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APPLICATION OF RAMANUJAN'S P-Q THETA FUNCTION IDENTITIES OF LEVEL 15 TOCOLOR PARTITION IDENTITIES

H.T. SHWETHA*[1] AND N. BHASKAR[2]

**Department of Mathematics,
Vidyavardhaka College of Engineering, Mysuru-570002, India.**

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ABSTRACT

M.Somos discovered around 6277 theta-function identities of different levels using computer and offered no proof for them and these identities highly resembles Ramanujan's recordings.

The purpose of this paper is to establish colorpartition identities to three Somostheta function identities of level 15.

Keywords*: Theta-functions, Dedekindη - functions, color partitions.*

2010 Mathematics Subject Classification*: 11F11, 11F20, 11P83.*

1. INTRODUCTION

Throughout this paper, we assume $|q| < 1$. Let

$$f(-q) = \prod_{n=1}^{\infty}(1 - q^n)$$

For $q=e^{2\pi i\tau}$,$f(-q) = e^{\frac{-\pi i\tau}{12}}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind η-function for $Im(\tau) > 0$. For convenience we set $f(-q^k) = f_k$.

Ramanujan recorded several identities which involve $f(-q), f(-q^n), f(-q^m)$ and $f(-q^{mn})$ called level mn in his second notebook [3] and Lost Notebook [4].

For example

$$f_1{}^4 f_2{}^4 f_5{}^2 f_{10}{}^2 + 5 f_1{}^2 f_2{}^2 f_5{}^4 f_{10}{}^4 = f_2{}^6 f_5{}^6 + f_1{}^6 f_{10}{}^6.$$

Michael Somos recently used a computer to discover several new elegant theta-function identities in the spirit of Ramanujan and offered no proof for them. Somos has a large list of η-product identities and he runs PARI/GP scripts to look at each identity in P -Q forms. Recently B. Yuttanan [5] has proved certain Somostheta-function identities of different levels by employing

Ramanujan'smodular equations and K.R.Vasuki and R.G.Veeresha [6] proved η-function identities of level 14 discovered by Somos.

The purpose of this paper, is to establish certain interesting colorpartition of Ramanujan's identities of level 15 conjectured by Somos.

Corresponding Author: H.T. Shwetha*[1],
Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru-570002, India.

1. Somoside n tities of level 15

In this section we state Ramanujan's identities of level15.For

$$X=\frac{f_1}{q^{\frac{1}{12}}f_3}\ ,\ Y=\frac{f_5}{q^{\frac{5}{12}}f_{15}}\ ,P=\frac{f_1}{q^{\frac{1}{6}}f_5}\ \text{ and } Q=\frac{f_3}{q^{\frac{1}{2}}f_{15}}$$

We have

Theorem 2.1: *[1,p.221][3,p.325]*

$$(XY)^2+5+\frac{9}{(XY)^2}=\left(\frac{Y}{X}\right)^3-\left(\frac{X}{Y}\right)^3, \tag{2.1}$$

Theorem 2.2: *[1,p.223][3,p.323]*

$$(PQ)^3+\frac{125}{(PQ)^3}=\left(\frac{Q}{P}\right)^6-9\left(\frac{Q}{P}\right)^3-9\left(\frac{P}{Q}\right)^3-\left(\frac{P}{Q}\right)^6, \tag{2.2}$$

Theorem 2.3: *[1,p.226]*

$$(PQ)^3-\frac{125}{(PQ)^3}=(XY)^4+(XY)^2-\frac{9}{(XY)^2}-\frac{81}{(XY)^4}. \tag{2.3}$$

Somos also rediscovered the above three identities, a proof of these identities can be found in [1,pp.221–230].

From these identities, in Section 3 we deduce certain interesting color partition identities.

2. COLORPARTTTION

The Somos's identities, mentioned in Section 2, have interesting applications to color partition. Sen-Shan Huang introduced color partition in [2]. A positive integer n has k colors if there are k copies of n and all of them are viewed as distinct objects. Partition of a positive integer into parts with colors are called *"colored partitions"*.

For example, if 1 is allowed to have 2 colors, the n all the (colored) partitions of 2 are $2, 1_r+1_r, 1_g+1_g$ and 1_r+1_g where we use the indices r (red) and g (green) to distinguish two copies of 1.

The generating function for the number of partitions of n, where all the parts are congruent to u (mod$v$) and have k color is

$$\frac{1}{(q^u;q^v)_\infty{}^k},$$

Where

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^n).$$

Definition 3.1: *Let P (n, k, l, m) denote the number of partition of n into parts not congruent to 0(*mod*15), with parts congruent to 0(*mod*3) having k colors and parts congruent to 0(*mod *5) having l colors and parts not congruent to 0(*mod*3) or 0(*mod*5) having m colors.*

Definition 3.2: *We define*

$$(a_1,a_2,a_3,\dots a_n;q)_\infty=\prod_{k=1}^{\infty}(a_k;q)_\infty$$

And

$$(q^{r_1\pm},q^{r_2\pm},q^{r_3\pm},\dots,q^{r_n\pm};q^s)_\infty:=(q^{r_1},q^{r_2},\dots,q^{r_n},q^{s-r_1},q^{s-r_2}\dots,q^{s-r_n};q^s)_\infty$$

where $r_i<s$ with $1\le i\le n$.

For example,

$$(q^{1\pm},q^{2\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{6\pm},q^{7\pm};q^{15})_\infty:=(q^1,q^2,q^3,q^4,q^5,q^6,q^7;q^{15})_\infty$$

Theorem 3.3: We have, for $n\ge 2$

$$\mathrm{P(n+2,6,2,7)+5P(n,6,6,9)+9(n-2,6,10,11)=P(n+1,6,12,12)+P(n-1,6,6,6)},$$

Where $P(0,k,l,m)=1$.

Proof: Dividing (2.1) by ${f_1}^{12}$, we find that

$$\frac{1}{q^2}\left(\frac{f_3}{f_1}\right)\left(\frac{f_5}{f_1}\right)^5\left(\frac{f_{15}}{f_1}\right)+5\left(\frac{f_3}{f_1}\right)^3\left(\frac{f_5}{f_1}\right)^3\left(\frac{f_{15}}{f_1}\right)^3+9q^2\left(\frac{f_3}{f_1}\right)^5\left(\frac{f_5}{f_1}\right)\left(\frac{f_{15}}{f_1}\right)^5-\frac{1}{q}\left(\frac{f_3}{f_1}\right)^6\left(\frac{f_{15}}{f_1}\right)^6-q\left(\frac{f_{15}}{f_1}\right)^6=0.$$

This implies

$$\begin{aligned}\frac{1}{q^2}\frac{1}{({q_7}^{1\pm},{q_7}^{2\pm},{q_6}^{3\pm},{q_7}^{4\pm},{q_2}^{5\pm},{q_6}^{6\pm},{q_7}^{7\pm};q^{15})_\infty}+\frac{5}{({q_9}^{1\pm},{q_9}^{2\pm},{q_6}^{3\pm},{q_9}^{4\pm},{q_6}^{5\pm},{q_6}^{6\pm},{q_9}^{7\pm};q^{15})_\infty}\\ +\frac{9q^2}{({q_{11}}^{1\pm},{q_{11}}^{2\pm},{q_6}^{3\pm},{q_{11}}^{4\pm},{q_{10}}^{5\pm},{q_6}^{6\pm},{q_{11}}^{7\pm};q^{15})_\infty}\\ -\frac{1}{q}\frac{1}{({q_{12}}^{1\pm},{q_{12}}^{2\pm},{q_6}^{3\pm},{q_{12}}^{4\pm},{q_{12}}^{5\pm},{q_6}^{6\pm},{q_{12}}^{7\pm};q^{15})_\infty}\\ +\frac{q}{({q_6}^{1\pm},{q_6}^{2\pm},{q_6}^{3\pm},{q_6}^{4\pm},{q_6}^{5\pm},{q_6}^{6\pm},{q_6}^{7\pm};q^{15})_\infty}=0.\end{aligned}$$

Employing the definition of $P(n,k,l,m)$ in the above , we obtain

$$\sum_{n=0}^{\infty}P(n+2,6,2,7)q^n+5\sum_{n=0}^{\infty}P(n,6,6,9)q^n+9\sum_{n=0}^{\infty}P(n-2,6,10,11)q^n-\sum_{n=0}^{\infty}P(n+1,6,12,12)q^n$$
$$-\sum_{n=0}^{\infty}P(n-2,6,10,11)q^n=0.$$

On comparing the coefficient of q^n, we obtain the required result.

Similarly as above we can deduce from (2.2) and (2.3) the following color partition identities respectively.

Theorem 3.4: we, have, for $n\geq 2$

$$\begin{aligned}&P(n+2,6,12,15)+125\,P(n-2,19,12,21)\\&=P(n+2,12,12,24)-9P(n+1,12,12,21)-9P(n-1,12,12,15)-P(n-2,12,12,12).\end{aligned}$$

We have, for $n\geq 2$

$$\begin{aligned}&P(n+2,2,8,9)-125P(n-2,14,8,15)\\&=P(n+2,8,0,8)+P(n+1,8,4,10)-9(n-1,12,12,14)-81P(n-2,6,16,16).\end{aligned}$$

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