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COMMON FIXED POINT THEOREM FOR A CLASS OF MAPPINGS

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ABSTRACT

In this paper, We established Some Common Fixed Point Theorems for a class mapping in metric space.

Key Words: *Common Fixed Point, Metric Space, Self Mapping, Commuting Mapping, Continuous Mappings.*

AMS Subject Classification: *47H10, 54H25,*

1. INTRODUCTION

The following generalization of the well known Banach Contraction Principle is due to Jungck (1976)

Theorem: A Let f be a continuous self mapping of a complete metric space (X, d). If there exists a mapping $g: X \to X$ and a constant $0 \leq \alpha < 1$ such that,

(a) f and g commute,
(b) $g(X) \subset f(X)$,
(c) $d(gx, gy) \leq \alpha d(fx, fy)$ for all $x, y \in X$

Then f and g have a unique common fixed point.

Throughout this section (X, d) denotes a metric space, and R^+ the set of non negative real numbers. φ denotes the family of mapping such that each $\phi \in \varphi$, $\phi: (R^+)^5 \to R^+$, and ϕ is upper semi continuous and non decreasing in each co- ordinate variable , also for a mapping $\gamma: R^+ \to R^+$,

let $= \phi(t, t, a_1 t, a_2 t, t) < t$, where $a_1 + a_2 = 3$. The following lemma of Singh and Meade (1977) is the key in proving of various result,

Lemma: 1.1 For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the composition of γ with itself n times. In 1979, Yeh proved an interesting extension of a common fixed point theorem due to Jungck (1976), which as follows,

Theorem: B Let E, F, and T be three continuous self mapping of a complete metric space (X, d) satisfying condition:
(C_1) $ET = TE$, $FT = TF$, $E(X) \subset T(X)$ and $F(X) \subset T(X)$

(C_2) there exists an $\phi \in \varphi$ such that for all $x, y \in X$,

$d(Ex, Fy) \leq \phi\,(d(Tx, Ty), d(Tx, Ex), d(Tx, Fy), d(Ty, Ex), d(Ty, Fy))$, where ϕ, satisfies the condition:

(C_3) $g(t) \equiv \phi(t, t, at, bt, t) < t$ for each t in $R^+ - \{0\}$, where $a + b = 2$,

Then E, F, T have a unique common fixed point.

Definition: 1.2 (Sessa 1982) Let A and S be two self mapping on X , then {A,S} is said to be 'weakly commuting pair' if $d(ASx, SAx) \leq d(Ax, Sx)$ for all $x \in X$. It is clear that, commuting pair is weakly commuting, but not conversely as shown in the following example,

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Example: 1.3 Consider $X = [0,1]$ with the usual metric. Let us define $Ax = \frac{1}{2}x$ and $Sx = \frac{x}{x+2}$ for every $x \in X$, then for all $x \in X$ one gets,

$$d(SAx, ASx) = \left| \frac{x}{4+x} - \frac{x}{4+2x} \right| = \frac{x^2}{(4+x)(4+2x)}$$

$$d(SAx, ASx) \le \frac{x^2}{(4+2x)} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Ax)$$

So $\{A, S\}$ is a weakly commuting pair, However, for any non zero $x \in X$ we have,

$$SAx = \frac{x}{4+x} > \frac{x}{4+2x} = ASx$$

Thus A and S are not commuting mappings.

2. MAIN RESULT

Theorem 2.1: Let X be a complete metric space and A, B. S, T, and P be continuous mapping from X into itself, such that satisfying the following conditions:

$1C_1 - P(X) \subseteq AB(X) \cap ST(X)$

$1C_2 -$ The pair $\{P, AB\}$ and $\{P, ST\}$ are compatible.

$1C_3 -$ there exists a, $\phi \in \varphi$ such that for all $x, y \in X$,

$$d(ABx, STy) \le \phi \begin{Bmatrix} d(Px, ABx), d(Py, STy), d(Px, STy) \\ , d(Py, ABx), d(Px, Py) \end{Bmatrix}$$

Where ϕ satisfies the condition:

$1C_4 -$ for any $t > 0$, $\phi(t, t, a_1 t, a_2 t, t) < t$, where $a_1 + a_2 = 3$.

Then A, B, S, T and P have a unique common fixed point in X.

Proof: Let x_0 be and arbitrary point in X, then $Px_0 \in X$, since $P(X)$ is contained in $AB(X)$, there exists a point $x_1 \in X$, such that, $Px_0 = ABx_1$. Since $P(X)$ is also Contained in $ST(X)$, we can choose a point $x_2 \in X$, such that $Px_1 = STx_2$. In general we construct the sequence of elements of X such that,

$$ABx_{2n} = Px_{2n+1} \quad \text{and} \quad STx_{2n+1} = Px_{2n+2}$$

For all $n = 0, 1, 2, 3 \ldots\ldots$

Now

$$d(Px_{2n+1}, Px_{2n+2}) = d(ABx_{2n}, STx_{2n+1})$$

From $1C_3$, we have,

$$d(ABx_{2n}, STx_{2n+1}) \le \phi \left[\begin{pmatrix} d(Px_{2n}, ABx_{2n}), d(Px_{2n+1}, STx_{2n+1}), \\ d(Px_{2n}, STx_{2n+1}), d(Px_{2n+1}, ABx_{2n}), d(Px_{2n}, Px_{2n+1}) \end{pmatrix} \right]$$

$$d(Px_{2n+1}, Px_{2n+2}) \le \phi \left[\begin{pmatrix} d(Px_{2n}, Px_{2n+1}), d(Px_{2n+1}, Px_{2n+2}), \\ d(Px_{2n}, Px_{2n+2}), d(Px_{2n+1}, Px_{2n+1}), d(Px_{2n}, Px_{2n+1}) \end{pmatrix} \right]$$

Let us assume that, $d(Px_{2n+1}, Px_{2n+2}) = d_{2n+1}$ then

$$d_{2n+1} \le \phi\, (d_{2n}, d_{2n+1}, d_{2n} + d_{2n+1}, 0, d_{2n+1})$$

$$d_{2n+1} \le d_{2n}$$

Consequently, $\{d_{2n}\}$ is a non decreasing sequence of non negative reals, hence

$$d_1 = d(P_1, P_2) = d(ABx_0, STx_1)$$

$$d_1 \le \phi \begin{pmatrix} d(Px_0, Px_1), d(Px_1, Px_2), d(Px_0, Px_2) \\ , d(Px_1, Px_1), d(Px_0, Px_1) \end{pmatrix}$$

$$d_1 \le \phi\,(d_0, d_1, d_0 + d_1, 0, d_1)$$

$$d_1 \le \phi\,(d_0, d_0, 2d_0, d_0, d_0)$$

$$d_1 \le \gamma\,(d_0)$$

in general, we have $d_n \le \gamma^n(d_0)$ so if $d_0 > 0$, *then by lemma* 1.1 gives

$$\lim_{n\to\infty} d_n = 0$$

Since then $d_n = 0$ for each n.

Now we wish to prove that the sequence $\{Px_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d_n = 0$. It is sufficient to show that the sequence $\{Px_n\}$ is a Cauchy sequence, suppose that $\{Px_n\}$ is not a Cauchy sequence. then there is an $\varepsilon > 0$ such that for each even integers 2k , k = 0,1, 2, ……. There exists even integers 2n(k) and 2m(k) with $2k \le 2n(k) \le 2m(k)$ such that,

$$d(Px_{2n(k)}, Px_{2m(k)}) > \varepsilon \tag{2.1.1}$$

Let for each even integer 2k, 2m(k) be the least integer exceeding 2n(k) and satisfying 2.1.2,

Therefore

$$d(Px_{2n(k)}, Px_{2m(k)-2}) \le \varepsilon \quad \text{and} \quad d(Px_{2n(k)}, Px_{2m(k)}) > \varepsilon \tag{2.1.2}$$

Then , for each even integer 2k we have,

$$\varepsilon < d(Px_{2n(k)}, Px_{2m(k)}) \le d(Px_{2n(k)}, Px_{2m(k)-2}) + d(Px_{2m(k)-2}, Px_{2m(k)-1}) + d(Px_{2n(k)-1}, Px_{2n(k)})$$

So by 2.1.2 and $d_n \to 0$, we obtain

$$\lim_{n\to\infty} d(Px_{2n(k)}, Px_{2m(k)}) = \varepsilon$$

It follows immediately from the triangular inequality that,

$$|\, d(Px_{2n(k)}, Px_{2m(k)-1}) - d(Px_{2n(k)}, Px_{2m(k)}) \,| \le d_{2m(k)-1}$$

$$|\, d(Px_{2n(k)+1}, Px_{2m(k)-1}) - d(Px_{2n(k)}, Px_{2m(k)}) \,| \le d_{2m(k)-1} + d_{2n(k)}$$

Hence by 2.1.2, as $k \to \infty$

$$d(Px_{2n(k)}, Px_{2m(k)-1}) \to \varepsilon \text{ and } d(Px_{2n(k)+1}, Px_{2m(k)-1}) \to \varepsilon \tag{2.1.3}$$

Now,

$$d(Px_{2n(k)}, Px_{2m(k)}) \le d(Px_{2n(k)}, Px_{2n(k)+1}) + d(Px_{2n(k)+1}, Px_{2m(k)})$$

$$d(Px_{2n(k)}, Px_{2m(k)}) \le d_{2n(k)} + \phi \begin{pmatrix} d(Px_{2n(k)}, P_{2m(k)-1}), d_{2n(k)}, d(Px_{2n(k)}, Px_{2m(k)}), \\ d(Px_{2m(k)-1}, Px_{2n(k)+1}), d_{2m(k)-1} \end{pmatrix}$$

Using 2.1.3 $\lim_{n\to\infty} d_n = 0$, and upper semicountinuity and non decreasing property of ϕ in each co- ordinate variable , we have

$$\varepsilon \le \phi(\varepsilon, 0, \varepsilon, \varepsilon, 0) \le \gamma(\varepsilon) < \varepsilon$$

As $k \to \infty$, which contradiction. Thus $\{Px_n\}$ is a Cauchy sequence and hence by completeness of X, there is a, $u \in X$ such that $Px_n \to u$. since the sequence $\{ABx_n\}$ and $\{STx_n\}$ are Subsequence of $\{Px_n\}$ which follows $\{ABx_{2n}\}$ and $\{STx_{2n+1}\}$ also converges to the same point 'u' in X, i.e

$$\lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} STx_{2n+1} = u \tag{2.1.4}$$

$$Pu = ABu = STu$$

Let us assume that $Bu \neq u$, then we take from $1C_3$

$$d(AB(Bu), STu) \leq \phi\begin{pmatrix} d\big(P(Bu), AB(Bu)\big), d(Pu, STu), \\ d(P(Bu), STu), d\big(Pu, AB(Bu)\big), d(P(Bu), Pu) \end{pmatrix}$$

$$d(AB(Bu), STu) \leq \phi(0, 0, d(P(Bu), STu), d(Pu, AB(Bu)), d(P(Bu), Pu))$$

$$d(\ (Bu), u) \leq \gamma(d(\ (Bu), u)\ )$$

Which contradiction

$$Bu = u = ABu = A(Bu) = Au$$

Similarly we can show that,

$$Tu = u = STu = S(Tu) = Su$$

i.e, u is a common fixed point of A, B, S, T, and P in X.

Uniqueness: Let us assume that 'w' is another fixed point of A, B, S, T, and P in X, different from 'u'. i.e $u \neq w$, then

$$d(u, w) = d(Pu, Pw) = d(ABu, STw)$$

By using $1C_3$, we get

$$d(ABu, STw) \leq \phi\begin{pmatrix} d(Pu, ABu), d(Pw, STw), d(Pu, STw), \\ d(Pw, ABu), d(Pu, Pw) \end{pmatrix}$$

$$d(u, w) \leq \phi(0, 0, d(u, w), d(w, u), d(u, w))$$

$$d(u, w) \leq \gamma . d(u, w)$$

Which contradiction.

u is unique common fixed point of A, B, S, T and P in X.

Theorem: 2.2 Let X be a complete metric space and A, B. S, T, and P be continuous mapping from X into itself, such that satisfying the following conditions:

$2C_1 -$ $P(X) \subseteq AB(X) \cap ST(X)$

$2C_2 -$ The pair $\{P, AB\}$ and $\{P, ST\}$ are compatible.

$2C_3 -$ there exists a, $\phi \in \varphi$ such that for all $x, y \in X$,

$$[d(ABx, STy)]^2 \leq \phi\begin{pmatrix} \big(d(Px, ABx)\big)^2, \big(d(Py, STy)\big)^2, d(Px, STy)d(Px, ABx), \\ d(Py, STy)d(Py, ABx), d(ABx, STy)d(Px, Py) \end{pmatrix}$$

Where ϕ satisfies the condition:

$2C_4 -$ for any $t > 0$, $\phi(t, t, a_1 t, a_2 t, t) < t$, where $a_1 + a_2 = 3$.

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Proof: Let x_0 be and arbitrary point in X, then $Px_0 \in X$, since P(X) is contained in AB(X), there exists a point $x_1 \in X$, such that, $Px_0 = ABx_1$. Since P(X) is also Contained in ST(X), we can choose a point $x_2 \in X$, such that

$$Px_1 = STx_2.$$

In general we construct the sequence of elements of X such that,

$ABx_{2n} = Px_{2n+1}$ and $STx_{2n+1} = Px_{2n+2}$

For all $n = 0, 1, 2, 3 \dots\dots$

Now

$$d(Px_{2n+1}, Px_{2n+2}) = d(ABx_{2n}, STx_{2n+1})$$

From $2C_3$, we have,

$$[d(ABx_{2n}, STx_{2n+1})]^2 \leq \phi\begin{pmatrix} \left(d(Px_{2n}, ABx_{2n})\right)^2, \left(d(Px_{2n+1}, STx_{2n+1})\right)^2, \\ d(Px_{2n}, STx_{2n+1})d(Px_{2n}, ABx_{2n}) \\ , d(Px_{2n+1}, STx_{2n+1})d(Px_{2n+1}, ABx_{2n}), \\ d(ABx_{2n}, STx_{2n+1})d(Px_{2n}, Px_{2n+1}) \end{pmatrix}$$

$$[d(Px_{2n+1}, Px_{2n+2})]^2 \leq \phi\begin{pmatrix} \left(d(Px_{2n}, Px_{2n+1})\right)^2, \left(d(Px_{2n+1}, Px_{2n+2})\right)^2, \\ d(Px_{2n}, Px_{2n+2})d(Px_{2n}, Px_{2n+1}), \\ d(Px_{2n+1}, Px_{2n+2})d(Px_{2n+1}, Px_{2n+1}), \\ d(Px_{2n+1}, Px_{2n+2})d(Px_{2n}, Px_{2n+1}) \end{pmatrix}$$

Let us assume that, $d(Px_{2n+1}, Px_{2n+2}) = d_{2n+1}$ then

$$d_{2n+1} \leq [\phi\, ((d_{2n}^2, d_{2n+1}^2, (d_{2n} + d_{2n+1})^2, 0, d_{2n+1}d_{2n})]^{\frac{1}{2}}$$

$$d_{2n+1} \leq d_{2n}$$

Consequently, $\{d_{2n}\}$ is a non decreasing sequence of non negative real's, hence

$$d_1 \leq \gamma\, (d_0)$$

in general, we have $d_n \leq \gamma^n(d_0)$ so if $d_0 > 0$, *then by lemma* 1.1 gives

$$\lim_{n\to\infty} d_n = 0$$

Since then $d_n = 0$ for each n.

Now we wish to prove that the sequence $\{Px_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d_n = 0$. It is sufficient to show that the sequence $\{Px_n\}$ is a Cauchy sequence, suppose that $\{Px_n\}$ is not a Cauchy sequence. then there is an $\varepsilon > 0$ such that for each even integers 2k , k = 0,1, 2, ……. There exists even integers 2n(k) and 2m(k) with $2k \leq 2n(k) \leq 2m(k)$ such that,

$$d\left(Px_{2n(k)}, Px_{2m(k)}\right) > \varepsilon \qquad (2.2.1)$$

Let for each even integer 2k, 2m(k) be the least integer exceeding 2n(k) and satisfying (2.2.1) therefore

$$d\left(Px_{2n(k)}, Px_{2m(k)-2}\right) \leq \varepsilon \quad \text{and} \quad d\left(Px_{2n(k)}, Px_{2m(k)}\right) > \varepsilon \qquad (2.2.2)$$

Then , for each even integer 2k we have,

$$\varepsilon < d\left(Px_{2n(k)}, Px_{2m(k)}\right) \leq d\left(Px_{2n(k)}, Px_{2m(k)-2}\right) + d\left(Px_{2m(k)-2}, Px_{2m(k)-1}\right) + d\left(Px_{2n(k)-1}, Px_{2n(k)}\right)$$

So by 2.2.2, and $d_n \to 0$, we obtain

$$\lim_{n\to\infty} d\left(Px_{2n(k)}, Px_{2m(k)}\right) = \varepsilon$$

It follows immediately from the triangular inequality that,

$$\left| d\left(Px_{2n(k)}, Px_{2m(k)-1}\right) - d\left(Px_{2n(k)}, Px_{2m(k)}\right) \right| \leq d_{2m(k)-1}$$

$$\left| d\left(Px_{2n(k)+1}, Px_{2m(k)-1}\right) - d\left(Px_{2n(k)}, Px_{2m(k)}\right) \right| \leq d_{2m(k)-1} + d_{2n(k)}$$

Hence by 2.2.2, as $k \to \infty$

$$d\left(Px_{2n(k)}, Px_{2m(k)-1}\right) \to \varepsilon \text{ and } d\left(Px_{2n(k)+1}, Px_{2m(k)-1}\right) \to \varepsilon \tag{2.2.3}$$

Now,

$$d\left(Px_{2n(k)}, Px_{2m(k)}\right) \leq d\left(Px_{2n(k)}, Px_{2n(k)+1}\right) + d\left(Px_{2n(k)+1}, Px_{2m(k)}\right)$$

By using $2C_3$ and 2.2.3 $\lim_{n\to\infty} d_n = 0$, and upper semicountinuity and non decreasing property of ϕ in each co- ordinate variable , we have

$$\varepsilon \leq \phi(\varepsilon, 0, \varepsilon, \varepsilon, 0) \leq \gamma(\varepsilon) < \varepsilon$$

As $k \to \infty$, which contradiction. Thus $\{Px_n\}$ is a Cauchy sequence and hence by completeness of X , there is a, $u \in X$ such that $Px_n \to u$. since the sequence $\{ABx_n\}$ and $\{STx_n\}$ are Subsequence of $\{Px_n\}$ which follows $\{ABx_{2n}\}$ and $\{STx_{2n+1}\}$ also converges to the same point 'u' in X, i.e

$$\lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} STx_{2n+1} = u \tag{2.2.4}$$

$$Pu = ABu = STu$$

Let us assume that $Bu \neq u$, then we take from $2C_3$

$$[d(AB(Bu), STu)]^2 \leq \phi \begin{pmatrix} \left(d\left(P(Bu), AB(Bu)\right)\right)^2, \left(d(Pu, STu)\right)^2, \\ d(P(Bu), STu)d(P(Bu), ABu), \\ d(Pu, STu)d\left(Pu, AB(Bu)\right), \\ d(AB(Bu), STu)d(P(Bu), Pu) \end{pmatrix}$$

Which follows,

$$d(Bu, u) \leq \gamma^{\frac{1}{2}} d(Bu, u)$$

Which contradiction,

Similarly we can show that,

$$Tu = u = STu = S(Tu) = Su$$

i.e, u is a common fixed point of A, B, S, T, and P in X.

Uniqueness: Let us assume that 'w' is another fixed point of A, B, S, T, and P in X, different from 'u'. i.e $u \neq w$, then

$$d(u, w) = d(Pu, Pw) = d(ABu, STw)$$

By using $2C_3$, we get

$$[d(ABu, STw)]^2 \leq \phi \begin{pmatrix} \left(d(Pu, ABu)\right)^2, \left(d(Pw, STw)\right)^2, \\ d(Pu, STw)d(Pu, ABu), \\ d(Pw, STw)d(Pw, ABu), \\ d(ABu, STw)d(Pu, Pw) \end{pmatrix}$$

$$d(u, w) \leq \gamma^{\frac{1}{2}} . d(u, w)$$

Which contradiction.

u is unique common fixed point of A, B, S, T and P in X.

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