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ON gπ US SPACES

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Abstract

In this paper we introduce and study the notion of gπ –US spaces, gπ -convergency, sequentially gπ -compactness, sequentially gπ -continuity and sequentially gπ -sub-continuity by utilizing gπ -open sets.

***Keywords and phrases**: topological spaces, gπ – open sets, gπ US – spaces, gπ – convergence, sequentially GπO –compactness.*

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1. Introduction and Preliminaries:

In 1967, A. Wilansky [11] introduced and studied the concept of US -spaces. Levine [8] introduced the concept of generalized closed sets of a topological space. Since the advent of these notions, several research papers with interesting results in different respects came to existence (see, [1], [2], [3], [4], [5], [6], [9]). This paper is devoted to deal with the concepts of g π -US spaces, gπ -convergence, sequentially G π O-compactness, sequentially gπ -continuity and sequentially gπ -sub-continuity.

Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by Int(A) and Cl(A), respectively. A subset A is said to be regular open (resp. regular closed) if A = int(cl(A) (resp.A = cl(int(A)). The finite union of regular open sets is said to be π -open. The complement of a π-open set is said to be π-closed. A ⊂ X is called a generalized closed set (briefly g-closed set) of X [8] if Cl(A) ⊂ G holds whenever A ⊂ G and G is open in X. A subset A of X is called a g -open set of X, if its complement A^c is g -closed in X. A ⊂ X is called a gπ-closed set of X [10] if πCl(A) ⊂ G holds whenever A ⊂ G and G is open in X. A subset A of X is called a gπ -open set of X, if its complement A^c is gπ -closed in X.

A space X is G Ocompact if every g-open cover of X has a finite subcover. A subset A of a space X is said to be GO-compact if A is GO-compact as a subspace of X. The product space of two non-empty spaces is GO-compact, if each factor space is GO compact [1]. If A is g-open in X and B is g-open in Y, then A× B is g-open in X ×Y [8]. A function f: X → Y is said to be g-continuous (resp.gπ continuous) [1] ([10]) if the inverse image of every closed set in Y is g-closed (resp.gπ-closed) in X.

2. gπ –US spaces:

Definition: 1 A sequence $\{x_n\}$ in a space X gπ-converges to a point x ∪ X if $\{x_n\}$ is eventually in every gπ-open set containing x.

Definition: 2 A space X is said to be gπ-US if every sequence in X gπ –converges to a point of X.

Definition: 3 A space X is said to be:

(1) gπ-T_1 if for each pair of distinct points x and y in X there exist a gπ-open set U in X such that x ∪U and y ∉ U and a gπ-open set V in X such that y ∈V and x ∉ V .

(2) gπ-T_2 if for each pair of distinct points x and y in X there exist gπ-open sets U and V such that U ∩V =Ø and x ∈U, y ∈V.

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Theorem: 2.1 Every gπ-US space is gπ-T_1.

Proof: Let X be a gπ-US space and x, y be two distinct points of X. Consider the sequence $\{x_n\}$, where $x_n = x$ for any $n \in N$. Clearly, $\{x_n\}$ gπ converges to x. Since $x \neq y$ and X is gπ-US, $\{x_n\}$ does not gπ-converge to y, i.e., there exists a gπ-open set U containing x but not y. Similarly, we obtain a gπ-open set V containing y but not x. Thus, X is gπ $-T_1$.

Theorem: 2.2 Every gπ-T_2 space is gπ- US.

Proof: Let X be a gπ-T_2 space and $\{x_n\}$ a sequence in X. Assume that $\{x_n\}$ gπ-converges to two distinct points x and y. Then $\{x_n\}$ is eventually in every gπ-open set containing x and also in every gπ-open set containing y. Since X is gπ-T_2 then $\{x_n\}$ is eventually in two disjoint gπ-open sets. This is a contradiction. Therefore, X is gπ-US.

Definition: 4 A subset A of a space X is said to be:

(1) Sequentially gπ-closed if every sequence in A gπ-converges to a point in A,

(2) Sequentially GπO-compact if every sequence in A has a subsequence which gπ converges to a point in A.

Theorem: 2.3 A space is gπ-US if and only if the diagonal set Δ is a sequentially gπ-closed subset of the product space $X \times X$.

Proof: Suppose that X is a gπ-US space and $\{(x_n, x_n)\}$ is a sequence in the diagonal Δ. If follows that $\{x_n\}$ is a sequence in X. Since X is gπ-US, the sequence $\{x_n\}$ gπ-converges to a unique point, say $x \in X$. This implies that the sequence $\{(x_n, x_n)\}$ gπ-converges to (x, x) which clearly belongs to Δ. Therefore, Δ is a sequentially gπ-closed subset of $X \times X$.

Conversely, suppose that the diagonal Δ is a sequentially gπ-closed subset of $X \times X$. Assume that a sequence $\{x_n\}$ is gπ-converging to x and y. Then it follows that $\{(x_n, x_n)\}$ gπ-converges to (x, y). By hypothesis, since Δ is sequentially gπ-closed, we have $(x, y) \in \Delta$. Thus, x = y. Therefore, X is gπ-US.

Theorem: 2.4 If a space X is gπ-US and a subset M of X is sequentially GπO compact, then M is sequentially gπ-closed.

Proof: Assume that $\{x_n\}$ is any sequence in M which gπ-converges to a point $x \in X$. Since M is sequentially GπO-compact, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\{x_{nk}\}$ gπ-converges to $m \in M$. Since X is gπ-US, we have x = m. This shows that M is sequentially gπ-closed.

Theorem: 2.5 The product space of an arbitrary family of gπ-US topological spaces is a gπ-US topological space.

Proof: Let $\{X_\lambda: \lambda \in \Delta\}$ be a family of gπ-US topological spaces with the index set Δ. The product space of $\{X_\lambda: \lambda \in \Delta\}$ is denoted by ΠX_λ .Let $\{x_n(\lambda)\}$ be a sequence in ΠX_λ. Suppose that $\{x_n(\lambda)\}$ gπ-converges to two distinct points x and y in ΠX_λ. Then there exists a $\lambda_0 \in \Delta$ such that $x(\lambda_0) \neq y(\lambda_0)$. Then $\{x_n(\lambda_0)\}$ is a sequence in $X_{\lambda 0}$. Let $V_{\lambda 0}$ be any gπ-open set in $X_{\lambda 0}$ containing $x(\lambda_0)$. Then $V= V_{\lambda 0} \times \Pi_{\lambda \neq \lambda 0} X_\lambda$ is a gπ-open set of ΠX_λ containing x. Therefore, $\{x_n(\lambda)\}$ is eventually in V. Thus, $\{x_n(\lambda_0)\}$is eventually in $V_{\lambda 0}$ and it gπ-converges to $x(\lambda_0)$. Similarly, the sequence $\{x_n(\lambda_0)\}$ gπ-converges to $y(\lambda_0)$. This is a contradiction as $X_{\lambda 0}$ is a gπ-US space. Therefore, the product space ΠX_λ is gπ-US.

3. Sequentially GπO-compact preserving functions:

Definition: 5 A function f: X →Y is said to be:

(1) Sequentially gπ-continuous at $x \in X$ if the sequence $\{f(x_n)\}$ gπ- converges to f (x) whenever a sequence $\{x_n\}$ gπ-converges to x. If f is sequentially gπ-continuous at each $x \in X$, then it is said to be sequentially gπ-continuous.

(2) Sequentially nearly gπ-continuous, if for each sequence $\{x_n\}$ in X that gπ-converges to $x \in X$, there exists subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that the sequence $\{f(x_{nk})\}$ gπ-converges to $\{(x_n)\}$.

(3) Sequentially sub gπ-continuous if for each point $x \in X$ and each sequence $\{x_n\}$ in X gπ-converging to x, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ and a point $y \in Y$ such that the sequence $\{f(x_{nk})\}$ gπ-converge to y.

(4) Sequentially GπO-compact preserving if the image f(M) of every sequentially GπO-compact set M of X is a sequentially GπO-compact subset of Y.

Theorem: 3.1 Let f_1: X →Y and f_2: X →Y be two sequentially gπ- continuous functions. If Y is gπ-US, then the set E = {x ∈ X: $f_1(x) = f_2(x)$} is sequentially gπ-closed.

Proof: Suppose that Y is gπ-US and {x_n} is any sequence in E that f_1-converges to x ∈ X. Since f_1 and f_2 are sequentially gπ-continuous functions, the sequence {$f_1(x_n)$} (respectively, {$f_n(x_n)$}) converges to $f_1(x)$ respectively, $f_2(x)$). Since x_n ∈ E for each n ∈ N and Y is gπ-US, $f_1(x) = f_2(x)$ and hence x ∈ E. This shows that E is sequentially gπ-closed.

Lemma: 3.2 Every function f: X →Y is sequentially sub-gπ-continuous if Y is sequentially GπO-compact.

Proof: Let {x_n} be a sequence in X that gπ converges to x ∈ X. It follows that {f (x_n)} is a sequence in Y. Since Y is sequentially GπO compact, there exists a subsequence {(x_{nk})} of {f(x_n)} that gπ-converges to a point y ∈Y .

Therefore, f: X →Y is sequentially sub-gπ-continuous.

Theorem 3.3 Every sequentially nearly gπ-continuous function is sequentially GπO-compact preserving.

Proof: Let f: X →Y be a sequentially nearly gπ-continuous function and M be any sequentially GπO-compact subset of X. We will show that f (M) is a sequentially GπO-compact subset of Y. So, assume that {y_n} is any sequence in f (M). Then for each n ∈ N, there exists a point x_n ∈ M such that f (x_n) = y_n. Now M is sequentially GπO-compact, so there exist a subsequence {x_{nk}} of {x_n} that gπ-converges to a point x ∈M. Since f is sequentially nearly gπ-continuous, there exists a subsequence {x_{nk} (i)} of {x_{nk}} such that {f (x_{nk} (i))} gπ-converges to f (x) . Therefore, there exists a subsequence {y_{nk} (i)} of {y_n} that gπ-converges to f (x) . This implies that f (M) is a sequentially GπO-compact set of Y.

Theorem: 3.4 Every sequentially GπO-compact preserving function is sequentially sub-gπ-continuous.

Proof: Suppose that f: X → Y is a sequentially GπO-compact preserving function. Let x be any point of X and {x_n} a sequence that gπ converges to x. We denote the set {x_n : n ∈ N} by A and put M = A ∪{x}. Since {x_n} gπ-converges to x, M is sequentially GπO-compact. By hypothesis, f is sequentially GπO-compact preserving and hence f (M) is a sequentially GπO-compact subset of Y. Now in f (M) there exists a subsequence {f (x_{nk}) } of {f (x_n)} that gπ-converges to a point y ∈ f (M) . This implies that f sequentially sub-gπ-continuous.

Theorem 3.5 A function f: X →Y is sequentially GπO-compact preserving if and only if f $|_M$: M → f (M) is sequentially sub gπ-continuous for each sequentially GπO-compact set M of X.

Proof: Necessity: suppose that f: X →Y is a sequentially GπO-compact preserving function. Then f (M) is sequentially GπO-compact in Y for each sequentially GπO-compact subset M of X. Therefore, by Theorem 3.4 f $|_M$: M → f (M) is sequentially sub-gπ-continuous.

Sufficiency: Let M be any sequentially GπO-compact set of X. We will show that f (M) is sequentially GπO-compact subset of Y. Let {y_n} be any sequence in f (M). Then for each n ∈ N , there exists a point x_n ∈ M such that f (x_n) = y_n

Since {x_n} is a sequence in the sequentially GπO-compact set M there exists a subsequence {x_{nk}} of {x_n} that gπ-converges to a point in M. By hypothesis f $|_M$: M → f (M) is sequentially sub-gπ-continuous hence there exists a subsequence {y_{nk}} of {y_n} that gπ-converges to y ∈ f (M). This implies that f (M) is sequentially GπO-compact in Y.

Corollary: 3.6 If a function f: X →Y is sequentially sub-gπ-continuous and f (M) is sequentially gπ-closed in Y for each sequentially GπO-compact set M of X, then f is sequentially GπO-compact preserving.

Proof: It will suffice to show that is sequentially sub-gπ-continuous for each sequentially GπO-compact set M of X, and by Lemma 3.2 we are done. So, let {x_n} be any sequence in M that gπ-converges to a point x ∈ M. Then, since f is sequentially sub-gπ-continuous there exists a subsequence {x_n} of {x_n} and a point y ∈ Y such that {f (x_{nk})} gπ converges to y.

Since {f(x_{nk})} is a sequence in the sequentially gπ-closed set f (M) of Y, we obtain y ∈ f (K) . This implies that f $|_M$: M → f (M) is sequentially sub-gπ-continuous.

REFERENCES:

[1] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Univ (Math.), 12 (1991), 5-13.

[2] M. Caldas, On g-closed sets and g-continuous mappings, Kyungpook Math. J., 33(1993), 205-209.

[3] M. Caldas, Further results on generalized open mappings in topological spaces, Bull. Cal. Math. Soc. 88(1996), 37-42.

[4] W. Dunham and N. Levine, Further results on generalized closed sets in topology, Kyungpook Math. J., 20(1980), 169-175.

[5] W. Dunham, 1/ 2 T -spaces, Kyungpook Math. J., 17(1977), 161-169.

[6] W. Dunham, A new closure operator for non- 1 T topologies, Kyungpook Math. J., 22(1982), 55-60.

[7] J. Dontchev and T. Noiri. Quasi-normal spaces and gπ-closed sets, Acta Math Hungar., 89 (2000) 211-219.

[8] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), 19(1970), 89-96.

[9] S. R. Malghan, Generalized closed sets maps, J. Karnatak Univ. Sci., 27(1982), 82-88.

[10] S. Saranya, A.Parvathi, Gπ-closed sets in topological spaces (Submitted).

[11] A. Wilanski, Between 1 T and 2 T, Amer. Math. Monthly Vol. (1967), 261-266.

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