Ωγθ-CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we offer a new class of sets called Ωγθ -closed sets in topological spaces and we study some of its basic properties. The family of Ωγθ -closed sets of a topological space forms a topology and is denoted by τΩγθ . Notice that this class of sets lies between the class of θ-closed sets and the class of θω-closed sets. Using these sets, we obtain a decomposition of θ-continuity and we introduce new spaces called TΩγθ and sTΩγθ . Using these spaces we obtain another decomposition of T1/2-spaces.

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1. INTRODUCTION:

In 1963 Levine [17] introduced the notion of semi-open sets. Velicko [30] introduced the notion of θ-closed sets and it is well known that the collection of all θ-closed sets of a topological space forms a topology and is denoted by τθ . Levine [16] also introduced the notion of g-closed sets and investigated its fundamental properties. This notion was shown to be productive and very useful. Dontchev and Maki [11] introduced the notion of Θ-generalized closed sets.

After the advent of g-closed sets, Arya and Nour [4], Sheik John [25], Ravi and Ganesan [23] and Dontchev [10] introduced gs-closed sets, ω-closed sets, g -closed sets and gsp-closed sets respectively.

Quite recently, Ganesan et al. [13] have introduced the notion of Ωω-closed sets which lies between the θ-closed sets and the Ωg -closed sets.

In this paper, we introduce a new class of sets called Ωγθ -closed sets in topological spaces. This class lies between the class of θ-closed sets and the class of Ωω-closed sets. We study some of its basic properties and characterizations. Interestingly it turns out that the family of Ωγθ -closed sets of a topological space forms a topology. This collection is denoted by τΩγθ . From the definitions, it follows immediately that τθ ⊆ τΩγθ . Using these sets, we obtain a decomposition of θ-continuity and we introduce new type of spaces called TΩg -spaces and sTΩg -spaces. Using these spaces, we obtain another decomposition of T1/2-spaces.
2. PRELIMINARIES:

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c or X \ A denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition 2.1 A subset A of a space (X, τ) is called:

(i) a semi-open set [17] if A ⊆ cl(int(A));
(ii) a preopen set [19] if A ⊆ int(cl(A));
(iii) α-open set [20] if A ⊆ int(cl(int(A)));
(iv) β-open set [1] (= semi-preopen set [2]) if A ⊆ cl(int(A));
(v) a regular open set [26] if A = int(cl(A)).

The complements of the above mentioned open sets are called their respective closed sets.

The preclosure [21] (resp. semi-closure [8], α-closure [20], semi-pre-closure [2]) of a subset A of X, denoted by pcl(A) (resp. scl(A), α cl(A), spcl(A)), is defined to be the intersection of all preclosed (resp. semi-closed, α-closed, semi-pre-closed) sets of (X, τ) containing A. It is known that pcl(A) (resp. scl(A), α cl(A), spcl(A)) is a preclosed (resp. semi-closed, α-closed, semi-pre-closed) set.

Definition 2.2 [30]

A point x of a space X is a called a θ-adherent point of a subset A of X if cl(U) ∩ A ≠ ∅ for every open set U containing x. The set of all θ-adherent points of A is called the θ-closure of A and is denoted by cl_θ(A). A subset A of a space X is called θ-closed if and only if A = cl_θ(A). The complement of a θ-closed set is called θ-open. Similarly, the θ-interior of a set A in X, written int_θ(A), consists of those points x of A such that for some open set U containing x, cl(U) ⊆ A. A set A is θ-open if and only if A = int_θ(A), or equivalently, X \ A is θ-closed.

A point x of a space X is called a δ-adherent point of a subset A of X if int(cl(U)) ∩ A ≠ ∅ for every open set U containing x. The set of all δ-adherent points of A is called the δ-closure of A and is denoted by cl_δ(A). A subset A of a space X is called δ-closed if and only if A = cl_δ(A). The complement of a δ-closed set is called δ-open. Similarly, the δ-interior of a set A in X, written int_δ(A), consists of those points x of A such that for some regularly open set U containing x, U ⊆ A. A set A is δ-open if and only if A = int_δ(A), or equivalently, X \ A is δ-closed.

The family of all θ-open (resp. δ-open) subsets of (X, τ) forms a topology on X and is denoted by τ_θ (resp. τ_δ). From the definitions it follows immediately that τ_θ ⊆ τ_δ ⊆ τ. [7].

Definition 2.3 A point x ∈ X is called a semi θ-cluster [9] point of A if A ∩ scl(U) ≠ ∅ for each semi-open set U containing x.

The set of all semi θ-cluster points of A is called the semi θ-cluster of A and is denoted by scl θ(A). Hence, a subset A is called semi-θ-closed if scl θ(A) = A. The complement of a semi-θ-closed set is called semi-θ-open set.

Recall that a subset A of a space (X, τ) is said to be δ-semi-open [22] if A ⊆ cl(int δ(A)).

Definition 2.4 A subset A of a space (X, τ) is called:

(i) a generalized closed (briefly, g-closed) set [16] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
(ii) a generalized semi-closed (briefly, gs-closed) set [4] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
(iii) a semi-generalized closed (briefly, sg-closed) set [5] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ).
(iv) an α-generalized closed (briefly, α g-closed) set [18] if α cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
a generalized semi-preclosed (briefly, gsp-closed) set [10] if \( \text{spcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(vi) a generalized preclosed (briefly, gp-closed) set [21] if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(vii) a \( \theta \)-closed set [27] (\( \omega \)-closed set [25]) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\).

(viii) a \( g \)-closed set [23] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is sg-open in \((X, \tau)\).

(ix) a \( \psi \)-closed set [29] if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is sg-open in \((X, \tau)\).

(x) a \( \theta \)-generalized closed set (briefly, \( g \theta \)-closed) [11] if \( \theta \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(xi) a \( \theta \omega \)-closed set [13] if \( \theta \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi open in \((X, \tau)\).

Remark: 2.5 The collection of all \( g \theta \)-closed (resp. \( g \)-closed, \( \theta \omega \)-closed, sg-closed, \( \psi \)-closed, \( \omega \)-closed, g-closed, \( \theta \)-closed, \( \alpha \)-closed, semi-closed) sets of \( X \) is denoted by \( G \theta C(X) \) (resp. \( G \theta C(X), \theta \omega C(X), SG C(X), \psi C(X), \omega C(X), G C(X), \theta C(X), \alpha C(X), S C(X) \)).

We denote the power set of \( X \) by \( P(X) \).

Remark: 2.6 [6] We have the following diagram in which the converses of the implications need not be true.

Remark: 2.7 [25]

(i) Every \( \theta \)-closed set is \( g \theta \)-closed.

(ii) \( g \theta \)-closed sets and \( \omega \)-closed sets are independent.

Remark: 2.8 [7] \((X, \tau)\) is regular if and only if \( \tau_\theta = \tau \).

Remark: 2.9 [24] A space \( X \) is called \( \tau \theta \)-space if every \( g \)-closed set in \( X \) is closed.

Definition: 2.10 A topological space \((X, \tau)\) is called a \( R_1 \)-space [12] if every two different points with distinct closures have disjoint neighborhoods.

Proposition: 2.11 [7] Let \((X, \tau)\) be a space. Then,

(i) if \( A \subseteq X \) is preopen then \( \text{cl}(A) = \text{cl}_\theta (A) \).

(ii) \((X, \tau)\) is \( R_1 \) if and only if \( \text{cl}(\{x\}) = \text{cl}_\theta (\{x\}) \) for each \( x \in X \).

Proposition: 2.12 [12, 14] Let \((X, \tau)\) be a space. If \( A \subseteq X \) is preopen then \( \text{cl}(A) = \alpha \text{cl}(A) = \text{cl}_\theta (A) \).

Definition: 2.13 [16] A space \((X, \tau)\) is called \( T_{1\theta} \)-space if every \( g \)-closed set is closed.

Lemma: 2.14 [13] In any space, if a singleton is \( \theta \)-open then it is regular open.

Lemma: 2.15 [13] In a regular space, singleton is \( \theta \)-open if and only if it is regular open.

Lemma: 2.16 [13] If \( A \) is both closed and preopen of a topological space \( X \), then the following are equivalent.

(i) \( A \) is \( \theta \)-closed.

(ii) \( A \) is \( \delta \)-closed.

(iii) \( A \) is \( \alpha \)-closed.
Lemma: 2.17 [13] If a subset A of a space $(X, \tau)$ is clopen, then the following are equivalent.

(i) $A$ is $\theta$-closed.
(ii) $A$ is $\delta$-closed.
(iii) $A$ is $\alpha$-closed.
(iv) $A$ is regular closed.

3. $\theta G$ -CLOSED SETS:

We introduce the following definition.

Definition: 3.1 A subset $A$ of $X$ is called a $\theta G$-closed set if $\theta cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is sg-open in $(X, \tau)$.

The complement of $\theta G$-closed set is called $\theta G$-open set.

The collection of all $\theta G$-closed sets of $X$ is denoted by $G \theta C(X)$.

Proposition: 3.2 Every $\theta$-closed set is $\theta G$-closed.

Proof: Let $A$ be a $\theta$-closed set and $G$ be any sg-open set containing $A$ in $(X, \tau)$. Since $A$ is $\theta$-closed, $cl_\theta(A) = A$ for every subset $A$ of $X$. Therefore $cl_\theta(A) \subseteq G$ and hence $A$ is $\theta G$-closed.

The converse of Proposition 3.2 need not be true as seen from the following example.

Example: 3.3 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$. Then $G \theta C(X) = \{\phi, \{b, c\}, X\}$ and $\theta C(X) = \{\phi, X\}$. Here, $A = \{b, c\}$ is $\theta G$-closed but not $\theta$-closed set in $(X, \tau)$.

Proposition: 3.4 Every $\theta G$-closed set is $\theta G$-closed.

Proof: Let $A$ be a $\theta G$-closed set and $G$ be any open set containing $A$ in $(X, \tau)$. Since every open set is sg-open and $A$ is $\theta G$-closed, $\theta cl(A) \subseteq G$. Therefore $cl_\theta(A) \subseteq G$ and $G$ is open. Hence $A$ is $\theta G$-closed.

The converse of Proposition 3.4 need not be true as seen from the following example.

Example: 3.5 Let $X$ and $\tau$ be as in the Example 3.3. Then $G \theta C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $\theta C(X) = \{\phi, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, c\}, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Here, $A = \{a, c\}$ is $\theta G$-closed but not $\theta G$-closed set in $(X, \tau)$.

Proposition: 3.6 Every $\theta G$-closed set is $\theta \omega$-closed.

Proof: Let $A$ be a $\theta G$-closed set and $G$ be any semi open set containing $A$ in $(X, \tau)$. Since every semi open set is sg-open and $A$ is $\theta G$-closed, $cl_\theta(A) \subseteq G$. Therefore $cl_\theta(A) \subseteq G$ and hence $A$ is $\theta \omega$-closed.

The converse of Proposition 3.6 need not be true as seen from the following example.

Example: 3.7 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then $G \theta C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $\theta C(X) = \{\phi, \{a\}, \{b, c\}, X\}$. Here, $A = \{a, b\}$ is $\theta G$-closed but not $\theta G$-closed set in $(X, \tau)$.

Proposition: 3.8 Every $\theta G$-closed set is g-closed.

Proof: Let $A$ be a $\theta G$-closed set and $G$ be any open set containing $A$ in $(X, \tau)$. Since every open set is sg-open and $A$ is $\theta G$-closed, $cl(A) \subseteq G$. Therefore $cl(A) \subseteq G$ and hence $A$ is g-closed.
Example: 3.9 Let X and τ be as in the Example 3.3. Then $\theta\tilde{G} C(X) = \{\phi, \{b, c\}, X\}$ and $G C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Here, $A = \{a, b\}$ is g-closed but not $\theta\tilde{G}$-closed set in $(X, \tau)$.

**Proposition: 3.10** Every $\theta\tilde{G}$-closed set is $\omega$-closed.

**Proof:** Let $A$ be a $\theta\tilde{G}$-closed set and $G$ be any semi open set containing $A$ in $(X, \tau)$. Since every semi open set is sg-open and $A$ is $\theta\tilde{G}$-closed, $cl_\theta (A) \subseteq G$. Since $cl(A) \subseteq cl_\theta (A) \subseteq G$, $cl(A) \subseteq G$ and hence $A$ is $\omega$-closed.

The converse of Proposition 3.10 need not be true as seen from the following example.

Example: 3.11 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\theta\tilde{G} C(X) = \{\phi, \{b, c\}, X\}$ and $\omega C(X) = \{\phi, \{c\}, \{b, c\}, X\}$. Here, $A = \{c\}$ is $\omega$-closed but not $\theta\tilde{G}$-closed set in $(X, \tau)$.

**Proposition: 3.12** Every $\theta\tilde{G}$-closed set is $g\theta$-closed.

**Proof:** Let $A$ be a $\theta\tilde{G}$-closed and $G$ be any sg-open set containing $A$. Since $cl(A) \subseteq cl_\theta (A) \subseteq G$, $cl(A) \subseteq G$ and hence $A$ is $g\theta$-closed.

The converse of Proposition 3.12 need not be true as seen from the following example.

Example: 3.13 Let $X$ and $\tau$ be as in the Example 3.11. Then $\theta\tilde{G} C(X) = \{\phi, \{b, c\}, X\}$ and $G C(X) = \{\phi, \{c\}, \{b, c\}, X\}$. Here, $A = \{c\}$ is $g\theta$-closed but not $\theta\tilde{G}$-closed set in $(X, \tau)$.

**Proposition: 3.14** Every $\theta\tilde{G}$-closed set is sg-closed.

**Proof:** Let $A$ be a $\theta\tilde{G}$-closed and $G$ be any semi open set containing $A$ in $(X, \tau)$. Since every semi open set is sg-open, $cl_\theta (A) \subseteq G$. Since $scl(A) \subseteq cl_\theta (A) \subseteq G$, $scl(A) \subseteq G$ and hence $A$ is sg-closed.

The converse of Proposition 3.14 need not be true as seen from the following example.

Example: 3.15 Let $X$ and $\tau$ be as in the Example 3.7. Then $\theta\tilde{G} C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $SG C(X) = P(X)$. Here, $A = \{a, b\}$ is sg-closed but not $\theta\tilde{G}$-closed set in $(X, \tau)$.

**Proposition: 3.16** Every $\theta\tilde{G}$-closed set is $\psi$-closed.

**Proof:** It is true that $scl(A) \subseteq cl_\theta (A)$ for every subset $A$ of $(X, \tau)$.

The converse of Proposition 3.16 need not be true as seen from the following example.

Example: 3.17 Let $X$ and $\tau$ be as in the Example 3.3. Then $\theta\tilde{G} C(X) = \{\phi, \{b, c\}, X\}$ and $\psi C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Here, $A = \{b\}$ is $\psi$-closed but not $\theta\tilde{G}$-closed set in $(X, \tau)$.

**Remark: 3.18** The following examples show that $\theta\tilde{G}$-closedness is independent of closedness, semi-closedness and $\alpha$-closedness.

Example: 3.19 Let $X$ and $\tau$ be as in the Example 3.3. Then $\theta\tilde{G} C(X) = \{\phi, \{b, c\}, X\}$ and $\alpha C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Here, $A = \{b\}$ is $\alpha$-closed as well as semi-closed in $(X, \tau)$ but it is not $\theta\tilde{G}$-closed set in $(X, \tau)$.

Example: 3.20 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$. Then $\theta\tilde{G} C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\alpha C(X) = S C(X) = \{\phi, \{c\}, X\}$. Here, $A = \{a, c\}$ is $\theta\tilde{G}$-closed but it is neither $\alpha$-closed set nor semi-closed set in $(X, \tau)$.  

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Example: 3.21 In Example 3.11, \{c\} is closed but not \(\bar{\theta}g\) -closed set.

In Example 3.20, \{b, c\} is \(\bar{\theta}g\) -closed but not closed set.

Remark: 3.22 \(\theta\omega\) -closed sets and \(g\) -closed sets are independent.

Example: 3.23 In Example 3.7, \{a, b\} is \(\theta\omega\) -closed but not \(g\) -closed set.

In Example 3.11, \{c\} is \(g\) -closed but not \(\theta\omega\) -closed set.

Remark: 3.24 From the above discussions and known results in [10, 12, 13, 23, 25, 28, 29], we obtain the following diagram, where \(A \rightarrow B\) (resp. \(A \leftrightarrow B\)) represents \(A\) implies \(B\) but not conversely (resp. \(A\) and \(B\) are independent of each other).

\[
\begin{array}{c}
\text{\(\sigma\)-closed} \\
\downarrow \\
\text{\(\text{\(g\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(\omega\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(g\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(\omega\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(g\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(s\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(s\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(s\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(s\)}\)-closed} \\
\downarrow \\
\text{\(\text{\(s\)}\)-closed} \\
\end{array}
\]

4. PROPERTIES OF \(\bar{\theta}g\) -CLOSED SETS:

Definition: 4.1 [23] The intersection of all sg-open subsets of \((X, \tau)\) containing \(A\) is called the sg-kernel of \(A\) and is denoted by \(\text{sg-ker}(A)\).

Lemma: 4.2 A subset \(A\) of \((X, \tau)\) is \(\bar{\theta}g\) -closed if and only if \(\text{\(\theta\)}\text{cl}(A) \subseteq \text{sg-ker}(A)\).

Proof: Suppose that \(A\) is \(\bar{\theta}g\) -closed. Then \(\text{\(\theta\)}\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is sg-open. Let \(x \in \text{\(\theta\)}\text{cl}(A)\). If \(x \notin \text{sg-ker}(A)\), then there is a sg-open set \(U\) containing \(A\) such that \(x \notin U\). Since \(U\) is a sg-open set containing \(A\), we have \(x \notin \text{\(\theta\)}\text{cl}(A)\) and this is a contradiction.

Conversely, let \(\text{\(\theta\)}\text{cl}(A) \subseteq \text{sg-ker}(A)\). If \(U\) is any sg-open set containing \(A\), then \(\text{\(\theta\)}\text{cl}(A) \subseteq U\). Therefore, \(A\) is \(\bar{\theta}g\) -closed.

Remark: 4.3 The collection of all \(\bar{\theta}g\) -closed sets of a topological space forms a topology and is denoted by \(\tau\bar{\theta}g\).

Remark: 4.4 If \(A\) is a \(\bar{\theta}g\) -closed set and \(F\) is a \(\theta\) -closed set, then \(A \cap F\) is a \(\bar{\theta}g\) -closed set.

Proof: Since \(F\) is \(\theta\) -closed, it is \(\bar{\theta}g\) -closed. Therefore by Remark 4.3, \(A \cap F\) is also a \(\bar{\theta}g\) -closed set.

Proposition: 4.5 If a set \(A\) is \(\bar{\theta}g\) -closed in \((X, \tau)\), then \(\text{\(\theta\)}\text{cl}(A) \cap A\) contains no nonempty sg-closed set in \((X, \tau)\).

Proof: Suppose that \(A\) is \(\bar{\theta}g\) -closed. Let \(F\) be a sg-closed subset of \(\text{\(\theta\)}\text{cl}(A) \cap A\). Then \(A \subseteq F\). Therefore \(\text{\(\theta\)}\text{cl}(A) \subseteq F\). Consequently, \(F \subseteq (\text{\(\theta\)}\text{cl}(A))^\circ\). We already have \(F \subseteq \text{\(\theta\)}\text{cl}(A)\). Thus \(F \subseteq \text{\(\theta\)}\text{cl}(A) \cap (\text{\(\theta\)}\text{cl}(A))^\circ\) and \(F\) is empty.
The converse of Proposition 4.5 need not be true as seen from the following example.

**Example: 4.6** Let \( X \) and \( \tau \) be as in the Example 3.20. Then \( \theta G C(X) = SG C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\} \). If \( A = \{c\} \), then \( cl_{\theta}(A) - A = \{a, b\} \) does not contain any nonempty sg-closed set. But \( A \) is not \( \theta - \)closed in \((X, \tau)\).

**Proposition: 4.7** Let \( A \subseteq Y \subseteq X \) where \( Y \) is open and suppose that \( A \) is \( \theta G \) -closed in \((X, \tau)\). Then \( A \) is \( \theta G \) -closed relative to \( Y \).

**Proof:** Let \( A \subseteq Y \cap G \), where \( G \) is sg-open in \((X, \tau)\). Then \( A \subseteq G \) and hence \( cl_{\theta}(A) \subseteq G \). This implies that \( Y \cap cl_{\theta}(A) \subseteq Y \cap G \). Thus \( A \) is \( \theta G \) -closed relative to \( Y \).

**Proposition: 4.8** If \( A \) is a sg-open and \( \theta G \) -closed in \((X, \tau)\), then \( A \) is \( \theta \) -closed in \((X, \tau)\).

**Proof:** Since \( A \) is sg-open and \( \theta G \) -closed, \( cl_{\theta}(A) \subseteq A \) and hence \( A \) is \( \theta \) -closed in \((X, \tau)\).

**Theorem: 4.9** Let \( A \) be a subset of a regular space \((X, \tau)\). Then,

(i) \( A \) is \( \theta G \) -closed if and only if \( A \) is \( \theta G \) -closed.

(ii) if \((X, \tau)\) is \( \tau \) -regular, then \( A \) is \( \theta G \) -closed if and only if \( A \) is closed.

**Proof:**

(i) It follows from Remark 2.8.

(ii) It follows from Remark 2.9.

**Theorem: 4.10** Let \( A \) be a preopen subset of a topological space \((X, \tau)\). Then the following conditions are equivalent.

(i) \( A \) is \( \theta G \) -closed.

(ii) \( A \) is \( \theta G \) -closed.

(iii) \( A \) is \( \theta G \) -closed.

(iv) \( A \) is g-closed.

(v) \( A \) is \( \alpha g \) -closed.

**Proof:**

(i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v). It is obvious from Remark 3.24.

(v) \( \Rightarrow \) (i). It follows from Propositions 2.11 and 2.12.

Recall that a partition space [12] is a topological space where every open set is closed.

**Corollary: 4.11** Let \( A \) be a subset of the partition space \((X, \tau)\). Then the following conditions are equivalent.

(i) \( A \) is \( \theta G \) -closed.

(ii) \( A \) is \( \theta G \) -closed.

(iii) \( A \) is \( \theta G \) -closed.

(iv) \( A \) is g-closed.

(v) \( A \) is \( \alpha g \) -closed.

**Proof:** A topological space is a partition space if and only if every subset is preopen. Then the claim follows straight from Theorem 4.10.

**Theorem: 4.12** For a singleton subset \( A \) of an \( R_1 \) topological space \((X, \tau)\), the following conditions are equivalent.

(i) \( A \) is \( \theta G \) -closed.

(ii) \( A \) is \( \theta G \) -closed.

**Proof:** (i) \( \Rightarrow \) (ii) is clear.

(ii) \( \Rightarrow \) (i). Note that in \( R_1 \)-spaces, the concepts of closure and \( \theta \) -closure coincide for singleton sets: see Proposition 2.11.
Theorem: 4.13 For a subset $A$ of a topological space $(X, \tau)$, the following conditions are equivalent.

(i) $A$ is clopen.
(ii) $A$ is $\theta G$-closed, preopen and semi-closed.
(iii) $A$ is $\theta G$-closed and (regular) open.
(iv) $A$ is $\alpha g$-closed and (regular) open.

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i). It follows from Theorem 3.13 [12].

Definition: 4.14 A space $(X, \tau)$ is called locally sg-$\theta$-indiscrete space if every sg-open set is $\theta$-closed.

Theorem: 4.15 For a topological space $(X, \tau)$, the following conditions are equivalent.

(i) $X$ is locally sg-$\theta$-indiscrete.
(ii) Every subset of $X$ is $g \theta$-closed.

Proof: (i) $\Rightarrow$ (ii). Let $A \subseteq U$, where $U$ is sg-open and $A$ is an arbitrary subset of $X$. Since $X$ is locally sg-$\theta$-indiscrete, then $U$ is $\theta$-closed. We have $cl_\theta (A) \subseteq cl_\theta (U) = U$. Thus $A$ is $\theta G$-closed.

(ii) $\Rightarrow$ (i). If $U \subseteq X$ is sg-open, then by (ii) $cl_\theta (U) \subseteq U$ or equivalently $U$ is $\theta$-closed. Hence $X$ is locally sg-$\theta$-indiscrete.

5. DECOMPOSITION OF $\theta$-CONTINUITY:

In this section, we obtain a decomposition of continuity called $\theta$-continuity in topological spaces.

To obtain a decomposition of $\theta$-continuity, we first introduce the notion of $\theta G$ lc*-continuous functions in topological spaces and by using $\theta G$ -continuity, prove that a function is $\theta$-continuous if and only if it is both $\theta G$ -continuous and $\theta G$ lc*-continuous.

We introduce the following definition.

Definition: 5.1 A subset $A$ of a space $(X, \tau)$ is called $\theta G$ lc*-set if $A = M \cap N$, where $M$ is sg-open and $N$ is $\theta$-closed in $(X, \tau)$.

Example: 5.2 Let $X$ and $\tau$ be as in the Example 3.3. Then $\{a, b\}$ is $\theta G$ lc*-set in $(X, \tau)$.

Remark: 5.3 Every $\theta$-closed set is $\theta G$ lc*-set but not conversely.

Example: 5.4 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, X\}$. Then $\{b, c\}$ is $\theta G$ lc*-set but not $\theta$-closed in $(X, \tau)$.

Remark: 5.5 $\theta G$ -closed sets and $\theta G$ lc*-sets are independent of each other.

Example: 5.6 Let $X$ and $\tau$ be as in the Example 3.20. Then $\{a, c\}$ is an $\theta G$ -closed set but not $\theta G$ lc*-set in $(X, \tau)$.

Example: 5.7 Let $X$ and $\tau$ be as in the Example 5.4. Then $\{a, b\}$ is an $\theta G$ lc*-set but not $\theta G$ -closed set in $(X, \tau)$.

Proposition: 5.8 Let $(X, \tau)$ be a topological space. Then a subset $A$ of $(X, \tau)$ is $\theta$-closed if and only if it is both $\theta G$ -closed and $\theta G$ lc*-set.
Proof: Necessity is trivial. To prove the sufficiency, assume that \( A = M \cap N \), where \( M \) is sg-open and \( N \) is \( \theta \)-closed in \((X, \tau)\). Therefore, \( A \subseteq M \) and \( A \subseteq N \) and so by hypothesis, \( cl_{\theta}(A) \subseteq M \) and \( cl_{\theta}(A) \subseteq N \). Thus \( cl_{\theta}(A) \subseteq M \cap N = A \) and hence \( cl_{\theta}(A) = A \) i.e., \( A \) is \( \theta \)-closed in \((X, \tau)\).

We introduce the following definition

Definition: 5.9 A function \( f: (X, \tau) \to (Y, \sigma) \) is said to be \( \theta \)-continuous if for each closed set \( V \) of \((Y, \sigma)\), \( f^{-1}(V) \) is a \( \theta \)-closed set in \((X, \tau)\).

Example: 5.10 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \theta \)-continuous and \( \theta \)-closed.

Definition: 5.11 A function \( f: (X, \tau) \to (Y, \sigma) \) is called

(i) \( \theta \)-continuous if for each closed set \( V \) of \((Y, \sigma)\), \( f^{-1}(V) \) is \( \theta \)-closed.

(ii) \( \theta \)-continuous function but not \( \theta \)-closed.

Proposition: 5.12 Every \( \theta \)-continuous function is \( \theta \)-continuous but not conversely.

Proof: It follows from Proposition 3.2.

Example: 5.13 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, Y\} \). We have \( \theta C(X) = \{\emptyset, X\} \) and \( \theta C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\} \). Define \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \theta \)-continuous but not \( \theta \)-closed, since \( f^{-1}(\{a, c\}) = \{a, c\} \) is not \( \theta \)-closed in \((X, \tau)\).

Remark: 5.14 Every \( \theta \)-continuous function is \( \theta \)-continuous but not conversely.

Example: 5.15 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\emptyset, \{b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{a, c\}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \theta \)-continuous function but not \( \theta \)-closed since for the closed set \( \{b\} \) in \((Y, \sigma)\), \( f^{-1}(\{b\}) = \{b\} \), which is not \( \theta \)-closed in \((X, \tau)\).

Remark: 5.16 \( \theta \)-continuity and \( \theta \)-continuity are independent of each other.

Example: 5.17 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \theta \)-continuous function but not \( \theta \)-continuous.

Example: 5.18 Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{b, c\}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \theta \)-continuous function but not \( \theta \)-continuous.

We have the following decomposition for continuity.

Theorem: 5.19 A function \( f: (X, \tau) \to (Y, \sigma) \) is \( \theta \)-continuous and \( \theta \)-continuous if and only if it is both \( \theta \)-continuous and \( \theta \)-continuous.

Proof: Assume that \( f \) is \( \theta \)-continuous. Then by Proposition 5.12 and Remark 5.14, \( f \) is both \( \theta \)-continuous and \( \theta \)-continuous.

Conversely, assume that \( f \) is both \( \theta \)-continuous and \( \theta \)-continuous. Let \( V \) be a closed subset of \((Y, \sigma)\). Then \( f^{-1}(V) \) is both \( \theta \)-closed and \( \theta \)-closed. By Proposition 5.8, \( f^{-1}(V) \) is a \( \theta \)-closed set in \((X, \tau)\) and so \( f \) is \( \theta \)-continuous.

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6. DECOMPOSITION OF $T_{1/2}$-SPACES:

We introduce the following definition:

**Definition: 6.1** A space $(X, \tau)$ is called a $T_{\theta \theta}$-space if every $\theta \theta$-closed set in it is closed.

**Example: 6.2** Let $X$ and $\tau$ be as in the Example 3.3. Then $\theta \theta G C(X) = \{\phi, \{b, c\}, X\}$ and the sets in $\{\phi, \{b, c\}, X\}$ are closed. Thus $(X, \tau)$ is a $T_{\theta \theta}$-space.

**Example: 6.3** Let $X$ and $\tau$ be as in the Example 3.20. Then $\theta \theta G C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ and the sets in $\{\phi, \{c\}, X\}$ are closed. Thus $(X, \tau)$ is not a $T_{\theta \theta}$-space.

**Theorem: 6.4** For a topological space $(X, \tau)$, the following properties are equivalent:

(i) $(X, \tau)$ is a $T_{\theta \theta}$-space.
(ii) Every singleton of $(X, \tau)$ is either open or sg-closed.

**Proof:** (i) $\rightarrow$ (ii). If $\{x\}$ is not sg-closed, then $X - \{x\}$ is not sg-open. Hence $X$ is only sg-open set containing $X - \{x\}$. Therefore $cl_\theta (X - \{x\}) \subseteq X$. Thus $X - \{x\}$ is $\theta \theta$-closed. By (i) $X - \{x\}$ is closed, i.e. $\{x\}$ is open.

(ii) $\rightarrow$ (i). Let $A \subseteq X$ be a $\theta \theta$-closed. Let $x \in cl_\theta (A)$. We consider the following two cases:

**Case:** (a) Let $\{x\}$ be open. Since $x$ belongs to the closure of $A$, then $\{x\} \cap A \neq \phi$. This shows that $x \in A$.

**Case:** (b) Let $\{x\}$ be sg-closed. If we assume that $x \notin A$, then we would have $x \in cl_\theta (A) - A$ which cannot happen according to Proposition 4.5. Hence $x \in A$.

So in both cases we have $cl_\theta (A) \subseteq A$. Since the reverse inclusion is trivial, then $A = cl_\theta (A)$ or equivalently $A$ is $\theta$-closed. It implies that $A$ is closed.

**Definition: 6.5** A space $(X, \tau)$ is called a $g T_{\theta \theta}$-space if every $g$-closed set is $\theta \theta$-closed.

**Example: 6.6** Let $X$ and $\tau$ be as in the Example 3.20. Then $G C(X) = \theta ^{G} G C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Thus $(X, \tau)$ is a $g T_{\theta \theta}$-space.

**Example: 6.7** Let $X$ and $\tau$ be as in the Example 3.3. Then $G C(X) = \phi \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\theta \theta G C(X) = \{\phi, \{b, c\}, X\}$. Thus $(X, \tau)$ is not a $g T_{\theta \theta}$-space.

**Proposition: 6.8** Every $T_{1/2}$-space is $T_{\theta \theta}$-space but not conversely.

**Proof:** Follows from Proposition 3.8.

The converse of Proposition 6.8 need not be true as seen from the following example.

**Example: 6.9** Let $X$ and $\tau$ be as in the Example 3.3. Then $G C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\theta \theta G C(X) = \{\phi, \{b, c\}, X\}$. Thus $(X, \tau)$ is $T_{\theta \theta}$-space but it is not a $T_{1/2}$-space.

**Proposition: 6.10** Every regular $T_{1/2}$-space is $g T_{\theta \theta}$-space but not conversely.

**Proof:** Obvious.

The converse of Proposition 6.10 need not be true as seen from the following example.
Example: 6.11 Let X and $\tau$ be as in the Example 3.20. Then $G C(X) = \theta G C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Thus $(X, \tau)$ is a $\theta T_{1/2}$-space but not a $T_{1/2}$-space.

Remark: 6.12 $\theta T_{1/2}$-spaces and $\theta T_{1/2}$-spaces are independent.

Example: 6.13 Let X and $\tau$ be as in the Example 3.20. Thus $(X, \tau)$ is a $\theta T_{1/2}$-space but it is not a $T_{1/2}$-space.

Example: 6.14 Let X and $\tau$ be as in the Example 3.3. Thus $(X, \tau)$ is a $\theta T_{1/2}$-space but it is not a $\theta T_{1/2}$-space.

Theorem: 6.15 A regular space $(X, \tau)$ is $T_{1/2}$ if and only if it is both $\theta T_{1/2}$ and $\theta T_{1/2}$.

Proof: Necessity. Follows from Propositions 6.8 and 6.10.

Sufficiency. Assume that $(X, \tau)$ is both $\theta T_{1/2}$ and $\theta T_{1/2}$. Let A be a $\theta$-closed set of $(X, \tau)$. Then A is $\theta$-closed, since $(X, \tau)$ is $\theta T_{1/2}$. Again since $(X, \tau)$ is a $\theta T_{1/2}$, A is closed set in $(X, \tau)$ and so $(X, \tau)$ is $T_{1/2}$.

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\[ S. \text{Ganesan}, O. \text{Ravi}^*, \text{K. Mahaboob Hassain Sherieff and S.Pious Missier/} \Theta^g -\text{CLOSED SETS IN TOPOLOGICAL SPACES/ IJMA- 2(11), Nov.-2011, Page: 2358-2369} \]


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