. If $a \in F$ and $b \in B$ such that $a \leq b$, then $b \in F$.

Theorem 1.9: [10] Let B be a Boolean algebra and $I, F \subseteq B$, then

1. If I is an ideal, then $I' = \{a' : a \in I\}$, and,
2. If F is a filter, then $F' = \{a' : a \in F\}$.

Remark: If B be a Boolean algebra, then

1. An ideal I is proper if and only if $1 \notin I$.
2. If $a \neq 0 \in B$, then there is a maximal ideal M not containing the element a.

Theorem 1.10: [10, 14] An ideal I in a Boolean algebra B is maximal if and only if either $p \in I$ or $p' \in I$, but not both, for each $p \in B$.

Definition 1.11: [3, 10, 14] Let B_1 and B_2 be two Boolean algebras. A mapping $f: B_1 \longrightarrow B_2$ is called a Boolean homomorphism if the following axioms are satisfying:

1. $f(a \vee b) = f(a) \vee f(b)$;
2. $f(a \wedge b) = f(a) \wedge f(b)$;
3. $f(0) = 0$ and
4. $f(1) = 1, \forall a, b \in B_1$.

Proposition 1.12: [3, 10, 14] Let $f: B_1 \longrightarrow B_2$ be a Boolean homomorphism between two Boolean algebras B_1 and B_2. Then for any $a \in B_1$, then $f(a') = (f(a))'$ in B_2 .

Definition 1.13: [3, 10] Let $f: B_1 \longrightarrow B_2$ be a Boolean homomorphismbetween two Boolean algebras. The Kernel of f, written Ker f, is defined to be the set ker $(f) = \{a \in B_1 : f(a) = 0\}$.

Definition 1.14: [3, 10] Let I be a Boolean ideal of Boolean algebra B .Define a binary congruence relation between the objects of B thus: $a \equiv b (mod\ I) \Leftrightarrow a + b = (a - b) \vee (b - a) \in I$.

Proposition 2.8: The congruence relation. $\equiv .(\ mod\ I)$ is an equivalence relation on$B$.

Now, to construct quotient algebra via above congruence relation $.\equiv .(\ mod\ I)$ on B. Let $B/_I$ be denotes the quotient set $B/_{.\equiv .(\ mod\ I)}$. Let I be a Boolean ideal of Boolean algebra B. Define operations $\vee, \wedge$ and ' on $B/_I$ as follows:

i. $[a] \vee [b] = [a \vee b]$;
ii. $[a] \wedge [b] = [a \wedge b]$ and
iii. $[a]' = [a']$.

These operations are well-defined. Then $\left(B/_I, \vee, \wedge, ', [0], [1]\right)$ is a Boolean algebra see [3]. This is called the quotient Boolean Algebra.

2. MONADIC BOOLEAN ALGEBRAS

This section is an extension of monadic algebras and their properties given in [7]. Monadic algebra was introduced by Halmos, see [7].

Definition 2.1: A monadicalgebra is a pair $\mathrm{M} = (B, \exists)$, where B is a Boolean algebra and $\exists$ is a quantifier operators on$B$.

Definition 2.2: Let A be a sub set of monadic algebra $\mathrm{M} = (B, \exists)$. Then A is called a monadic sub-algebra of B, if

i. A is a Boolean sub algebra of B,and,
ii. If $a \in A$, then $\exists(a) \in A$.

Definition 2.3: Let I be a sub set of monadic algebra $\mathrm{M} = (B, \exists)$. Then I is called a monadic ideal of B, if

i. I is an ideal of B, and,
ii. If $b \in I$, then $\exists(b) \in I$.

Definition 2.4: Let F be a sub set of monadic algebra $\mathrm{M} = (B, \exists)$. Then F is called a monadic filter of B, if

i. F is a filter of B,and,
ii. If $b \in F$, then $\exists(b) \in F$.

Definition 2.5: Let $\mathrm{M}_1 = (B_1, \exists)$ and $\mathrm{M}_2 = (B_2, \exists)$ be two monadic algebras. A monadic homomorphism is a mapping $f: B_1 \longrightarrow B_2$ such that:

i. f is a Boolean homomorphism, and
ii. If $a \in B_1$,then $f(\exists(a)) = \exists(f(a))$. i.e, $f\exists = \exists f$.

Theorem 2.6: Let $f: B_1 \longrightarrow B_2$ be a monadic homomorphism between two monadic algebras. Then the following are true:

i. $f(0) = 0$,
ii. $f(1) = 1$,
iii. If A is a monadic sub-algebra of B_1, then $f(A)$ is a monadic sub-algebra of B_2,
iv. If A is a monadic sub-algebra of B_2 then $f^{-1}(A)$ is a monadic sub-algebra of B_1,
v. If I is a monadic ideal of B_1, then $f(I)$ is a monadic ideal of B_2,
vi. If I is a monadic ideal of $f(B_1)$ then $f^{-1}(I)$ is a monadic ideal of B_1.
vii. If F is a monadic filter of B_1, then $f(F)$ is a monadic filter of B_2 ,and,
viii. If F is a monadic filter of $f(B_1)$, then $f^{-1}(F)$ is a monadic filter of B_1.

Proof:

i. $f(0) = f(a \wedge a') = f(a) \wedge f(a') = f(a) \wedge (f(a))' = 0$.
ii. $f(1) = f(a \vee a') = f(a) \vee f(a') = f(a) \vee (f(a))' = 1$.
iii. Let A be a monadic sub-algebra of B_1. Define $f(A) = \{f(a): a \in A\}$. Assume that $f(a)$ and $f(b) \in f(A)$, therefore a and $b \in A$, hence $(a \vee b) \in A$, so $f(a \vee b) = f(a) \vee f(b) \in f(A)$,therefore $f(A)$ is a closed under operator join $\vee$. Now Suppose that $f(a) \in f(A)$ for some $a \in A$, therefore $a' \in A$, but
$f(a') = (f(a))' \in f(A)$ and $f(\exists(a)) = \exists(f(a))$. Hence $f(A)$ is a monadic sub-algebra of B_2.
iv. Let A is a monadic sub-algebra of B_2. Define $f^{-1}(A) = \{a: f(a) \in A\}$. Consider a and $b \in f^{-1}(A)$,
implies that $f(a)$ and $f(b) \in A$, hence $f(a) \vee f(b) \in A$, but $f(a) \vee f(b) = f(a \vee b) \in A$, so
$a \vee b \in f^{-1}(A)$,, therefore $f^{-1}(A)$ is a closed under operator join $\vee$. Now let $a \in f^{-1}(A)$, implies that
$f(a) \in A$, therefore $(f(a))' \in A$, but $f(a) = (f(a))' \in A$, so $a' \in f^{-1}(A)$. Hence $f^{-1}(A)$ is a Boolean sub-algebra of B_1. Finally, let $a \in f^{-1}(A)$, therefore $f(a) \in A$,so $\exists\,(f(a)) \in A$ and
$f(\exists(a)) = \exists(f(a)) \in A$, hence $\exists(a) \in f^{-1}(A)$. So that $f^{-1}(A)$ is a monadic sub-algebra of B_1.
v. Let I be a monadic ideal of B_1. Define $f(I) = \{f(a): a \in I\}$. Suppose that $f(a)$ and $f(b) \in f(I)$, therefore a and $b \in I$, hence $(a \vee b) \in I$, so $f(a \vee b) = f(a) \vee f(b) \in f(I)$, therefore $f(I)$ is a closed under operator join $\vee$. Now Suppose that $f(a) \in f(I)$ and $f(b) \in f(B_1)$ for some $a \in I$ and $b \in B_1$ therefore $a \wedge b \in I$, but $f(a) \wedge f(b) = f(a \wedge b) \in f(I)$, therefore $f(I)$ is a monadic ideal of $f(B_1)$
vi. Let I is a monadic ideal of $f(B_1)$. Define $f^{-1}(I) = \{a: f(a) \in I\}$. Suppose that
$a, b \in f^{-1}(I) \Rightarrow f(a), f(b) \in I \Rightarrow f(a) \vee f(b) \in I \Rightarrow f(a \vee b) = f(a) \vee f(b) \in I \Rightarrow a \vee b \in f^{-1}(I)$.
Let $a \in f^{-1}(I)$ and , $b \in f^{-1}(B_1) \Rightarrow f(a) \in I$ and $f(b) \in B_2 \Rightarrow f(a) \wedge f(b) \in I$
$\Rightarrow f(a \wedge b) = f(a) \wedge f(b) \in I \Rightarrow a \wedge b) \in I \Rightarrow f^{-1}(I)$ is an ideal of B_1 . Since $f(a) \in I$ and $\exists(f(a)) \in I$, we have $f(\exists(a)) = \exists(f(a))$. Therefore $\exists(a) \in f^{-1}(I)$. Hence $f^{-1}(I)$ is a monadic ideal of B_1.
vii. Let F be a monadic filter of B_1. Define $f(F) = \{f(a): a \in F\}$.
Suppose that $f(a)$ and $f(b) \in f(F) \Rightarrow a$ and $b \in F \Rightarrow a \wedge b \in F \Rightarrow f(a \wedge b) \in f(F)$. Therefore $f(F)$ is a closed under operator meet $\wedge$. Let $f(a) \in F$ and $f(b) \in f(B_1) \Rightarrow a \in F$ & $b \in B_1 \Rightarrow a \vee b \in F$, implies that
$f(a \vee b) = f(a) \vee f(b) \in f(F)$. Hence $f(F)$ is a Boolean filter of B_2. Finally, suppose that $f(a) \in f(F)$, then $a \in F$, therefore $\exists(a) \in F$. Hence $f(\exists(a) \in f(F)$. But $f(\exists(a)) = \exists\,(f(a)) \in f(F)$. So that $f(F)$ is a monadic filter of B_2.

viii. Consider F is a monadic Boolean filter of $f(B_1)$. Define $f^{-1}(F)=\{a: f(a)\in F\}$.
Suppose that $a,b\in f^{-1}(F)\Rightarrow f(a),f(b)\in F\Rightarrow f(a)\wedge f(b)\in F\Rightarrow f(a\vee b)=f(a)\wedge f(b)\in F$
$\Rightarrow a\wedge b\in f^{-1}(F)$. Therefore $f^{-1}(F)$ is a closed under operator meet $\wedge$. Let $a\in f^{-1}(F)$ and
$b\in f^{-1}(B_1)\Rightarrow f(a)\in F$ and $f(b)\in B_1\Rightarrow f(a)\vee f(b)\in F\Rightarrow f(a\vee b)=f(a)\vee f(b)\in F$
$\Rightarrow (a\vee b)\in F\Rightarrow f^{-1}(F)$ is a Boolean filter of B_1.Now let $b\in f^{-1}(F)\Longrightarrow f(b)\in F\Longrightarrow \exists(f(b))\in F$. But $\exists\big(f(b)\big)=f\big(\exists(b)\big)\Longrightarrow \exists(b)\in f^{-1}(F)$. Hence $f^{-1}(F)$ is a monadic Boolean filter of B_1.

Definition 2.7: Let $f: B_1\longrightarrow B_2$ be a monadic homomorphism between two monadic algebras. The Kernel of f, written $\ker f$, is defined to be the set $\ker(f)=\{a\in B_1: f(a)=0\}$.

Theorem 2.8: Let $f: B_1\longrightarrow B_2$ be a monadic homomorphism between two monadic algebras B_1 and B_2. Then

i. $\ker(f)=\{a\in B_1: f(a)=0\}$ is a monadic sub-algebra of B_1.

ii. $ker\,(f)=\{a\in B_1: f(a)=0\}$ is a monadic ideal of B_1. But $ker\,(f)$ is not a monadic filter of B_1.

Proof:

i. Since $0\in ker\,f$, $Ker(f)\neq\emptyset$. Let a and $b\in ker\,f\Rightarrow f(a)=0$ and $f(a)=0$
$\Rightarrow f(a\vee b)=f(a)\vee f(b)=0\vee 0=0\Rightarrow a\vee b\in\ker(f)$. Now suppose that $a\in Ker\,(f)\Longrightarrow f(a)=0$.
$\Longrightarrow f(a)=f(0)\Longrightarrow f(a)\wedge f(a^{'})=f(0)\wedge f(a^{'})\Longrightarrow f(a)\wedge\big(f(a)\big)^{'}=f(0)\wedge f(a^{'})\Longrightarrow 0=f(a^{'})$. Hence $a^{'}\in\ker(f)$, consequently, $\ker(f)$ is a Boolean sub-algebra of B_1. Now let $a\in\ker f\Rightarrow f(a)=0$
$\Longrightarrow \exists\big(f(a)\big)=\exists(0)\Longrightarrow f\big(\exists(a)\big)=0\Longrightarrow \exists(a)\in\ker(f)$. Hence $Ker(f)$ is a monadic sub-algebra of B_1.

ii. Let a and $b\in\ker(f)\Longrightarrow f(a)=0$ and $f(b)=0\Longrightarrow f(a)\vee f(b)=0\vee 0\Longrightarrow f(a\vee b)=0$
$\Longrightarrow a\vee b\in\ker(f)$. Now, Assume that $a\in\ker(f)$ and $b\in B_1\Longrightarrow f(a)=0$ and $f(b)\in B_2$. But
$f(a\wedge b)=f(a)\wedge f(b)=0\wedge f(b)=f(0)\wedge f(b)=0\Longrightarrow a\wedge b\in\ker(f)$. Hence $\ker(f)$ is a Boolean ideal of B_1. Suppose that $b\in\ker(f)\Longrightarrow f(b)=0\Longrightarrow\exists(f(b))=\exists(0)\Longrightarrow f(\exists(b))=0$. Hence
$\exists(b)\in ker(f)$, So $ker(f)$ is a monadic Boolean ideal of B_1.

Let B be a monadic algebra and I be a monadic ideal of B. Let $B/_I=\{[a]: a\in B\}$ be denotes the quotient set.

Define $\eta: B\longrightarrow B/_I$ such that $\eta(a)=[a],\forall a\in B$. η is called canonical (natural) map and it is an epimorphism.

Consider b_1 and $b_2\in B$. such that $\eta(b_1)=\eta(b_2)$. Therefore $\eta(b_1)+\eta(b_2)=0\Longrightarrow\eta(b_1+b_2)=0\Longrightarrow b_1$ and $b_1+b_2\in I\Longrightarrow\exists(b_1+b_2)\in I\Longrightarrow\exists(b_1)+\exists(b_2)\in I\Longrightarrow\eta\big(\exists(b_1)+\exists(b_2)\big)=\eta\big(\exists(b_1)\big)+\eta\big(\exists(b_2)\big)\in I$
$\Longrightarrow\eta\big(\exists(b_1)\big)=\eta\big(\exists(b_2)\big)$. According to above argument, define existential quantifier $\exists$ on $B/_I$ as follows:
$\exists([a])=\eta\big(\exists(a)\big)=[\exists(a)]$. $\exists$ is well-defined and satisfies the following axioms,

1. $\exists[0]=\eta\big(\exists(0)\big)=[\exists(0)]=[0]$.
2. Since $a\le\exists(a)$,therefore $a\wedge\exists(a)=a\Longrightarrow\eta\big(a\wedge\exists(a)\big)=\eta(a)\Longrightarrow\eta(a)\wedge\eta\big(\exists(a)\big)=\eta(a)$
$\Longrightarrow [a]\wedge\exists(a)=[a]\Longrightarrow [a]\le\exists(a)$ and
3. $\exists([a])\wedge\exists([b])=\eta(\exists(a\wedge\exists(b))=\eta(\exists(a)\wedge\exists(b))$
$=\eta\big(\exists(a)\big)\wedge\eta(\exists(b))$.
$=[\exists(a)]\wedge[\exists(b)]$.
$=\exists[a]\wedge\exists[b]]$ and consequently $B/_I$ is a monadic quotient algebra.

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Definition 2.9: Let $\mathrm{M}=(B,\exists)$ be a monadic algebra. A mapping $c:\mathrm{M}\longrightarrow\mathrm{M}$ is called constant, if c is a Boolean endomorphism such that $c\exists=\exists$ and $\exists c=c$. The following theorems tells us about the main properties of constant mapping on monadic algebra $\mathrm{M}=(B,\exists)$.

Theorem 2.10: Consider $\mathrm{c}:\mathrm{M}\longrightarrow\mathrm{M}$ is a constant on monadic algebra $\mathrm{M}=(B,\exists)$. Then:

1. $c_{|\exists(B)}=id$.
2. $c(B)\subseteq\exists(B)$.
3. If c is a Boolean endomorphism satisfying 1 and 2, then c is constant.
4. c is idempotent function i.e., $\mathrm{c}^2=c$.
5. $\mathrm{c}(a)\le\exists(a),\forall a\in B$.
6. $\mathrm{c}\forall=\forall$.
7. $\forall c=c$.
8. If c is a Boolean endomorphism satisfying 6 and 7, then c is constant.
9. $\forall(a)\le c(a)$ for all $a\in B$.

Proof:

1. Let $\exists(b) \in \exists(B)$, implies that $c\big(\exists(b)\big) = (c\exists)(b) = \exists(b)$. Hence $c_{|\exists(B)} = id$.
2. Suppose that $c(b) \in c(B) \Longrightarrow c(b) = (\exists c)(b) = \exists\big(c(b)\big) \in \exists(B)$.
3. $(c\exists)(a) = c\big(\exists(a)\big) = \exists(a)$. Hence $c\exists = \exists$
   $(\exists c)(b) = \exists\big(c(b)\big) = \exists\big(\exists(a)\big)$, where $c(b) = \exists(a)$, since $c(B) \subseteq \exists(B)$.
   $= \exists(a) = c\exists(a) = c(b)$. Hence $\exists(c) = c$.
4. $c^2(a) = c\big(c(a)\big) = c\big(\exists c(a)\big) = (c\exists)\big(c(a)\big) = \exists\big(c(a)\big) = c(a)$. Hence $c^2 = c$.
5. Since $a \leq \exists(a), \forall\, a \in B$. Therefore $a \wedge \exists(a) = a \Longrightarrow c\big(a \wedge \exists(a)\big) = c(a) \Longrightarrow c(a) \wedge c\exists(a) = c(a)$
   $\Longrightarrow c(a) \wedge \exists(a) = c(a) \Longrightarrow c(a) \leq \exists(a)$ for all $a \in B$.
6. $(c\forall)(a) = c\big(\forall(a)\big) = c((\exists(a'))' = (c(\exists(a'))' = ((c\exists)(a'))' = (\exists(a'))' = \forall(a)$. Hence $c\forall = \forall$.
7. $\forall c(a) = \forall\big(c(a)\big) = (\exists(c(a))')' = \Big(\exists\big(c(a')\big)\Big)' = \big(c(a')\big)' = c(a')' = c(a)$. Therefore $\forall c = c$.
8. To prove that $\exists c = c$ and $c\exists = \exists$. Since $\forall c = c \Longrightarrow (\forall c)(a') = c(a') \Longrightarrow \forall\big(c(a')\big) = c(a')$
   $\Longrightarrow \Big(\exists\big(c(a')\big)'\Big)' = c(a') \Longrightarrow \exists(c(a')' = \big(c(a')\big)' \Longrightarrow \exists\big(c(a)\big) = c(a) \Longrightarrow (\exists c)(a) = c(a)$. Hence $\exists c = c$.
   Also since $c\forall = \forall \Longrightarrow (c\forall)(a) = \forall(a) \Longrightarrow c\big(\forall(a)\big) = \forall(a) \Longrightarrow c\Big(\big(\exists(a')\big)'\Big) = \big(\exists(a')\big)'$
   $\Longrightarrow \Big(c\big(\exists(a')\big)'\Big) = \big(\exists(a')\big)' \Longrightarrow c\big(\exists(a')\big) = \exists(a') \Longrightarrow (c\exists)(a') = \exists(a')$. Hence $c\exists = \exists$.
9. Since $\forall(a) \leq a$ for all $a \in B$. Therefore, $\forall(a) \wedge a = \forall(a) \Longrightarrow c(\forall(a) \wedge a) = c\big(\forall(a)\big)$
   $\Longrightarrow (c\forall)(a) \wedge c(a) = (c\forall)(a) \Longrightarrow \forall(a) \wedge c(a) = \forall(a) \Longrightarrow \forall(a) \leq c(a)$ for all $a \in B$.

3. SIMPLE MONADIC ALGEBRAS

Definition 3.1: A monadic algebra is called simple, if $\{0\}$ is the only proper ideal in it.

Definition 3.2: Let M be a monadic ideal of monadic algebra $\mathrm{M} = (B, \exists)$, then M is called maximal monadic ideal if for any monadic ideal I with $M \subseteq I \subseteq B$ either $M = I$ or $I = B$.

Theorem 3.3: A monadic Algebra$\mathrm{M} = (B, \exists)$ is called simple if and only if its quantifier is simple.

Proof: Let $\mathrm{M} = (B, \exists)$ be a simple monadic algebra. To show that the quantifier $\exists$ on B is simple. If $b = 0$, then $\exists(b) = \exists(0) = 0$. Now, if $b \in B$ and $b \neq 0$. Define $I = \{a : a \leq \exists(b)\}$. To prove that I is a monadic ideal of B.

Let $b_1, b_2 \in I \Longrightarrow b_1 \leq \exists(b)$ and $b_2 \leq \exists(b) \Longrightarrow b_1 \vee b_2 \leq \exists(b) \vee \exists(b) = \exists(b) \Longrightarrow b_1 \vee b_2 \in I$.

Let $a \in I \Longrightarrow a \leq \exists(b)$, but $b \leq \exists(b)$ for all $b \in B \Longrightarrow a \wedge b \leq \exists(b) \vee \exists(b) = \exists(b) \Longrightarrow a \wedge b \in B$. Hence I is an ideal in B. Suppose that $a \in I \Longrightarrow a \leq \exists(b) \Longrightarrow \exists(a) \leq \exists\big(\exists(b)\big) = \exists(b) \Longrightarrow \exists(a) \in I$. Therefore I is a monadic ideal of B.
Since $\mathrm{M} = (B, \exists)$ is simple monadic algebra, therefore $I = B$.

Conversely, let $\exists$ be a simple quantifier, therefore $\exists(b) = 1$, whenever $b \neq 0$. Suppose that I is a monadic ideal of B.

If $b \in I \Longrightarrow \exists(b) \in I \Longrightarrow 1 \in I$. Hence $I = B$, consequently, B is a monadic simple algebra.

Definition 3.4: A Booleansub-algebra M of B^Xwhich given by definition (1.10) and theorem (1.11) in [1]is called **a** B-valued functional monadic algebra with domain X (or a functional monadic algebra) if (i). for any $p \in M$,then sup $R(p)$ and inf $R(p)$ exists in B, where $R(p) = \{p(x) : x \in X\}$,and (ii). $\exists\big(p(x)\big) = \sup R(p)$ and $\forall\big(p(x)\big) = \inf R(p) \in M$

Remark: The symbol O will be used to denote the 2-elments Boolean algebra$\{0,1\}$. O is a Boolean subalgebra of every Boolean algebra, see example (1.11) in [1].

Theorem 3.5: A monadic algebra is simple if and only if it is isomorphic to an O-valued functional monadic algebra.

Proof: Let B an O-valued functional monadic algebra with domain X. To prove that B is simple monadic algebra. Let $p \in B$ such that $p \neq 0 \Longrightarrow p(x_0) = 1$ for some $x_0 \in X \Longrightarrow 1 \in R(p) \Longrightarrow SupR(p) = 1 \Longrightarrow \exists(b) = 1$ (by definition 3.4). Hence $\exists$ is a simple monadic algebra on B. Therefore B is simple monadic (by theorem 3.3). Conversely, let B be a simple monadic algebra. Since by Stone's theorem there exists:

i. A set X,
ii. Boolean sub-algebra A of O^X, and
iii. A Boolean isomorphism $f : \mathrm{B} \longrightarrow \mathrm{A}$. Also $\exists$ on B is simple, since $\exists$ on A is simple (by first part of proof). Therefore $f\big(\exists(a)\big) = \exists(f(a))$ for all $a \in B$. Therefore f is an isomorphism. Hence $B \cong A$.

Definition 3.6: A monadic algebra B is semisimple if the intersection of all maximal ideals in B is$\{0\}$.

Theorem 3.7: Every Monadic algebra is semisimple**.**

Proof: Let B a monadic algebra Assume that $a \neq 0$. To prove that there is a monadic maximal ideal I in B such that $a \notin I$. In particular, B is a Boolean algebra, then there is a maximal ideal I_0 in B such that $a \notin I_0$. Define $I = \exists^{-1}(I_0) = \{b \in B : \exists(b) \in I_0\}$. Let b and $\in I$, therefore $\exists(b) \in I_0$ and $\exists(c) \in\in I_0$. Therefore $\exists(b) \vee \exists(c) \in I_0$ (since I_0 is maximal ideal), But $\exists(b \vee c) = \exists(b) \vee \exists(b)$,implies that $b \vee c \in I$. Likewise, let $b \in I$ and $d \in \exists^{-1}(B)$.

Therefore, $\exists(b) \in I_0$ and $\exists(d) \in B \Longrightarrow \exists(b) \wedge \exists(d) \in I_0$ (Since I_0 is maximal ideal) But $\exists(b \wedge d) \leq \exists(b) \wedge \exists(d)$. Hence $b \wedge d \in I$. Therefore I is an ideal of B. Let $b \in I \Longrightarrow \exists(b) \in I_0$, since $\exists(b) = \exists(\exists(b)) \Longrightarrow \exists(b) \in I$. Therefore I is a monadic ideal of B. Finally, to prove that I is a maximal ideal. Suppose that J is a monadic ideal properly including I (i.e. $I \subseteq J$). Therefore J contain an element b such that $\Longrightarrow \exists(b) \notin I_0$, therefore $\Longrightarrow \exists(b) \in J$ (by assume) and $(\exists(b))' \in I_0$) (Since $I_{0,}$ is maximal ideal) ,but $\exists(\exists(b))' = (\exists(b))' \Longrightarrow (\exists(b))' \in I \subseteq J$. Hence $(\exists(b))' \in J$, therefore $1 = \exists(b) \vee (\exists(b))' \in J\ Hence\ J = B$.

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Source of support: Nil, Conflict of interest: None Declared.