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SOLUTION OF BINARY QUADRATIC DIOPHANTINEEQUATION: $y^2 = 63x^2 + 37$

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ABSTRACT

In this paper, the binaryquadratic equation $y^2=63x^2+37$, which representsa hyperbola, also known as positive form of Pell equation is analyzed for finding its non-zero distinct integer solutions. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. A few relations between the solutions are presented, Using the solutions, a special Pythagorean triangle is constructed.

Keywords: *Binaryquadratic, Hyperbola, Parabola, Pell equation, Integral solutions.*

INTRODUCTION

Diophantine equations are rich in variety. The binary quadratic Diophantine equation of the form $y^2 = Dx^2 + N$ have been analyzed by many researchers for their integer solutions for particular values.

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $y^2 = 6x^2 + 37$ representing hyperbola. A few interesting relations among its solutions are presented. Also, a general formula to obtain sequence of integer solutions to the given hyperbola based on its known solution is presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Using the solutions, a special Pythagorean triangle is constructed.

Consider the binary quadratic equation

$$y^2 = 63x^2 + 37 \quad (1)$$

This is similar to the equation

$$y^2 = Dx^2 + N, \text{ where } D = 63, N = 37$$

Then the smallest positive integer solution of (1) is given by $x_0 = 1$, $y_0 = 10$

We find the other solutions of Pell equation (1) by considering the corresponding pellian equation with N = 1

$$y^2 = 63x^2 + 1 \quad (2)$$

The smallest solution positive integer of equation (2) is, $\tilde{x}_0 = 1$, $\tilde{y}_0 = 8$,

The general solution of equation (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{D}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{63}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n \quad (3)$$

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Where $f_n = (\tilde{y}_0+\sqrt{D}\,\tilde{x}_0)^{n+1} + (\tilde{y}_0-\sqrt{D}\,\tilde{x}_0)^{n+1}$

$g_n = (\tilde{y}_0+\sqrt{D}\,\tilde{x}_0)^{n+1} - (\tilde{y}_0-\sqrt{D}\,\tilde{x}_0)^{n+1}$

$$f_n = (8+(\sqrt{63}))^{n+1} + (8-(\sqrt{63}))^{n+1} \quad (4)$$

$$g_n = (8+(\sqrt{63}))^{n+1} - (8-(\sqrt{63}))^{n+1} \quad (5)$$

Put n = 0 in equation (3) we get, $(\tilde{x}_0, \tilde{y}_0) = (1, 8)$

Which is same as the solution found by inspection.

Put n = 1 in equation (4)

$f_1 = 254$, $g_1 = 32\sqrt{63}$

$\tilde{x}_1 = 16$, $\tilde{y}_1 = 127$

To find other solutions of equation (1),

We use Brahmagupta lemma, stated as follows

Brahmagupta Lemma: If (x_0, y_0) & (x_1, y_1) represents solutions of Pell equations $y^2 = Dx^2+k_1$ & $y^2 = Dx^2+k_2$, D > 0 then $(x_0y_1 + y_0x_1, y_0y_1 + Dx_0x_1)$

Represents solutions of Pell equation $y^2 = Dx^2 + k_1k_2$

Now we have $y^2 = 63x^2 + 37$ $k_1 = -37$, D = 63

$y^2 = 63x^2 + 1$ $k_2 = 1$, D = 63

Solution of (1) is $(x_0, y_0) = (1, 10)$

Solution of (2) is $(\tilde{x}_n, \tilde{y}_n) = (\frac{1}{2\sqrt{D}} g_n, \frac{1}{2} f_n)$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the general solution (x_{n+1}, y_{n+1}) of (1) is found to be

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = \frac{1}{2}f_n + \frac{5}{\sqrt{63}}g_n \quad (6)$$

$$y_{n+1} = y_0\tilde{y}_n + Dx_0\tilde{x}_n = (5)\, f_n + \frac{\sqrt{63}}{2}\, g_n \quad (7)$$

Therefore, $$x_{n+1} = \frac{1}{2}f_n + \frac{5}{\sqrt{63}}g_n \quad (8)$$

Therefore, $$y_{n+1} = 5f_n + \frac{\sqrt{63}}{2}g_n \quad (9)$$

In f_n & g_n replace n by n+1,

$$f_{(n+1)} = 8f_n + \sqrt{63}g_n \quad (10)$$

Similarly, $$g_{n+1} = 8g_n + \sqrt{63}f_n \quad (11)$$

Replacing n by n+1 in equation (8) and (9)

$$x_{n+2} = \frac{1}{2}f_{n+1} + \frac{5}{\sqrt{63}}g_{n+1} \quad (12)$$

$$y_{n+2} = 5f_{n+1} + \frac{\sqrt{63}}{2}g_{n+1} \quad (13)$$

Substituting (10) and (11) in (12), (13)

$$x_{n+2} = 9f_n + \frac{143}{2\sqrt{63}}g_n \quad (14)$$

$$y_{n+2} = \frac{143}{2}f_n + 9\sqrt{63}g_n \quad (15)$$

Similarly replace n by n+1 in equation (8) & (9)

$f_{n+2} = 127f_n + 16\sqrt{63}g_n$

$g_{n+2} = 16\sqrt{63}\, f_n + 127\, g_n$

Replace n by n+2 in equation (6) and (7) and a substituting $f_{(n+2)}$ and g_{n+2}

$$x_{n+3} = \frac{287}{2}f_n + \frac{1139}{\sqrt{63}}g_n \quad (16)$$

$$y_{n+3} = 1139f_n + \frac{287\sqrt{63}}{2}g_n \quad (17)$$

Now consider the equation (8), (14), (16)

$$x_{n+1} = \frac{1}{2}f_n + \frac{5}{\sqrt{63}}g_n$$
$$x_{n+2} = 9f_n + \frac{143}{2\sqrt{63}}g_n$$
$$x_{n+3} = \frac{287}{2}f_n + \frac{1139}{\sqrt{63}}g_n$$

Solving for f_n and g_n using (8) and (14), we get

$$g_n = \sqrt{63}\,[\frac{36}{37}x_{n+1} - \frac{2}{37}x_{n+2}]$$
$$f_n = \frac{20}{37}x_{n+2} - \frac{286}{37}x_{n+1}$$

Substitute f_n and g_n in equation (16), we get

$$x_{n+3} = 16x_{n+2} - x_{n+1} \quad (18)$$

Similarly solving (8) and (14) for f_n and g_n and substituting in (17), we get,

$$y_{n+3} = 16\,y_{n+2} - y_{n+1} \quad (19)$$

Therefore the recurrence relation are given by

$$x_{n+3} = 16x_{n+2} - x_{n+1}$$
$$y_{n+3} = 16\,y_{n+2} - y_{n+1}$$

which gives infinite number of solutions of (1) for n = - 1, 0, 1, 2, 3, 4……etc,

Thus, (16) and (17) represent the integer solutions of the hyperbola (1). After getting first two solutions, the other solutions are found using the recurrence relations (18) and (19) and a few numerical examples are given in the following Table: 1

Table-1: Solutions

n	x_{n+1}	y_{n+1}
-1	1	10
0	18	143
1	287	2278
2	4574	36305
3	72897	578602
4	1161778	9221327

Now, among the equations (8), (9), (14), (15), (18), (19)

We take any two equations and solve for f_n & g_n

First we consider equation (8) and (9) and solve for f_n & g_n. We get,

$$f_n = \frac{2}{37}[10y_{n+1} - 63x_{n+1}] = \frac{2}{37}X \qquad \text{where } X = 10y_{n+1} - 63x_{n+1}$$
$$g_n = \frac{2\sqrt{63}}{37}[10x_{n+1} - y_{n+1}] = \frac{2\sqrt{63}}{37}Y \qquad \text{where } Y = 10x_{n+1} - y_{n+1}$$

we have, $f_n = (8+\sqrt{63})^{n+1} + (8-\sqrt{63})^{n+1}$ $g_n = (8+\sqrt{63})^{n+1} - (8-\sqrt{63})^{n+1}$

So, $f_n^2 - g_n^2 = 4$

$$(\frac{2}{37}X)^2 - (\frac{2\sqrt{63}}{37}Y)^2 = 4 \text{ or } X^2 - 63Y^2 = 1369 \quad \text{(I)}$$

Which represents a hyperbola, where

$$X = 10y_{n+1} - 63x_{n+1},\; Y = 10x_{n+1} - y_{n+1}$$

Similarly by considering equations (14) and (15)

We get another hyperbola $X^2 - 63Y^2 = 1369$ (II)

Where, $X = 143y_{n+2} - 1134x_{n+2}$, $Y = 143x_{n+2} - 18y_{n+2}$

Considering equation (16) & (17), we get,

$$X^2 - 63Y^2 = 1369 \quad \text{(III)}$$

Where $X = 2278y_{n+3} - 18081x_{n+3}$, $Y = 2278x_{n+3} - 287y_{n+3}$

Considering equation (16) and (15)

$$4X^2 - 63Y^2 = 87616 \qquad \text{(IV)}$$

Where $X=1139y_{n+2} - 567x_{n+3}$, $Y=143x_{n+3} - 287y_{n+2}$

So, Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 2 below:

Table-2: Hyperbolas

SN	Hyperbola	(X,Y)
1	$X^2-63Y^2=1369$	$(10y_{n+1} - 63x_{n+1}, 10x_{n+1} - y_{n+1})$
2	$X^2-63Y^2=1369$	$(143y_{n+2} -1134x_{n+2}, 143x_{n+2} -18y_{n+2})$
3	$X^2-63Y^2=1369$	$(2278y_{n+3} -18081x_{n+3}, 2278x_{n+3} - 287y_{n+3})$
4	$4X^2-63Y^2=87616$	$(1139y_{n+2} - 567x_{n+3}, 143x_{n+3} - 287y_{n+2})$

To obtain Parabola, Consider,

$$f_n = (8+\sqrt{63}))^{n+1} + (8-\sqrt{63})^{n+1}$$
$$g_n = (8+\sqrt{63})^{n+1} - (8-\sqrt{63})^{n+1}$$

From which we get, $f_{2n+1}=f_n^2-2$

Considering the pair of equations as in the case of hyperbola and solving for f_n and substituting for fn and f_{2n+1} in $f_{2n+1} = f_n^2-2$, we get different Parabolas
Which are presented in the Table: 3 below:

Table-3: Parabolas

S.N	Parabola	(X,Y)
1	$Y^2 = \frac{37}{2}(X+37)$	$(10y_{2n+2} - 63x_{2n+2}, 10y_{n+1}-63x_{n+1})$
2	$Y^2 = \frac{37}{2}(X+37)$	$(143y_{2n+3} -1134x_{2n+3}, 143y_{n+2} -1134x_{n+2})$
3	$Y^2 = \frac{37}{2}(X+37)$	$(2278y_{2n+4} -18081x_{2n+4}, 2278y_{n+3}-18081x_{n+3})$
4	$Y^2=\frac{148}{2}(X+148)$	$(1139y_{2n+3} -567x_{2n+4}, 1139y_{n+2}-567x_{n+3})$

Nasty number:

(A positive integer n is a Nasty number if n = ab = cd and a + b = c - d or a – b = c + d where a, b, c, d are non-zero distinct positive integers (Bert Miller 1980)

Example: 6, 24, 54, 96 etc

Note that 6n^2 is always a nasty number as the Diophantine Equation $z^2 = 6x^2 + y^2$ always has the solution z = 6r^2+s^2, y = 6r^2-s^2, x = 2rs

Since $f_{2n+1} = f_n^2-2$,

Consider equation (7) & (8) solving for f_n we get

$$f_n = \frac{2}{37}[10y_{n+1} -63x_{n+1}], \; f_{2n+1} = \frac{2}{37}[10y_{2n+2} -63x_{2n+2}], \; f_n^2 = f_{2n+1}+2$$

$6f_n^2 = 6f_{2n+1}+12 =12 + \frac{12}{37}[(10y_{2n+2} -63x_{2n+2})$ is a Nasty Number.

Similarly by using f_n from the Pair of equations solved in the case of Parabola, we get different Nasty numbers. Each of the following expressions represents the Nasty number:

$$N_1 = 12 + \frac{12}{37}(10y_{2n+2} - 63x_{2n+2})$$
$$N_2 = 12 + \frac{12}{37}(143y_{2n+3} - 1134x_{2n+3})$$
$$N_3 = 12 + \frac{12}{37}(2278y_{2n+4} - 18081x_{2n+4})$$
$$N_4 = 12 + \frac{12}{148}(1139y_{2n+3} -567x_{2n+4})$$

Construction of special Pythagorean triangle

Let p and q be two non-zero distinct integers such that $P > q > 0$ .Treat P, q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$ where $\alpha = 2pq$, $\beta = P^2 - q^2$, $\gamma = p^2 + q^2$, $P > q > 0$

Taking $p = x_{n+1} + y_{n+1}, q = x_{n+1}$, it is observed that $T(\alpha,\beta,\gamma)$ is satisfied by the following relations:

- $63\beta - 2\alpha - 61\gamma = 37$
- $\alpha - \gamma + \beta = \frac{4A}{P}$
- $\frac{2A}{P} = x_{n+1}y_{n+1}$

❖ Each of the following is a Nasty number:

- ✓ $3\left(\alpha - \frac{4A}{p}\right)$
- ✓ $\frac{1}{21}[2(\gamma - \alpha) - 37]$

where A, P represent the area and perimeter of $T(\alpha,\beta,\gamma)$.

CONCLUSION

In this paper, we have presented infinitely many integer solutions for the Diophantine equation, represented by hyperbola given by $y^2 = 63x^2 + 37$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties. The researchers in the subject of Diophantine equations may also search for integer solutions of higher degree Diophantine equations with multiple variables and search for relations among solutions and special number patterns.

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