SOME FIXED POINT THEOREM SATISFYING GENERAL CONTRACTION CONDITION OF INTEGRAL TYPE

1Animesh Gupta*, 2Dilip Jaiswal and 3S. S. Rajput

1Department of Mathematics, Govt. Benazir Science & Commerce College, Bhopal, (M.P.), India
2Department of Mathematics, Moti Lal Vigny Mahavidhyalaya (MVM), Bhopal, (M.P.), India
3Head of Department,(Mathematics), Govt. P.G. Collage, Gadarwara, (M.P.), India

*E-mail: animeshgupta10@gmail.com

(Received on: 23-10-11; Accepted on: 07-11-11)

ABSTRACT

In this paper, we establish a fixed point theorem for a pair of self maps satisfying a general contractive condition of integral type. This theorem extends and generalizes some early results of Boikanyo [4]. And Jaggi and Doss [12]

2000 AMS Subject Classification: 54H25, 47H10.

Key Words: Lebesgue-integrable map, complete metric space, Common fixed point.

1. INTRODUCTION:

The first well known result on fixed points for contractive map was Banach fixed point theorem, published in 1922, [2]. In general setting of complete metric space, Smart [18], presented the following result.

Theorem: 1.1 Let \((X, d)\) be a complete metric space, \(\alpha \in [0,1)\) and let \(T: X \to X\) be a map such that for each \(x, y \in X\),

\[
d(Tx, Ty) \leq \alpha d(x, y)
\]

Then, \(T\) has a unique fixed point \(z \in X\) such that for each \(x \in X, \lim_{n \to \infty} T^nx = z\).

After the classical result, many theorems dealing with the maps satisfying various types of contractive inequalities have been established see in [4], [6], [20].

In 2002, Branciari [3], obtained the following theorem.

Theorem: 1.2 Let \((X, d)\) be a complete metric space, \(\alpha \in (0,1]\) and let \(T: X \to X\), be a mapping such that for each \(x, y \in X\),

\[
\int_0^d(Tx, Ty) \xi(t) \, dt \leq \int_0^d(xy) \xi(t) \, dt
\]

Where \(\xi: [0, +\infty) \to [0, +\infty]\) is a legesgue integrable mapping which is summable on each compact subset of \([0, +\infty]\), non negative, and such that, \(\forall \, \varepsilon > 0, \int_0^\varepsilon \xi(t) \, dt > 0\) Then, \(T\)

has unique fixed point \(z \in X\) such that for each \(x \in X, T^nx \to z\) as \(n \to \infty\).

It is mentioned in [7], that theorem 1.2 could be extended to more general contractive conditions, for example, in [15], Rhoades established that Theorem1.2, hold s

If we replace \(d(x, y)\) by \(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(Tx, Ty) + d(y, Tx)}{2}\}\) Other work in this direction include \([1, 8, 19, 20]\). In [17], Suzuki proved that Theorem 1.2 of Branciari is a particular case of the famous Meir-Keeler fixed point Theorem [14], More precisely, he proved that under hypotheses of Theorem 1.2, \(T\) is a MKC, that is for every \(\delta > 0\) such that,

*Corresponding author: 1Animesh Gupta*, *E-mail: animeshgupta10@gmail.com

International Journal of Mathematical Archive- 2 (11), Nov. – 2011 2451
\[ d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon \]

And then \( T \) has a unique fixed point.

In this paper, we obtain an extension of Theorem 1.2, through rational expression.

Our obtained result extends and improves the result of Jaggi and Dass [12]. Other results on fixed point theorems through rational expression can be found in [7, 9, 11].

2. MAIN RESULT:

**Theorem 2.1** Let \((X, d)\) be a complete metric space and let \( T: X \to X \) be a given mapping. We denote

\[ m(x, y) = \frac{d(x, Tx) + d(y, Ty) + d(xy)}{d(x, Tx) + d(y, Ty) + d(xy)} \quad \forall \ x, y \in X \]  

(1)

We assume that for each \( x, y \in X \),

\[ \int_0^1 d(Tx, Ty) \zeta(t) \, dt \leq \alpha \int_0^1 m(x, y) \zeta(t) \, dt + \beta \int_0^1 d(xy) \zeta(t) \, dt \]  

(2)

Where \( \alpha > 0, \beta > 0, 0 < \alpha + \beta < 1 \) and \( \zeta: (0,1) \to (0,1) \) is a Lebesgue- integrable mapping which is summable on each compact subset of \((0, \infty)\), nonnegative and such that

\[ \int_0^1 \zeta(t) \, dt > 0, \forall \varepsilon > 0 \]  

(3)

Then \( T \) has unique fixed point \( z \in X \) such that for each \( x \in X \), \( T^n x \to z \) as \( n \to +\infty \).

**Proof:** Let \( x \in X \) and we define the sequence \( \{x_n\} \) in \( X \), defined as, \( x_{n+1} = T^n x \) for each integer \( n \geq 1 \), from (2), we claim that

\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \]

To prove (4), we required to show that,

\[ \int_0^1 d(x_{n+1}, x_{n+2}) \zeta(t) \, dt \leq (\alpha + \beta) \int_0^1 d(x_0, x_1) \zeta(t) \, dt \]

Where, \( r = (\alpha + \beta) \)

By Using (2),

\[ \int_0^1 d(Tx_n, Tx_{n+1}) \zeta(t) \, dt \leq \alpha \int_0^1 m(x_n, x_{n+1}) \zeta(t) \, dt + \beta \int_0^1 d(x_n, x_{n+1}) \zeta(t) \, dt \]

By Using (1)

\[ m(x_n, x_{n+1}) = \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})} \]

\[ m(x_n, x_{n+1}) = d(x_n, x_{n+1}) \]

Which implies,

\[ \int_0^1 d(x_{n+1}, x_{n+2}) \zeta(t) \, dt \leq \alpha \int_0^1 d(x_n, x_{n+1}) \zeta(t) \, dt + \beta \int_0^1 d(x_n, x_{n+1}) \zeta(t) \, dt \]

\[ \int_0^1 d(x_{n+1}, x_{n+2}) \zeta(t) \, dt \leq (\alpha + \beta) \int_0^1 d(x_0, x_1) \zeta(t) \, dt \]

In general we can write,

\[ \int_0^1 d(x_{n+1}, x_{n+2}) \zeta(t) \, dt \leq (\alpha + \beta)^n \int_0^1 d(x_0, x_1) \zeta(t) \, dt \]

Since \( (\alpha + \beta) < 1 \) and, it follows that,

\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \]  

(4)
Now we show that, \( \{x_n\} \) is a Cauchy sequence in \( X \). Suppose that it is not then there exists an \( \varepsilon > 0 \) such that for each \( p \in \mathbb{N} \) there are \( m(p) \) and \( n(p) \) in \( \mathbb{N} \), with \( m(p) > n(p) > p \), such that,

\[
d(Tx_{m(p)}, Tx_{n(p)}) \geq \varepsilon, \quad d(Tx_{m(p)}-1, Tx_{n(p)}) < \varepsilon
\]

Hence

\[
\varepsilon \leq d(Tx_{m(p)}, Tx_{n(p)}) < d(Tx_{m(p)}, Tx_{m(p)-1}) + d(Tx_{m(p)-1}, Tx_{n(p)})
\]

\[
d(Tx_{m(p)}, Tx_{n(p)}) < d(Tx_{m(p)}, Tx_{m(p)-1}) + \varepsilon
\]

Using (4) and taking \( p \to +\infty \), we get

\[
d(Tx_{m(p)}, Tx_{n(p)}) \to \varepsilon^+
\]

This implies that there exists \( k \in \mathbb{N} \) such that \( p > k \Rightarrow d(Tx_{m(p)+1}, Tx_{n(p)+1}) < \varepsilon \)

In fact if there exists a subsequence \( (p_k) \in \mathbb{N}, \ p_k > k, \ d(Tx_{m(p)+1}, Tx_{n(p)+1}) \geq \varepsilon \)

We obtain from (2)

\[
\int^\varepsilon_0 \xi(t) \ dt \leq a \int_0^{m(x_{m(p)_k}, x_{n(p)_k})} \xi(t) \ dt + \beta \int_0^{d(x_{m(p)_k}, x_{n(p)_k})} \xi(t) \ dt
\]

On other hand we have,

\[
m(x_{m(p)_k}, x_{n(p)_k}) = \frac{d(x_{m(p)_k}, x_{n(p)_k})d(x_{n(p)_k}, x_{n(p)_k})}{d(x_{m(p)_k}, x_{m(p)_k}) + d(x_{n(p)_k}, x_{n(p)_k}) + d(x_{m(p)_k}, x_{n(p)_k})}
\]

\[
m(x_{m(p)_k}, x_{n(p)_k}) \to 0 \quad \text{as} \quad k \to \infty .
\]

Tacking \( k \to +\infty \) in (6) we get

\[
\int^\varepsilon_0 \xi(t) \ dt \leq \beta \int^\varepsilon_0 \xi(t) \ dt
\]

Which is contradiction being \( \beta \in (0,1) \) and the integral being positive. Let us prove now that there is \( \sigma \in (0,1) \), \( p_\sigma \in \mathbb{N} \) such that \( p > p_\sigma \Rightarrow d(Tx_{m(p)+1}, Tx_{n(p)+1}) < \varepsilon - \sigma \)

If it is not true, then there exists a subsequence \( (p_\sigma) \in \mathbb{N} \) such that

\[
d(Tx_{m(p)+1}, Tx_{n(p)+1}) \to \varepsilon^- \quad \text{as} \quad k \to +\infty .
\]

By (2), we obtain

\[
\int^\varepsilon_0 d(Tx_{m(p)_k-1}, x_{n(p)_k-1}) \xi(t) \ dt \leq a \int_0^{m(x_{m(p)_k}, x_{n(p)_k})} \xi(t) \ dt + \beta \int_0^{d(x_{m(p)_k}, x_{n(p)_k})} \xi(t) \ dt
\]

On taking as \( k \to +\infty \), we get

\[
\int^\varepsilon_0 \xi(t) \ dt \leq \beta \int^\varepsilon_0 \xi(t) \ dt
\]

Which contradiction, since \( \beta \in (0,1) \), now we can deduce the Cauchy character of \( \{Tx_n\} \). In fact for each natural number \( p > p_\sigma \), we have

\[
\varepsilon \leq d(Tx_{m(p)}, Tx_{n(p)}) < d(Tx_{m(p)}, Tx_{m(p)+1}) + d(Tx_{m(p)+1}, Tx_{n(p)+1}) + d(Tx_{n(p)+1}, Tx_{n(p)})
\]

\[
d(Tx_{m(p)}, Tx_{n(p)}) \to \varepsilon - \sigma \quad \text{as} \quad p \to +\infty
\]

Thus, \( \varepsilon \leq \varepsilon - \sigma \)

Which is contradiction. We conclude that \( \{Tx_n\} \) is Cauchy.
By the completeness of $X$, there is $z \in X$ such that $T_{x_n} \to z$ as $n \to \infty$. We shall show that $Tz = z$. Suppose by contradiction that $d(z, Tz) > 0$. We have

$$0 \leq d(z, Tz) - d(z, T_{x_{n+1}}) \leq d(Tz, T_{x_{n+1}})$$

(7)

First, let us prove that $d(T_{x_{n+1}}, z)$ is convergent sequence and it is converges to zero, then $d(T_{x_{n+1}}, Tz)$ is bounded sequence. Assume that there exists a subsequence

$$d(T_{x_{k+1}}, Tz) \rightarrow l \in (0, +\infty) \text{ as } k \to +\infty,$$

we obtain

$$\int_{0}^{l} \xi(t) \, dt \leq \alpha \int_{0}^{l} \xi(t) \, dt + \beta \int_{0}^{l} \xi(t) \, dt$$

from (1) and as $k \to +\infty$

$$\int_{0}^{l} \xi(t) \, dt \leq \beta \int_{0}^{l} \xi(t) \, dt$$

Which is contradiction the hypothsis,

Since $0 < \alpha < 1$, then $d(T_{x_{n+1}}, Tz) \to 0$ as $n \to +\infty$

Now letting $n \to +\infty$ in (7) we get

$$0 < d(z, Tz) \leq d(z, T_{x_{n+1}}) + d(Tz, T_{x_{n+1}}) \to 0$$

We deduce that $z$ is a fixed point of $T$.

**Uniqueness:**

Suppose that there is another fixed point of $T$ say $w$, different from $z$ in $X$, then from (2) we have

$$\int_{0}^{d(T_{x_{n+1}}, Tz)} \xi(t) \, dt \leq \alpha \int_{0}^{d(w, x)} \xi(t) \, dt + \beta \int_{0}^{d(w, x)} \xi(t) \, dt$$

$$\int_{0}^{d(T_{x_{n+1}}, Tz)} \xi(t) \, dt \leq \beta \int_{0}^{d(w, x)} \xi(t) \, dt$$

which contradiction of hypotthesis. so $T$ has unique fixed point in $X$

**REMARK:**

By taking $\xi(t) = 1$ in the above theorem then we get the result of Jaggi and Das [12].

**REFERENCES:**


***************