International Journal of Mathematical Archive-2(11), 2011, Page: 2113-2118
Available online through www.ijma.info ISSN 2229 – 5046

A NOTE ON SUFFICIENT CONDITION OF HAMILTONIAN PATH TO COMPLETE GRPHS

S. Venu Madava Sarma

Assistant Professor of Mathematics, K. L. UNIVERSITY, Vaddeswaram, India
E-mail: svm190675@gmail.com

N. V. Nagendram

Assistant Professor in Mathematics, Lakireddy Balirddy College of Engineering, Mylavaram, India

E-mail: nvn220463@yahoo.co.in

T. V. Pradeep Kumar*

Assistant Professor of Mathematics, A.N.U College of Engineering & Technology,
Acharya Nagarjuna University, India

E-mail: pradeeptv5@gmail.com

(Received on: 11-10-11; Accepted on: 30-10-11)

ABSTRACT

A Hamiltonian Path is a spanning path in a graph, i.e., the path passing through every vertex of the graph. In this paper we study and giving a sufficient condition for a complete graph to posses a Hamiltonian path and given some related examples.

Key Words: *Graph, complete graph, path, Hamiltonian path, Tree, Spanning tree.*

1. INTRODUCTION:

The origin of graph theory started with the problem of Koinsber bridge, in 1735. This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits. In 1852, Thomas Gutherie found the famous four color problem. Then in 1856, Thomas. P. Kirkman and William R. Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once. In 1913, H.Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken. This time is considered as the birth of Graph Theory. Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory. Any how the term “Graph” was introduced by Sylvester in 1878 where he drew an analogy between “Quantic invariants” and covariants of algebra and molecular diagrams. In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory. In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

In this paper we study the Hamiltonian graphs, Hamiltonian paths and given a sufficient condition of Hamiltonian path to complete graphs.

1.1 Definition: A graph – usually denoted G (V, E) or G = (V, E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

Corresponding author:** *T. V. Pradeep Kumar**,E-mail:** *pradeeptv5@gmail.com*

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge e = (u, v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set V= {a, b, c, d, e. f} and edge set E = {(a, b), (b, c), (c, d), (c, e), (d, e), (e, f)}.

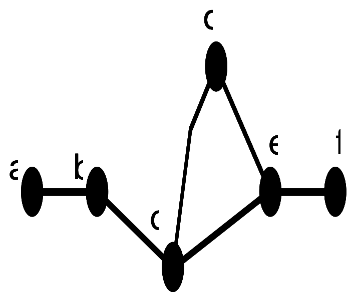


Figure: 1

1. 5 Definition: Two vertices *u* and *v* are *adjacent* if there exists an edge (*u,v*) that connects them.

1.6 Definition: An edge (u, v) is said to be *incident* upon nodes u and v.

1.7 Definition: An edge e = (u, u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

1.8 Definition: Every graph has associated with it an *adjacency matrix*, which is a binary $n \times n$ matrix A in which $a_{ij} = 1$ and $a_{ji} = 1$ if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Adjacency matrix for graph in Figure 1.

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

1.10 Definition: A *subgraph* of a graph *G* is a graph whose points and lines are contained in *G*. A complete subgraph of *G* is a section of *G* that is complete

1.11 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

1.12 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.

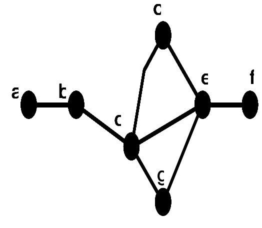


Figure: 2

1.13 Definition: A *component* of a graph is defined as a maximal subgraph in which a path exists from every node to every other (i.e., they are mutually reachable). The size of a component is defined as the number of nodes it contains. A connected graph has only one component.

1.14 Definition: A sequence of adjacent vertices $v_0, v_1, \ldots, v_n$ is known as a *walk*. In Figure 3, the sequence *a, b, c, b, c, g* is a walk. A walk can also be seen as a sequence of *incident* edges, where two edges are said to be incident if they share exactly one vertex.

1.15 Definition: A walk is closed if $v_o = v_n$.

1.16 Definition: A walk in which no vertex occurs more than once is known as a *path*. In Figure 3, the sequence *a, b, c, d, e, f* is a path.

1.17 Definition: A walk in which no edge occurs more than once is known as a *trail*. In Figure 3, the sequence *a, b, c, e, d, c, g* is a trail but not a path. Every path is a trail, and every trail is a walk.

1.18 Definition: A *cycle* can be defined as a closed path in which *n* >= 3. The sequence *c, e, d* in Figure 3 is a cycle.

1.19 Definition: A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

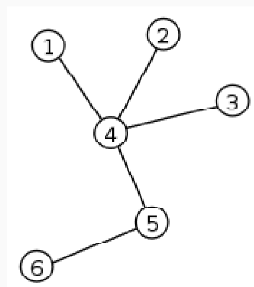


Figure 3: A labeled tree with 6 vertices and 5 edges

1.20 Definition: A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

1.21 Definition: The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a,b,c,d,e* has length 4.

1.22 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.23 Definition : In the mathematical field of graph theory, a **bipartite graph** (or **bigraph**) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

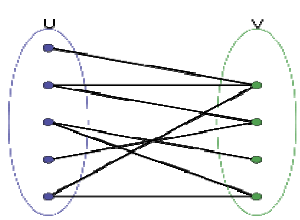


Figure 4: Example of a bipartite graph.

1.24 Definition: An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

1.25 Example: Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

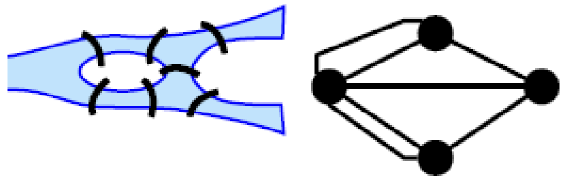


Figure 5: K¨onigsberg problem

2. HAMILTONIAN PATH AND HAMILTONIAN CIRCUIT:

In this section we proved main result related to the sufficient condition of Hamiltonian path to complete graph.

2.1 Definition: Another closely related problem is finding a **Hamilton path** in the graph (named after an Irish mathematician, Sir William Rowan Hamilton). Whereas an Euler path is a path that visits every edge exactly once, a Hamilton path is a path that visits every vertex in the graph 4 exactly once.

A **Hamilton circuit** is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example **Hamilton Circuit** would be

2.2 Example:

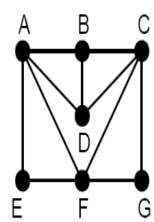


Figure 6: Hamilton Circuit would be AEFGCDBA.

2.3 Theorem (Dirac [2]). If G is a simple graph with n vertices where n ≥ 3 and δ (G) ≥ n/2, then G is Hamiltonian.

2.4 Theorem (Ore [4]). Let G be a simple graph with n vertices and u, v be distinct nonadjacent vertices of G with d(u) + d(v) ≥ n. Then G is Hamiltonian if and only if G + (u, v) is Hamiltonian.

2.5 Theorem (Bondy-Chvátal [1]). If G is a simple graph with n vertices, then G is Hamiltonian if and only if its closure is Hamiltonian..

2.6 Remark: The (Hamiltonian) closure of a graph G, denoted C(G), is the supergraph of G on V(G) obtained by iteratively adding edges between pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. Fortunately, the closure does not depend on the order in which we choose to add edges when more than one is available i.e. the closure of G is well-defined (For a proof of this statement see [5])

2.7 Theorem (Ore [4]). If d(u) + d(v) ≥ n for every pair of distinct nonadjacent vertices u and v of G, then G is Hamiltonian.

2.8 Theorem: 1.5. Let $G = (V, E)$ be a Complete graph with n vertices. If for all pair wise non- adjacent vertex-triples u, v, and w it holds that

$d(u) + d(v) + d(w) \geq ½ (3n - 5)$ then G has a Hamiltonian path.

To prove this theorem, we first state and prove the following useful Lemma.

2.9 Lemma: Let G = (V, E) be a connected graph with n vertices and P be a longest path in G. If P is contained in a cycle then P is a Hamiltonian path.

Proof: Suppose $P \equiv (u = u0, u1, u2\dots uk = v)$ of length k and P is contained in a cycle $C \equiv (u = u0, u1, u2, \dots, uk = v, u0 = u)$. Note that $V(C) = V(P)$, since otherwise P would be a part of a longer path, a contradiction. Assume for the sake of contradiction that $k < n - 1$, i.e. P is not Hamiltonian path. Since G is connected, there must be an edge of the form (x, y) such that $x \varepsilon V(P) = V(C)$ and $y \varepsilon V(G) - V(C)$. Let $x = ui$. Then there is a path $P' \equiv (y, x = ui, ui+1, \dots, uk, u0, u1, u2, \dots, ui\text{-}1)$ with length $k+1$, which is a contradiction, since P is a longest path in G.

Now we are ready to prove the main result of our paper i.e. Theorem 1.5 which is restated below for the sake of convenience.

Proof of Main Theorem 2.8: Let $P = (u_0, u_1, \dots, u_{p\text{-}1})$ be a longest path in G. And assume for the sake of contradiction that P is not a Hamiltonian path. Now since P is a longest path but not a Hamiltonian path, by the contra positive of Lemma 2.2, P cannot be contained in a cycle. And since P cannot be contained in a cycle, there cannot be any crossover edge involving u_0 and $u_{p\text{-}1}$. This essentially means that

$d(u_0) + d(u_{p\text{-}1}) \leq p - 1$. So we must have:

$$d(u_0) + d(u_{p\text{-}1}) + d(w) \geq ½\,(3n - 5)$$

$$\Rightarrow d(w) \geq ½\,(3n\text{-}5) - (d(u_0) + d(u_{p\text{-}1}))$$

$$\Rightarrow d(w) \geq ½\,(3n - 5) - (p - 1)$$

$$\Rightarrow d(w) \geq 3/2\, n - p - 3/2$$

Now we consider $dP(w)$. We calculate the upper limit of $dP(w)$ as follows. It is clear that $(w, u0), (w, up-1) \notin E$ since otherwise P would not be a longest path in G. Again, Note that w cannot be connected to ui and $ui+1$, since in that case we can easily get a path $P' = (u0, u1, \dots, ui, w, ui+1, \dots, up - 1)$ which is longer than P, leading to a contradiction. So

write that $d_P(w) \leq \frac{p-2}{2} + 1 = \frac{p}{2}$.

Now we have,

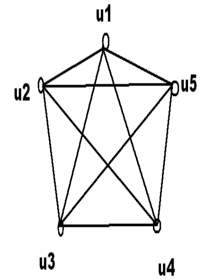
$$\begin{aligned}
d(w) &\geq \frac{3}{2}n - p - \frac{3}{2} \\
\Rightarrow d_P(w) + d_{\overline{P}}(w) &\geq \frac{3}{2}n - p - \frac{3}{2} \\
\Rightarrow \quad d_{\overline{P}}(w) &\geq \frac{3}{2}n - p - \frac{3}{2} - d_P(w) \\
&= \frac{3}{2}n - p - \frac{3}{2} - \frac{p}{2} \\
&= \frac{3}{2}(n - p - 1).
\end{aligned}$$

This leads to a contradiction since $|V(G)\backslash(V(P) \cup \{w\})| = n - p - 1 < \frac{3}{2}(n - p - 1)$, which we can

Completes the proof.

Now we will give an example for the above theorem:

2.10 Example: Let V = {u1, u2, u3, u4, u5}.



Here u1, u2, u3, u4, u5 are vertices

$d(u_1) = 4$, $d(u_2) = 4$, $d(u_3) = 4$, $d(u_4) = 4$ and $d(u_5) = 4$

Here number of vertices N = 5

$d(u_1) + d(u_3) + d(u_5) = 12$ (1)

$1 / 2\,(3n - 5) = 1 / 2\,(3*5 - 5) = 5$ (2)

From (1) and (2) 12 > 5

Theorem is satisfied.

REFERENCES:

[1] Bondy J.A. and V. Chvátal, A Method in Graph Theory, Discr. Math. 15 (1976), pp 111-136.

[2] Dirac G.A., Some Theorems on Abstract Graphs, Proc. Lond. Math. Soc. 2 (1952), pp 69-81.

[3] Garey M.R and D.S. Jhonson, Computers and Intractability: A Guide to the Theory of NP Completeness, W. H. Freeman and Company, New York.

[4] O. Ore, Note on Hamiltonian Circuits, Am. Mat. Monthly 67 (1960), pp 55.

[5]. West D.B, Introduction to Graph Theory, Prentice-Hall, Inc., New Jersey.

[6] Williams – Nash, C. St. J .A. "Hamiltonian Arcs and Circuits" in recent trends in graphtheory ed. By M. Capobianco et al., Springer – verlag, Berlin, 1971.

[7] V. Chvatal "New directions in Hamiltonian Graph theory" In New directions in graph theory ed. By F. Harary, Academic press, N. Y. London, 1973.

****