



On ρ -regular and ρ -normal spaces

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ABSTRACT

The concept of ρ -closed sets was introduced by jafari et al. The aim of this paper is to consider and characterize ρ -regular spaces and ρ -normal spaces via the concept of ρ -closed sets.

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1. INTRODUCTION AND PRELIMINARIES:

As a generalization of closed sets, ρ -closed sets were introduced and studied by jafari et al [4]. In this paper, we introduce ρ -regular spaces and ρ -normal spaces in topological spaces. We obtain several characterizations of ρ -regular and ρ -normal spaces and some preservation theorems for ρ -regular and ρ -normal spaces.

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $\text{cl}(A)$, $\text{int}(A)$, and A^c denote the closure of A , the interior of A and the complement of A in X , respectively.

We recall the following definitions, which are useful in the sequel.

Definition: 1.1. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. preopen [7] if $A \subseteq \text{int}(\text{cl}(A))$ and preclosed if $\text{cl}(\text{int}(A)) \subseteq A$.

Recall that the intersection of all preclosed sets containing A is called the preclosure of A and is denoted by $\text{pcl}(A)$.

Definition: 1.2. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. \hat{g} -closed [18] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X .
2. $*g$ -closed [16] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is \hat{g} -open in X .
3. $\#g$ -semi closed (briefly $\#g_s$ -closed) [17] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $*g$ -open in X .
4. \tilde{g} -closed set [3] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\#g_s$ -open in X .
5. ρ -closed set [4] if $\text{pcl}(A) \subseteq \text{Int}(U)$ whenever $A \subseteq U$ and U is \tilde{g} -open in (X, τ) .

The complements of the above mentioned sets are called their respective open set.

Recall that the intersection of all ρ -closed sets containing A is called the ρ -closure of A and is denoted by $\text{pcl}(A)$.

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Definition: 1.3 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called:

1. pre-continuous [7] if $f^{-1}(V)$ is preclosed in (X, τ) for every closed set V in (Y, σ) .
2. preirresolute [13] if $f^{-1}(V)$ is preclosed in (X, τ) for every preclosed set V in (Y, σ) .
3. \tilde{g} -irresolute [15] if $f^{-1}(V)$ is \tilde{g} -closed in (X, τ) for every \tilde{g} -closed set V in (Y, σ) .
4. M-Preopen [7] if $f(V)$ is Preopen in (Y, σ) for every preopen set V in (X, τ) .
5. ρ -irresolute [4] if $f^{-1}(V)$ is ρ -closed in (X, τ) for every ρ -closed set V in (Y, σ) .
6. ρ -closed [12] if $f(V)$ is ρ -closed in (Y, σ) for every closed set in (Y, σ) .
7. preclosed [7] if $f(V)$ is preclosed in (Y, σ) for every closed set V in (X, τ) .
8. \tilde{g}^* -open [2] if $f(V)$ is \tilde{g} -open in (Y, σ) for every \tilde{g} -open set V in (X, τ) .
9. weakly continuous [6] if for each point $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq \text{cl}(V)$.

Definition: 1.4. A space (X, τ) is called:

1. a ρ - $T_{1/2}$ space [5] if every ρ -closed set is preclosed.
2. a ρ - T_s space [5] if every ρ -closed set is closed.

Definition: 1.5 [5] Let x be a point of (X, τ) and V be a subset of X . Then V is called a ρ -neighbourhood of x in (X, τ) if there exists a ρ -open set U of (X, τ) such that $x \in U \subseteq V$.

Lemma: 1.6

- (i) [7] Let A and Y be subsets of a space X . If A is preopen in X and Y is open in X , then $A \cap Y$ is preopen in Y .
- (ii) [10] If A is ρ -closed in (X, τ) then $A = \rho\text{-cl}(A)$.

Definition: 1.7

- (i) [5] A topological space (X, τ) is ρ -compact if every ρ -open cover of X has a finite subcover.
- (ii) [4] A point x of X is called a pre- θ -cluster point of S if $\text{pcl}(U) \cap A \neq \emptyset$ for every preopen set U of x . The set of all pre- θ -cluster points of S is called the pre θ -closure of S and is denoted by $\text{pcl}_\theta(S)$.

Theorem: 1.8 [4] Suppose that $B \subseteq A \subseteq X$, B is a ρ -closed set relative to A and that A is open and ρ -closed in (X, τ) . Then B is ρ -closed in (X, τ) .

Definition: 1.9

- (i) [9] A space (X, τ) is called locally indiscrete space if every open subset of X is closed.
- (ii) [1] A space (X, τ) is called Submaximal space if every dense subset of X is open, or, equivalently, if every preopen subset is open.
- (iii) [19] A space (X, τ) is called extremally disconnected space if the closure of every open subset of X is open.
- (iv) [8] A space (X, τ) is called strongly compact space if every preopen cover of X has a finite subcover.
- (v) [12] A space (X, τ) is called ρ -hausdorff space if for each pair x, y of distinct points of X , there exists ρ -neighbourhoods U_1 and U_2 of x and y , respectively, that are disjoint.
- (vi) [10] A space (X, τ) is called prenormal if for any pair of disjoint closed sets A, B of X , there exist disjoint preopen sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Recall that a topological space (X, τ) is, 1. Compact if every open cover for X has a finite subcover. 2. Hausdorff space if for each pair of distinct points x and y in X there exist an open neighbourhood U of x and an open neighbourhood V of y such that $U \cap V = \emptyset$. 3. Regular space if for each point $x \in X$ and for each closed set F in X not containing x , there exist an open neighbourhood U of x and an open neighbourhood V of F such that $U \cap V = \emptyset$. 4. Normal space if for each pair of disjoint closed subsets F_1 and F_2 in X , there exist an open neighbourhood U_1 of F_1 and an open neighbourhood U_2 of F_2 such that $U_1 \cap U_2 = \emptyset$. 5. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is open if $f(V)$ is open in (Y, σ) for every open set V in (X, τ) .

Theorem: 1.10

- (i) Let X be a hausdorff space and F be a compact subset of X . Then for each x in $X \setminus F$, There exists disjoint open sets of F and x .
- (ii) If F_1 and F_2 are disjoint compact subsets of a Hausdorff space X , then there are disjoint open sets of F_1 and F_2 .
- (iii) [5] Every ρ -closed subset of a ρ -compact space is ρ -compact.

2. ρ -regular and ρ -normal spaces:

We introduce the following definition.

Definition: 2.1 A space (X, τ) is said to be ρ -regular if for every ρ -closed set F and each point $x \notin F$, there exists disjoint preopen sets U and V such that $F \subseteq U$ and $x \in V$.

Theorem: 2.2 If (X, τ) is an ρ -regular space and Y is open and ρ -closed subset of (X, τ) , then the subspace Y is ρ -regular.

Proof: Let F be any ρ -closed subset of Y and $y \in F^c$. By Theorem 1.8, F is ρ -closed in (X, τ) . Since (X, τ) is ρ -regular, there exist disjoint preopen sets U and V of (X, τ) such that $y \in U$ and $F \subseteq V$. By lemma 1.6(i), we get $U \cap Y$ and $V \cap Y$ are disjoint preopen sets of the subspace Y such that $y \in U \cap Y$ and $F \subseteq V \cap Y$. Hence the subspace Y is ρ -regular.

Theorem: 2.3 Let (X, τ) be a topological space. Then the following statements are equivalent:

1. (X, τ) is ρ -regular.
2. For each point $x \in X$ and for each ρ -open neighbourhood W of x , there exists a preopen set V of x such that $\text{pcl}(V) \subseteq W$.
3. For each point $x \in X$ and for each ρ -closed set F not containing x , there exists a preopen set V of x such that $\text{pcl}(V) \cap F = \emptyset$.

Proof: (1) \rightarrow (2)

Let W be any ρ -open neighbourhood of x . Then there exists an ρ -open set G such that $x \in G \subseteq W$. Since G^c is ρ -closed and $x \notin G^c$, by hypothesis there exist preopen sets U and V such that $G^c \subseteq U$, $x \in V$ and $U \cap V = \emptyset$ and so $V \subseteq U^c$. Now, $\text{pcl}(V) \subseteq \text{pcl}(U^c) = U^c$ and $G^c \subseteq U$ implies $U^c \subseteq G \subseteq W$. Therefore $\text{pcl}(V) \subseteq W$.

(2) \rightarrow (1)

Let F be any ρ -closed set and $x \notin F$. Then $x \in F^c$ and F^c is ρ -open and so F^c is an ρ -open neighbourhood of x . By hypothesis, there exists a preopen set V of x such that $x \in V$ and $\text{pcl}(V) \subseteq F^c$, which implies $F \subseteq (\text{pcl}(V))^c$. Then $(\text{pcl}(V))^c$ is a preopen set containing F and $V \cap (\text{pcl}(V))^c = \emptyset$. Therefore, X is ρ -regular.

(2) \rightarrow (3)

Let $x \in X$ and F be an ρ -closed set such that $x \notin F$. Then F^c is an ρ -open neighbourhood of x and by hypothesis, there exists a preopen set V of x such that $\text{pcl}(V) \subseteq F^c$ and hence $\text{pcl}(V) \cap F = \emptyset$.

(3) \rightarrow (2)

Let $x \in X$ and W be an ρ -open neighbourhood of x . Then there exists an ρ -open set G such that $x \in G \subseteq W$. Since G^c is ρ -closed and $x \notin G^c$, by hypothesis there exists a preopen set V of x such that $\text{pcl}(V) \cap G^c = \emptyset$. Therefore, $\text{pcl}(V) \subseteq G \subseteq W$.

Theorem: 2.4 Assume that $\text{pc}(X, \tau)$ (the set of all ρ -closed sets of (X, τ)) be closed under any intersection. Then the following are equivalent:

1. (X, τ) is ρ -regular.
2. $\text{pcl}_\theta(A) = \text{pcl}(A)$ for every subset A of (X, τ) .
3. $\text{pcl}_\theta(A) = A$ for every ρ -closed set A .

Proof: (1) \rightarrow (2)

For any subset A of (X, τ) , we always have $A \subseteq \text{pcl}(A) \subseteq \text{pcl}_\theta(A)$. Let $x \in (\text{pcl}(A))^c$. Then there exists a ρ -closed set F such that $x \in F^c$ and $A \subseteq F$. By assumption, there exists disjoint preopen sets U and V such that $x \in U$ and $F \subseteq V$. Now, $x \in U \subseteq \text{pcl}(U) \subseteq F^c \subseteq A^c$ and therefore $\text{pcl}(U) \cap A = \emptyset$. Thus, $x \in (\text{pcl}_\theta(A))^c$ and hence $\text{pcl}_\theta(A) = \text{pcl}(A)$.

(2) \rightarrow (3)

It is trivial follows by Lemma 1.6(ii) and by assumption.

(3) \rightarrow (1)

Let F be any ρ -closed set and $x \in F^c$. Since F is ρ -closed, by assumption $x \in (\text{pcl}_\theta(F))^c$ and so there exists an preopen set U such that $x \in U$ and $\text{pcl}(U) \cap F = \emptyset$. Then $F \subseteq (\text{pcl}(U))^c$. Let $V = (\text{pcl}(U))^c$. Then V is a preopen such that $F \subseteq V$. Also the sets U and V are disjoint and hence (X, τ) are ρ -regular.

Theorem: 2.5 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is bijective, \tilde{g} *-open and preirresolute, then f is ρ -irresolute.

Proof: Let A be any ρ -closed set in (Y, σ) . Let U be any \tilde{g} -open in (X, τ) such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$.

Since f is \tilde{g} *-open and A is ρ -closed in (Y, σ) , $\text{pcl}(A) \subseteq \text{int}(f(U))$.

Since f is bijective and preirresolute, $f^{-1}(\text{pcl}(A)) \subseteq f^{-1}(\text{int}(f(U))) \subseteq \text{int}(U)$ and $f^{-1}(\text{pcl}(A))$ is a preclosed set in (X, τ) . Now, $\text{pcl}(f^{-1}(A)) \subseteq \text{pcl}(f^{-1}(\text{pcl}(A))) = f^{-1}(\text{pcl}(A)) \subseteq \text{int}(U)$.

Therefore, $f^{-1}(A)$ is ρ -closed in (X, τ) and hence f is ρ -irresolute.

Theorem: 2.6 If (X, τ) is a ρ -regular space and $f: (X, \tau) \rightarrow (Y, \sigma)$ is bijective, \tilde{g} *-open, preirresolute and M -preopen then (Y, σ) is ρ -regular.

Proof: Let F be any ρ -closed subset of (Y, σ) and $y \notin F$. Since by Theorem 2.5, the map f is ρ -irresolute, thus we have $f^{-1}(F)$ is ρ -closed in (X, τ) . Since f is bijective, let $f(x) = y$, then $x \notin f^{-1}(F)$. By hypothesis, there exists disjoint preopen sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since f is M -preopen and bijective, we have $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. This shows that the space (Y, σ) is also ρ -regular.

Theorem: 2.7

- (i) [9] If $f: (X, \tau) \rightarrow (Y, \sigma)$ is \tilde{g} -irresolute, open, preclosed and A is a ρ -closed subset of (X, τ) then $f(A)$ is ρ -closed in (Y, σ) .
 (ii) If $f: (X, \tau) \rightarrow (Y, \sigma)$ is \tilde{g} -irresolute, open, preclosed, preirresolute injecton and (Y, σ) is ρ -regular, then (X, τ) is ρ -regular.

Proof: Let F be any ρ -closed set of (X, τ) and $x \notin F$. Since f is \tilde{g} -irresolute, open, preclosed, by (i), $f(F)$ is ρ -closed in (Y, σ) and $f(x) \notin f(F)$. Since (Y, σ) is ρ -regular and so there exists disjoint preopen sets U and V in (Y, σ) such that $f(x) \in U$ and $f(F) \subseteq V$. By hypothesis, $f^{-1}(U)$ and $f^{-1}(V)$ are preopen sets of (X, τ) such that $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore, (X, τ) is ρ -regular.

Lemma: 2.8 If Y is a ρ -compact subspace of the ρ - $T_{1/2}$ ρ -hausdorff space X and x_0 is not in Y , then there exists disjoint preopen sets U and V of X containing x_0 and Y respectively. Assume that $\rho o(X)$ (the set of all ρ -open sets of (X, τ)) be closed under any union and $\rho o(X)$ be also closed under finite intersection.

Proof: Since X is ρ -hausdorff and $x_0 \notin Y$, for each $x \in Y$ there exists disjoint ρ -open sets U_x and V_x such that $x_0 \in U_x$ and $x \in V_x$. The collection $\{V_x / x \in Y\}$ is evidently an ρ -open cover of Y . Since Y is ρ -compact subspace of X , there exists finitely many points x_1, x_2, \dots, x_n of Y such that $Y \subset \cup \{V_{x_i} : i = 1, 2, \dots, n\}$. Let $U = \cap \{U_{x_i} : i = 1, 2, \dots, n\}$ and $V = \cup \{V_{x_i} : i = 1, 2, \dots, n\}$. Then, by assumption U and V are disjoint ρ -open sets of X such that $x_0 \in U$ and $Y \subset V$. Since X is ρ - $T_{1/2}$, then the sets U and V are disjoint preopen sets of X containing x_0 and Y respectively.

Theorem: 2.9 Every ρ -compact ρ - $T_{1/2}$ ρ -hausdorff space is regular.

Proof: Let X be a ρ -compact ρ - $T_{1/2}$ ρ -hausdorff space. Let x be a point of X and B be a ρ -closed set in X not containing x , then by Theorem 1.12(iii), B is ρ -compact. Thus B is a ρ -compact subspace of the ρ - $T_{1/2}$ ρ -hausdorff space X . Let us assume that the set of all ρ -open sets be closed under any union and be also closed under finite intersection. Then, by Lemma 2.8, there exists disjoint preopen sets about x and B respectively. Hence X is regular.

Theorem: 2.10 Let (X, τ) be a ρ - T_s locally indiscrete space. Then the following are equivalent:

1. (X, τ) is regular.
2. For every closed set A and each $x \in X-A$, there exist disjoint ρ -open sets U and V such that $x \in U$ and $A \subseteq V$.

Proof: (1) \rightarrow (2) Let (X, τ) be a locally indiscrete space. Let A be a closed set and x be a point of $X-A$. Then by hypothesis, there exist disjoint open sets (and hence preopen sets) U and V such that $x \in U$ and $A \subseteq V$. Since the space (X, τ) is locally indiscrete space, then U and V are closed sets. Hence U and V are ρ -open sets. Therefore, for every closed set A and each $x \in X-A$, there exist disjoint ρ -open sets U and V such that $x \in U$ and $A \subseteq V$.

(2) \rightarrow (1) Let (X, τ) be a ρ - T_s space. Let the point x and the closed set A not containing x be given. Then by hypothesis, there exist disjoint ρ -open sets U and V such that $x \in U$ and $A \subseteq V$. Since the space is ρ - T_s , then there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$.

Theorem: 2.11 *The closure of a compact open subset of a compact hausdorff ρ - T_s space is ρ -regular.*

Proof: Let X be a compact hausdorff ρ - T_s space and A be a compact open subset of X . Then A is closed in X because compact subset of a hausdorff space is closed. Hence A is preclosed in X . Also A is open in X . By Theorem 3.1[4], A is ρ -closed in X .

Now to prove $\bar{A} = A$ is ρ -regular. Suppose the point x and the ρ -closed set F of A not containing x are given. By Theorem 1.8, F is ρ -closed in X . Since X is ρ - T_s , then F is closed in X . Hence F is compact in X because closed subset of a compact hausdorff space is compact. By Theorem 1.12[i], there exists disjoint open sets U and V of F and x respectively. Since every open set is preopen, then there exists disjoint preopen sets U and V of F and x respectively.

Therefore, X is ρ -regular.

Theorem: 2.12 *Let (X, τ) be a locally indiscrete submaximal space. Then the closure of a strongly compact subset of a ρ -regular space is compact.*

Proof: Let (X, τ) be a locally indiscrete ρ -regular space. Let A be a strongly compact subset of X . To prove: \bar{A} is compact. Let ξ be an open covering of \bar{A} . Then for each $x \in A \subseteq \bar{A}$ there exists an open set (and hence preopen set and by the definition 1.11(i), it is ρ -open set) W_x in ξ such that $x \in W_x$. Since X is ρ -regular and by Theorem 2.3, there exist a preopen set V_x of x such that $\text{pcl}(V_x) \subseteq W_x$ for each x . Now the family $\{V_x : x \in A\}$ is a preopen covering of A . Since A is strongly compact, then there exist finitely many points x_1, x_2, \dots, x_n such that $A \subseteq \bigcup_{i=1}^n V_{x_i}$, since X is submaximal and $\text{pcl}(V_{x_i}) = \text{cl}(V_{x_i})$ for every open set V_{x_i} , $A \subseteq \bigcup_{i=1}^n V_{x_i}$ implies $\text{cl}(A) \subseteq \text{cl}(\bigcup_{i=1}^n V_{x_i}) \subseteq \bigcup_{i=1}^n \text{cl}(V_{x_i}) = \bigcup_{i=1}^n \text{pcl}(V_{x_i}) \subseteq \bigcup_{i=1}^n W_{x_i}$.

This shows that every open covering of \bar{A} is reducible to a finite covering. Therefore, \bar{A} is compact.

We conclude this section with the introduction of ρ -normal space in topological space.

Definition: 2.13 A topological space (X, τ) is said to be ρ -normal if for any pair of disjoint ρ -closed sets A and B , there exist disjoint preopen sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem: 2.14 *If (X, τ) is a ρ -normal space and Y is an open and ρ -closed subset of (X, τ) , then the subspace Y is ρ -normal.*

Proof: Let A and B be any disjoint ρ -closed sets of Y . By Theorem 1.8, A and B be ρ -closed in (X, τ) . Since (X, τ) is ρ -normal, there exist disjoint preopen sets U and V of (X, τ) such that $A \subseteq U$ and $B \subseteq V$. By Lemma 1.6(i), $U \cap Y$ and $V \cap Y$ are disjoint preopen sets in Y and also $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$ and so the subspace Y is ρ -normal.

In the next theorem we characterize ρ -normal space.

Theorem: 2.15 *Let (X, τ) be a topological space. Then the following statements are equivalent:*

1. (X, τ) is ρ -normal.
2. For each ρ -closed set F and for each ρ -open set U containing F , there exist an preopen set V containing F such that $\text{pcl}(V) \subseteq U$.
3. For each pair of disjoint ρ -closed sets A and B in (X, τ) , there exists an preopen set U containing A such that $\text{pcl}(U) \cap B = \emptyset$.

Proof: (1) \rightarrow (2): Let F be a ρ -closed set and U be a ρ -open set such that $F \subseteq U$. Then $F \cap U^c = \emptyset$. By assumption, there exist preopen sets V and W such that $F \subseteq V$, $U^c \subseteq W$ and $V \cap W = \emptyset$, which implies $\text{pcl}(V) \cap W = \emptyset$. Now, $\text{pcl}(V) \cap U^c \subseteq \text{pcl}(V) \cap W = \emptyset$ and so $\text{pcl}(V) \subseteq U$.

(2) \rightarrow (3): Let A and B be disjoint ρ -closed sets of (X, τ) . Since $A \cap B = \emptyset$, $A \subseteq B^c$ and B^c is ρ -open. By assumption, there exists an preopen set U containing A such that $\text{pcl}(U) \subseteq B^c$ and so $\text{pcl}(U) \cap B = \emptyset$.

(3) \rightarrow (1): Let A and B be any two disjoint ρ -closed sets of (X, τ) . Then by assumption, there exists a preopen set U containing A such that $\text{pcl}(U) \cap B = \emptyset$. Again by assumption, there exists an preopen set V containing B such that $\text{pcl}(V) \cap A = \emptyset$. Also $\text{pcl}(U) \cap \text{pcl}(V) = \emptyset$. Thus we have $U \cap V = \emptyset$.

Theorem: 2.16 *Let (X, τ) be a submaximal extremally disconnected space. Then the following are equivalent.*

1. (X, τ) is ρ -normal.
2. For each pair of disjoint ρ -closed sets A and B in (X, τ) , there exists a preopen set U containing A and V containing B such that $\text{pcl}(U) \cap \text{pcl}(V) = \emptyset$.

Proof: (1) \rightarrow (2) Let A and B be any two disjoint ρ -closed sets of (X, τ) . Then, by Theorem 2.15, there exists a preopen set U containing A such that $\text{pcl}(U) \cap B = \emptyset$. Since $\text{pcl}(U)$ is preclosed and since (X, τ) is submaximal and extremally

disconnected, then, $\text{pcl}(U) = \text{cl}(U)$ and $\text{pcl}(U)$ is open and hence by Theorem 3.1 [4], $\text{pcl}(U)$ is ρ -closed and so B and $\text{pcl}(U)$ are disjoint ρ -closed sets in (X, τ) . Therefore, again by Theorem 2.15, there exists a preopen set V containing B such that $\text{pcl}(U) \cap \text{pcl}(V) = \emptyset$.

(2) \rightarrow (1) $\text{pcl}(U) \cap \text{pcl}(V) = \emptyset$ implies $U \cap V = \emptyset$ and by assumption (X, τ) is ρ -normal.

Theorem: 2.17 If (X, τ) is a ρ -normal space and $f: (X, \tau) \rightarrow (Y, \sigma)$ is bijective, \tilde{g} *-open, preirresolute and M -preopen then (Y, σ) is ρ -normal.

Proof: Let A and B be any disjoint ρ -closed sets of (Y, σ) . Since by Theorem 2.5, the map f is ρ -irresolute then we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint ρ -closed sets of (X, τ) . Since (X, τ) is normal, there exists disjoint preopen sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is M -preopen and bijective, we have $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. This shows that the space (Y, σ) is also ρ -normal.

Theorem: 2.18 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is \tilde{g} -irresolute, open, preclosed, preirresolute injecton and (Y, σ) is ρ -normal, then (X, τ) is ρ -normal.

Proof: Let A and B be any disjoint ρ -closed subsets of (X, τ) . Since f is \tilde{g} -irresolute, open, preclosed, by Theorem 2.7 (i), $f(A)$ and $f(B)$ are disjoint ρ -closed sets of (Y, σ) . Since (Y, σ) is ρ -normal and so there exists disjoint preopen sets U and V in (Y, σ) such that $f(A) \subseteq U$ and $f(B) \subseteq V$. By hypothesis, $f^{-1}(U)$ and $f^{-1}(V)$ are preopen sets of (X, τ) such that $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore, (X, τ) is ρ -normal.

Theorem: 2.19 Let (X, τ) be a ρ - T_s locally indiscrete space. Then the following are equivalent:

1. (X, τ) is normal.
2. For every disjoint closed sets A and B , there exist disjoint ρ -open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Proof: (1) \rightarrow (2) Let (X, τ) be a locally indiscrete space. Let A and B be disjoint closed subsets of (X, τ) . Then by hypothesis, there exist disjoint open sets (and hence preopen sets) U and V such that $A \subseteq U$ and $B \subseteq V$. Since the space (X, τ) is locally indiscrete space, then U and V are closed sets. Hence U and V are ρ -open sets. Therefore, for every disjoint closed sets A and B , there exist disjoint ρ -open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(2) \rightarrow (1) Let (X, τ) be a ρ - T_s space. Let A and B be disjoint closed subsets of (X, τ) . Then by assumption, there exist disjoint ρ -open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since U and V are ρ -open sets of X and X is a ρ - T_s space, then U and V are disjoint open sets of X . Therefore, for any pair of disjoint closed sets A and B , there exist disjoint open set U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem: 2.20 The closure of a compact open subset of a compact Hausdorff ρ - T_s space is ρ -normal.

Proof: Let X be a compact Hausdorff ρ - T_s space and A be a compact open subset of X . Then A is closed in X because compact subset of a Hausdorff space is closed. Hence A is preclosed in X . Also A is open in X . By Theorem 3.1[4], A is ρ -closed in X .

Now to prove $\bar{A} = A$ is ρ -normal. Suppose for any pair of disjoint ρ -closed sets F_1 and F_2 of A are given. By Theorem 1.8, F_1 and F_2 are ρ -closed sets in X . Since X is ρ - T_s , then F_1 and F_2 are closed sets in X . Hence F_1 and F_2 are compact sets in X because closed subset of a compact Hausdorff space is compact. By Theorem 1.12[ii], there exists disjoint open sets U and V of F_1 and F_2 respectively. Since every open set is preopen, then there exists disjoint preopen sets U and V of F_1 and F_2 respectively. Therefore, X is ρ -normal.

Theorem: 2.21 Let (Y, σ) be a submaximal extremally disconnected space. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is weakly continuous ρ -closed injection and (Y, σ) is ρ -normal, then (X, τ) is normal.

Proof: Let A and B be any two disjoint closed sets of (X, τ) . Since f is injective and ρ -closed, $f(A)$ and $f(B)$ are disjoint ρ -closed sets of (Y, σ) . Since (Y, σ) is ρ -normal, by Theorem 2.16, there exist preopen sets U and V such that $f(A) \subseteq U$ and $f(B) \subseteq V$ and $\text{pcl}(U) \cap \text{pcl}(V) = \emptyset$. Since the space (Y, σ) is submaximal, then there exist open sets U and V such that $f(A) \subseteq U$ and $f(B) \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since f is weakly continuous, it follows that [19, Theorem 1], $A \subseteq f^{-1}(U) \subseteq \text{int}(f^{-1}(\text{cl}(U)))$, $B \subseteq f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}(V)))$ and $\text{int}(f^{-1}(\text{cl}(U))) \cap \text{int}(f^{-1}(\text{cl}(V))) = \emptyset$. Therefore, (X, τ) is normal.

Theorem: 2.22 Let (Y, σ) be a submaximal extremally disconnected space. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is preirresolute ρ -closed injection and (Y, σ) is ρ -normal, then (X, τ) is prenormal.

Proof: Let A and B be any two disjoint closed sets of (X, τ) , Since f is injective and ρ -closed, $f(A)$ and $f(B)$ are disjoint ρ -closed sets of (Y, σ) . Since (Y, σ) is ρ -normal, by Theorem 2.16, there exist preopen sets U and V such that

$$f(A) \subseteq U \text{ and } f(B) \subseteq V \text{ and } \text{pcl}(U) \cap \text{pcl}(V) = \emptyset.$$

$$\text{Since } f \text{ is preirresolute, } A \subseteq f^{-1}(U) \subseteq \text{pint}(f^{-1}(\text{pcl}(U))), B \subseteq f^{-1}(V) \subseteq \text{pint}(f^{-1}(\text{pcl}(V))),$$

$$\text{Thus } \text{pint}(f^{-1}(\text{pcl}(U))) \cap \text{pint}(f^{-1}(\text{pcl}(V))) = \emptyset.$$

Therefore, (X, τ) is prenormal.

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