International Journal of Mathematical Archive-2(10), 2011, Page: 2005-2012
Available online through www.ijma.info ISSN 2229 – 5046

MODIFIED FOURIER'S METHOD FOR SOLVING QUADRATIC PROGRAMMING PROBLEMS

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(Received on: 29-09-11; Accepted on: 16-10-11)

ABSTRACT

We have applied Fourier's variables elimination method for solving the QPP. The K-T conditions are used to transform the QPP into linear inequalities and complementary slackness conditions play an important role to minimize the Lagrangian multipliers.

INTRODUCTION:

A Quadratic Programming Problem (QPP) is both a special case of non-linear programming problem and an extended case of linear programming problem, in which the objective function is a sum of a linear and a quadratic form and the constraints are linear.

The general mathematical model of a QPP may be written as:

$$\begin{aligned} &Maximize \quad Q(X) = C'X + X'DX \\ &Subject\ to \qquad AX \le b \\ &\qquad\qquad\qquad\quad X \ge 0 \end{aligned}$$

where D is an $n \times n$ symmetric matrix and all other symbols are usual notations.

Methods for solving a QPP are developed under the assumption that the quadratic form $X'DX$ is concave for maximization and convex for minimization. Wolfe (1959), using the Kuhn-Tucker (K-T) conditions, developed the original approach to QPP. The K-T conditions form a large linear program, with additional non-linear complementary slackness conditions. Wolfe then utilized a variant of the simplex algorithm which, incorporated provisions to enforce the complementary slackness conditions. Many available algorithms follow these principles. In this paper, we apply Fourier's Method to obtain the optimal solution to the QPP after deriving the problem into linear inequalities by using the Kuhn-Tucker (K-T) necessary conditions. Fourier (1826) discovered a method for solving the linear inequalities. Later Williams (1986) showed that the Fourier's method could be extended to solve the linear programming problems. In his method, the Fourier Variable Elimination Method generates a new set of constraints in which some constraints are redundant. Then Kanniappan and Thangavel (1998) modified this method by giving a technique to find that which variable should be eliminated first. This technique reduces the number of constraints generated.

DERIVATION OF K-T CONDITIONS:

The K-T necessary conditions are derived for identifying stationary points of a QPP subject to inequality constraints. This derivation is based on the Lagrangian method. These conditions are

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also sufficient for global optimum if the objective function is concave (convex) for maximization (minimization) and the solution space is convex set.
Let QPP is defined as

$$\begin{aligned} &Maximize \quad f(X) = C'X + X'DX \\ &Subject\ to \quad g(X) \le 0 \end{aligned}$$

The problem may be written as

$$\begin{aligned} &Maximize \quad f(X) = C'X + X'DX \\ &Subject\ to \quad g(X) + S^2 = 0 \end{aligned}$$

where S^2 is the slack variable to the constraints $g(X) \le 0$.

Let λ be the Lagrangian multipliers corresponding to the constraints $g(X) \le 0$

The Lagrangian function is thus given by:

$$L(X, \lambda, S) = f(X) - \lambda[g(X) + S^2]$$

A necessary condition for optimality is that, λ be non-negative for maximization problems. This result is justified as follows. The vector λ measures the rate of variation of f with respect to g, i.e., $\lambda = \frac{\partial f}{\partial g}$, as the right hand side of the constraint $g(X) \le 0$ changes from 0 to $\partial g (> 0)$, the solution space becomes less constrained and hence f cannot decrease, which implies that $\lambda \ge 0$. The remaining conditions will now be derived as the partial derivatives of L with respect to X, λ and S.

$$\frac{\partial L}{\partial X} = \nabla f(X) - \lambda \nabla g(X) = 0$$

$$\frac{\partial L}{\partial \lambda} = -[g(X) + S^2] = 0$$

$$\frac{\partial L}{\partial S} = -2\lambda S = 0$$

The conditions reduce to

$$-2X'D + \lambda' A = C \quad \ldots\ldots (i)$$

$$AX + S = b \quad \ldots\ldots (ii)$$

$$\lambda' S = 0 \quad \ldots\ldots (iii)$$

$$X, \lambda, S \ge 0$$

The conditions (i), (ii) and (iii) are respectively referred to as dual feasibility, primal feasibility and complementary slackness (CS) feasibility conditions.

The equation $\lambda_i S_i = 0$, $i = 1, 2, \ldots, m$ reveals the following results:

(1) If $\lambda_i \ne 0$, then $S_i = 0$. This means that the corresponding resource is scarce and, hence, it is consumed completely.

(2) If $S_i > 0$, then $\lambda_i = 0$. This means resource i is not scarce and, consequently, it has no effect on the value of $f(X)$.

The iterative procedure for the solution of a QPP by Modified Fourier's Method may be summarized as follows:

Step: 1 Convert the inequality constraints into equations by introducing slack variables S_i^2 in the i^{th} constraint $i = 1,2,\ldots,m$, and slack variables Y_j^2 in the j^{th} non-negativity constraint $j = 1,2,\ldots,n$.

Step: 2 Construct the Lagrangian function

$$L(X,S,\lambda,\mu,Y) = f(X) - \sum_{i=1}^{m} \lambda_i [\sum_{j=1}^{n} a_{ij} x_j - b_i + S_i^2] - \sum_{j=1}^{n} \mu_j (-x_j + Y_j^2)$$

where λ and μ are Lagrangian multipliers.

Step: 3 The Kuhn-Tucker (K-T) conditions may be derived as

$$\begin{aligned} &-2X'D + \lambda' A - \mu = C \\ &AX + S = b \\ &X, \lambda, S \geq 0 \\ &\lambda\, S = 0 = \mu\, X \end{aligned}$$

Step: 4 Arrange the above equations into a canonical form with slack variables S (> 0)

$$\begin{aligned} &-2X'D + \lambda' A - \mu \leq C \\ &-2X'D + \lambda' A - \mu \geq C \\ &AX \leq b \\ &X,\ \lambda, \mu \geq 0 \end{aligned}$$

Step: 5 In step 4, if $S > 0$, then to satisfy the complementary slackness conditions $(\lambda\, S = 0)$, minimize λ, the problem becomes as

$$\begin{aligned} &Maximize\ Z = -\lambda \\ &-2X'D + \lambda' A \leq C \\ &-2X'D + \lambda' A \geq C \\ &AX \leq b \\ &X,\ \lambda \geq 0 \end{aligned}$$

Step: 6 Arrange the inequalities into an equivalent system of linear constraints.

$$\begin{aligned} &Z + \lambda \leq 0 \\ &-2X'D + \lambda' A \leq C \\ &2X'D - \lambda' A \leq -C \\ &AX \leq b \\ &X \leq 0 \\ &-\lambda \leq 0 \end{aligned}$$

Step: 7 Construct the following pairwise disjoint sets for all variables i.e.,

$$I_j^+ = \{t : A_{t1} > 0\}$$
$$I_j^- = \{t : A_{t1} < 0\}$$
$$I_j^0 = \{t : A_{t1} = 0\}$$
$$where\ t = 1, 2, \ldots, m\ and\ \ j = 1, 2, \ldots, n$$

Step: 8 If any of the sets I_j^+ or I_j^- is empty for a variable, then the given problem is unbounded.

Step: 9 Otherwise, find the $\underset{1 \le j \le n}{Minimum}\{\left|I_j^+\right| \times \left|I_j^-\right|\}$, where $\left|C\right|$ denotes the number of constraints in the set C.

Step: 10 Choose the index j corresponding to the minimum in the above Step. Let it be X_j. Now we choose this variable to be eliminated first.

Step: 11 Apply the Fourier variable elimination method (1827) for eliminating the variable X_j as follows:

For each $k \in I_j^+$, $1 \in I_j^-$ we add A_{t1} times the inequality $A_k X \le b_k$ to $-A_{kt}$ time $A_1 X \le b_1$. In these new inequalities, the coefficient X_j is eliminated.

Repeat the above procedures until all the variables are eliminated except the variable Z. The least upper bound for Z will be considered as the optimal value of Z.

NUMERICAL ILLUSTRATION:

Consider the following QPP

$$\begin{aligned} &Maximize\ \ Q(X) = 2X_1 + 6X_2 - X_1^2 + 2X_1X_2 - 2X_2^2 \\ &Sub\ to. \qquad X_1 + X_2 \le 2 \\ &\qquad\qquad -X_1 + X_2 \le 2 \\ &\qquad\qquad X_1, X_2 \ge 0 \end{aligned}$$

We convert the inequality constraints into equations by introducing slack variables $S_1{}^2$ and $S_2{}^2$ respectively. Considering $X_1 \ge 0$ and $X_2 \ge 0$ also as the inequality constraints, we convert them also into equations by using slack variable $S_3{}^2$ and $S_4{}^2$. The problem thus becomes:

$$\begin{aligned} &Maximize\ \ Q(X) = 2X_1 + 6X_2 - X_1^2 + 2X_1X_2 - 2X_2^2 \\ &Sub\ to. \qquad X_1 + X_2 + S_1{}^2 - 2 = 0 \\ &\qquad\qquad -X_1 + X_2 + S_2{}^2 - 2 = 0 \\ &\qquad\qquad -X_1 + V_1{}^2 = 0\ ,\ -X_2 + V_2{}^2 = 0 \end{aligned}$$

constructing the Lagrangian function

$$L(X,S,\lambda) = 2X_1 + 6X_2 - X_1^2 + 2X_1X_2 - 2X_2^2 - \lambda_1(X_1 + X_2 + S_1{}^2 - 2)$$
$$- \lambda_2(-X_1 + X_2 + S_2{}^2 - 2) - \mu_1(-X_1 + V_1{}^2) + \mu_2(-X_2 + V_2{}^2)$$

To derive the necessary and sufficient conditions for maxima of L we equate the first-order partial derivatives of L equal to zero. Thus we have

$$\begin{aligned}
&2X_1 - 2X_2 + \lambda_1 - \lambda_2 - \mu_1 = 2\\
&-2X_1 + 4X_2 + \lambda_1 + 2\lambda_2 - \mu_2 = 6\\
&X_1 + X_2 + S_1 = 2\\
&-X_1 + 2X_2 + S_2 = 2\\
&\lambda_i S_i = 0 = \mu_i V_i\\
&X_i, \lambda_i, S_i, \mu_i, V_i \ge 0 \quad , i = 1,2
\end{aligned}$$

In the above equalities, if we assume the slack variables $S_1 > 0$ and $S_2 > 0$, then by complementary slackness criteria λ_1 and λ_2 must be equal to zero. Problem may thus be rewritten as follows:

$$\begin{aligned}
&Maximize\, Z = -\lambda_1 - \lambda_2\\
&2X_1 - 2X_2 + \lambda_1 - \lambda_2 - \lambda_3 \le 2\\
&2X_1 - 2X_2 + \lambda_1 - \lambda_2 - \lambda_3 \ge 2\\
&-2X_1 + 4X_2 + \lambda_1 + 2\lambda_2 - \lambda_4 \le 6\\
&-2X_1 + 4X_2 + \lambda_1 + 2\lambda_2 - \lambda_4 \ge 6\\
&X_1 + X_2 \le 2\\
&-X_1 + 2X_2 \le 2\\
&X_1, X_2, \lambda_1, \lambda_2 \ge 0
\end{aligned}$$

Arrange the above problem into an equivalent system of linear inequalities

$$\begin{aligned}
&Z + \lambda_1 + \lambda_2 \le 0 && (A_1)\\
&2X_1 - 2X_2 + \lambda_1 - \lambda_2 - \lambda_3 \le 2 && (A_2)\\
&-2X_1 + 2X_2 - \lambda_1 + \lambda_2 + \lambda_3 \le -2 && (A_3)\\
&-2X_1 + 4X_2 + \lambda_1 + 2\lambda_2 - \lambda_4 \le 6 && (A_4)\\
&2X_1 - 4X_2 - \lambda_1 - 2\lambda_2 + \lambda_4 \le -6 && (A_5)\\
&X_1 + X_2 \le 2 && (A_6)\\
&-X_1 + 2X_2 \le 2 && (A_7)
\end{aligned}$$

$$\begin{aligned}
&-X_1 \le 0 && (A_8)\\
&-X_2 \le 0 && (A_9)\\
&-\lambda_1 \le 0 && (A_{10})\\
&-\lambda_2 \le 0 && (A_{11})\\
&-\lambda_3 \le 0 && (A_{12})\\
&-\lambda_4 \le 0 && (A_{13})
\end{aligned}$$

Now we apply the Fourier Variable Elimination Method.

$I^{+}_{\lambda_4} = \{A_5\}$, $I^{-}_{\lambda_4} = \{A_4, A_{13}\}$; $I^{+}_{\lambda_3} = \{A_3\}$, $I^{-}_{\lambda_3} = \{A_2, A_{12}\}$;

$I^{+}_{\lambda_2} = \{A_1, A_3, A_4\}$, $I^{-}_{\lambda_2} = \{A_2, A_5, A_{11}\}$; $I^{+}_{\lambda_1} = \{A_1, A_2, A_4\}$, $I^{-}_{\lambda_1} = \{A_3, A_5, A_{10}\}$;

$I^{+}_{x_2} = \{A_3, A_4, A_6, A_7\}$, $I^{-}_{x_2} = \{A_2, A_5, A_9\}$; $I^{+}_{x_1} = \{A_2, A_5, A_6\}$, $I^{-}_{x_1} = \{A_3, A_4, A_7, A_8\}$

$$\underset{1\le j\le n}{Minimum}\{\left|I_j^{+}\right| \times \left|I_j^{-}\right|\} = \{1\times 2\} = 2$$

Eliminating first λ_3 and λ_4 simultaneously because, these are at independent positions.

$$\begin{array}{lll}
Z+\lambda_1+\lambda_2 \le 0 & (A_1) & (B_1)\\
0 \le 0 & (A_2+A_3) & (B_2)\\
-2X_1+2X_2-\lambda_1+\lambda_2 \le -2 & (A_3+A_{12}) & (B_3)\\
0\le 0 & (A_4+A_5) & (B_4)\\
2X_1-4X_2-\lambda_1-2\lambda_2 \le -6 & (A_5+A_{13}) & (B_5)\\
X_1+X_2 \le 2 & (A_6) & (B_6)\\
-X_1+2X_2 \le 2 & (A_7) & (B_7)\\
-X_1 \le 0 & (A_8) & (B_8)\\
-X_2 \le 0 & (A_9) & (B_9)\\
-\lambda_1 \le 0 & (A_{10}) & (B_{10})\\
-\lambda_2 \le 0 & (A_{11}) & (B_{11})
\end{array}$$

Similarly, eliminating λ_1, we get

$$\begin{array}{lll}
Z+2\lambda_2-2X_1+2X_2 \le -2 & (B_1+B_3) & (C_1)\\
Z-\lambda_2+2X_1-4X_2 \le -6 & (B_1+B_5) & (C_2)\\
Z+\lambda_2 \le 0 & (B_1+B_{10}) & (C_3)\\
X_1+X_2 \le 2 & (B_6) & (C_4)\\
-X_1+2X_2 \le 2 & (B_7) & (C_5)\\
-X_1 \le 0 & (B_8) & (C_6)\\
-X_2 \le 0 & (B_9) & (C_7)\\
-\lambda_2 \le 0 & (B_{11}) & (C_8)
\end{array}$$

Eliminating λ_2, we get

$$\begin{array}{lll}
3Z+2X_1-6X_2 \le -14 & (C_1+2C_2) & (D_1)\\
Z-2X_1+2X_2 \le -2 & (C_1+2C_8) & (D_2)\\
2Z+2X_1-4X_2 \le -6 & (C_2+C_3) & (D_3)\\
Z \le 0 & (C_3+C_8) & (D_4)\\
X_1+X_2 \le 2 & (C_4) & (D_5)\\
-X_1+2X_2 \le 2 & (C_5) & (D_6)\\
-X_1 \le 0 & (C_6) & (D_7)\\
-X_2 \le 0 & (C_7) & (D_8)
\end{array}$$

Eliminating X_1, we get

$$
\begin{array}{lll}
4Z-4X_2 \le -16 & (D_1+D_2) & (E_1) \\
3Z-2X_2 \le -10 & (D_1+2D_6) & (E_2) \\
3Z-6X_2 \le -14 & (D_1+2D_7) & (E_3) \\
3Z-2X_2 \le -8 & (D_2+D_3) & (E_4) \\
2Z \le -2 & (D_3+2D_6) & (E_5) \\
2Z-4X_2 \le -6 & (D_3+2D_7) & (E_6) \\
Z+4X_2 \le 2 & (D_2+2D_5) & (E_7) \\
3X_2 \le 4 & (D_5+D_6) & (E_8) \\
X_2 \le 2 & (D_5+D_7) & (E_9) \\
-X_2 \le 0 & (D_8) & (E_{10})
\end{array}
$$

Inequalities E_4 , E_6 and E_9 are redundant.

Eliminating X_2, we get

$$
\begin{array}{lll}
5Z \le -14 & (E_1+E_7) & (F_1) \\
12Z \le -32 & (3E_1+4E_8) & (F_2) \\
7Z \le -18 & (2E_2+E_7) & (F_3) \\
9Z \le -22 & (3E_2+E_8) & (F_4) \\
18Z \le -44 & (4E_3+6E_7) & (F_5) \\
3Z \le -6 & (E_3+2E_8) & (F_6) \\
Z \le 2 & (E_7+4E_{10}) & (F_7) \\
2Z \le -2 & (E_5) & (F_8)
\end{array}
$$

From the above inequalities, we find that the least upper bound for the variable Z is $-14/5$. This value involves the inequalities E_1 and E_7, and from these inequalities we can get the variable $X_2 = 6/5$. In the same way, we will obtain the variable $X_1 = 4/5$, $\lambda_1 = 14/5$ and $\lambda_2 = 0$. Now the results may be summarized as:

$$Q^*(X) = 36/5\,, \quad X_1^* = 4/5 \quad and \quad X_2^* = 6/5$$

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