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GENERALIZED BI-RECURRENT WEYL SPACES

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ABSTRACT

***I**n this paper, we have studied the generalized bi-recurrent Weyl spaces. The bi-recurrent properties of Weyl Conformal curvature, Weyl projective curvature and Weyl concircular curvature tensor have been studied and some relation between them has been derived.*

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1. Introduction:

An-n-dimensional differentiable manifold W_n is said to be Weyl space if it has a symmetric connection ∇^* and a symmetric conformal metric tensor g_{ij} preserved by ∇^*. Accordingly, in local coordinates there exists a covariant vector field T_k (complementary vector field) satisfying the condition [1], [2], [3],

$$\nabla^*{}_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.1}$$

The above equation can be extended to

$$\partial_k g_{ij} - g_{hj}\Gamma^h_{jk} - 2T_k g_{ij} = 0 \tag{1.2}$$

where Γ^h_{jk} are the connection coefficients of the symmetric connection ∇ and are defined as

$$\Gamma^h_{jk} = \begin{Bmatrix} h \\ jk \end{Bmatrix} - g^{hm}(g_{mj}T_k + g_{mk}T_j - g_{jk}T_m) \tag{1.3}$$

Moreover, under the renormalization condition

$$\tilde{g}_{ij} = \lambda^2 g_{ij} \tag{1.4}$$

of the metric tensor g_{ij}, the covariant vector field T_k is transformed by the law

$$\tilde{T}_k = T_k + \partial_k In\lambda \tag{1.5}$$

where λ is a scalar function defined on W_n. We denote such a Weyl space by $W_n(\Gamma^h_{jk}, g_{ij}, T_k)$ or $W_n(g, T)$.

An n-dimensional differential manifold having an anti-symmetric connection ∇ and anti-symmetric metric tensor g_{ij} preserved by ∇ is called generalized Weyl space [4]. It is denoted by $GW_n(g, T)$.

For such a space, in local co-ordinate system, the compatibility condition are

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.6}$$

Where T_k are the components of a covariant vector field, called the complementary vector field of the $GW_n(g, T)$ space. Using the concept of covariant differentiation ([5], [6]), the compatibility condition of (1.6) can be written as

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$$\partial_k g_{ik} - g_{hj} L_{ik}^h - g_{ih} L_{jk}^h - 2T_k g_{ij} = 0 \tag{1.7}$$

where L_{ik}^h are the connection coefficient of the anti-symmetric connection ∇ and are obtain from the compatibility condition as

$$L_{ij}^h = \Gamma_{ij}^h + \tfrac{1}{2}[\chi_{im}^h g_{jh} + \chi_{mj}^h g_{ih} + \chi_{ij}^h g_{hm}] g^{mi} \tag{1.8}$$

Now putting

$$\chi_{ij}^h = \tfrac{1}{2}[\chi_{im}^h g_{jh} + \chi_{mj}^h g_{ih} + \chi_{ij}^h g_{hm}] g^{mi} \tag{1.9}$$

We obtain

$$L_{ij}^h = \Gamma_{ij}^h + \chi_{ij}^h \tag{1.10}$$

Where Γ_{ij}^h and χ_{ij}^h are respectively the coefficient of a Weyl connection and the torsion tenor of $GW_n(g,T)$ space and are expressed as

$$\Gamma_{ij}^h = \tfrac{1}{2}\left[L_{ij}^h + L_{ji}^h\right] = L_{ij}^h \tag{1.11}$$

and

$$\chi_{kl}^h = \tfrac{1}{2}\left[L_{kl}^h - L_{lk}^h\right] = L_{[kl]}^h \tag{1.12}$$

square bracket stands for anti-symmetry.

The components of mixed curvature tensor and Ricci tensor of $GW_n(g,T)$ are respectively

$$L_{jki}^h = \partial_k L_{ji}^h - \partial_i L_{jk}^h + L_{lk}^h L_{ji}^l - L_{li}^h L_{jk}^l \tag{1.13}$$

$$L_{ij} = L_{ija}^a \tag{1.14}$$

on the other hand scalar curvature of $GW_n(g,T)$ is defined by

$$L = g^{ij} L_{ij} \tag{1.15}$$

It is easy to see that curvature tensor L_{jki}^h of $GW_n(g,T)$ can be written as

$$L_{jki}^h = B_{jki}^h + \chi_{jkl}^h \tag{1.16}$$

Where the tensors B_{jki}^h and χ_{jkl}^h are defined respectively

$$B_{jki}^h = \partial_k \Gamma_{ji}^h - \partial_i \Gamma_{jk}^h + \Gamma_{lk}^h \Gamma_{ji}^l - \Gamma_{lk}^h \Gamma_{jk}^l \tag{1.17}$$

$$\chi_{jki}^h = \nabla_k \chi_{ji}^h - \nabla_i \chi_{jk}^h + \chi_{li}^h \chi_{jk}^l - \chi_{lk}^h \chi_{ji}^l - 2\chi_{jl}^h \chi_{ki}^l \tag{1.18}$$

The curvature tensor of $GW_n(g,T)$ satisfies the relation [6].

$$L_{jkl}^h + L_{jlk}^h = 0 \tag{1.19}$$

$$L_{hlk}^j + L_{hkl}^j + L_{klh}^j = 2\left[\nabla_k \chi_{lh}^j + \nabla_h \chi_{kl}^j + 2\chi_{lm}^j \chi_{hk}^m + 2\chi_{hm}^j \chi_{kl}^m + 2\chi_{km}^j \chi_{lh}^m\right] \tag{1.20}$$

$$\nabla_m L_{jkl}^i + \nabla_k L_{jlm}^i + \nabla_l L_{jmk}^i = 2\left[L_{jpl}^i \chi_{mk}^p + L_{jpk}^i \chi_{lm}^p + L_{jpm}^i \chi_{kl}^p\right] \tag{1.21}$$

The Weyl Conformal curvatureC_{ijk}^h, Weyl projective curvature W_{ijk}^h and Weyl concircular curvature tensor Z_{ijk}^h in $GW_n(g,T)$ are respectively given by [7][8],

$$C_{ijk}^h = R_{ijk}^h + \frac{2}{n(n-2)}\left[\delta_k^h R_{[ij]} - \delta_j^h R_{[ik]} - g_{ik} g^{hm} R_{[mj]} + g_{ij} g^{hm} R_{[mk]} - (n-2)\delta_i^h R_{kj}\right]$$
$$-\tfrac{1}{(n-2)}\left[\delta_k^h R_{ij} - \delta_j^h R_{ik} - g_{ik} g^{hm} R_{mj} + g_{ij} g^{hm} R_{mj}\right] + \tfrac{R}{(n-1)(n-2)}\left(g_{ij}\delta_k^h - g_{ik}\delta_j^h\right) \tag{1.22}$$

$$W_{ijk}^h = R_{ijk}^h + \frac{1}{(n-1)}\left(R_{ik}\delta_j^h - R_{ij}\delta_k^h\right) \tag{1.23}$$

$$Z_{ijk}^h = R_{ijk}^h + \frac{R}{n(n-1)}\left(g_{ik}\delta_j^h - g_{ij}\delta_k^h\right) \tag{1.24}$$

From (1.22) and (1.24) we have

$$\begin{aligned} C_{ijk}^h = Z_{ijk}^h &+ \frac{2}{n(n-2)}\left[\delta_k^h Z_{[ij]} - \delta_j^h Z_{[ik]} - g_{ik}g^{hm}Z_{[mj]} + g_{ij}g^{hm}Z_{[mk]} - (n-2)\delta_i^h Z_{kj}\right] \\ &- \frac{1}{(n-2)}\left[\delta_k^h Z_{ij} - \delta_j^h Z_{ik} - g_{ik}g^{hm}Z_{mj} + g_{ij}g^{hm}Z_{mk}\right] \end{aligned} \tag{1.25}$$

where $Z_{jk} = R_{jk} - \frac{R}{n} g_{jk}$ (1.26)

2. Bi-Recurrent Weyl Spaces:

Definition: 2.1 If the curvature tensor L_{ijk}^h of GW_n satisfies the condition

$$L_{i\,jk,ab}^h = \lambda_{ab} L_{ijk}^h \tag{2.1}$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called bi-recurrent. Such a space is denoted by GW_n^*.

Definition: 2.2 If the curvature tensor C_{ijk}^h of GW_n satisfies the condition

$$C_{ijk,ab}^h = \lambda_{ab} C_{ijk}^h \tag{2.2}$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called generalized Weyl space with bi-recurrent Weyl Conformal curvature tensor. We denote such a space by $C^* - GW_n$.

Definition: 2.3 If the curvature tensor Z_{ijk}^h of GW_n satisfies the condition

$$Z_{i\,jk,ab}^h = \lambda_{ab} Z_{ijk}^h \tag{2.3}$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called generalized Weyl space with bi-recurrent Weyl Concircular curvature tensor. We denote such a space by $Z^* - GW_n$.

Definition: 2.4 If the curvature tensor W_{ijk}^h of GW_n satisfies the condition

$$W_{i\,jk,ab}^h = \lambda_{ab} W_{ijk}^h \tag{2.4}$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called generalized Weyl space with bi-recurrent Weyl Projective curvature tensor. We denote such a space by $W^* - GW_n$.

Theorem: 2.1 The necessary and sufficient condition for a Generalized Weyl space GW_n to be $W^* - GW_n$ is that it should be $Z^* - GW_n$.

Proof: Let GW_n satisfies the relation (2.3) then (2.3) in view of (1.24) reduces to

$$R_{ijk,ab}^h = \lambda_{ab} R_{ijk}^h + \frac{(R_{,ab} - \lambda_{ab} R)}{n(n-1)}\left(g_{ik}\delta_j^h - g_{ij}\delta_k^h\right) \tag{2.5}$$

contracting the indices h and k we have

$$R_{ij,ab}^h = \lambda_{ab} R_{ij}^h + \frac{(R_{,ab} - \lambda_{ab} R)}{n} g_{ij} \tag{2.6}$$

Now differentiating (1.23) covariantly with respect to a and then again differentiating the result thus obtained with respect to b we have

$$W_{ijk,ab}^h = R_{ijk,ab}^h + \frac{1}{(n-1)}\left(R_{ik,ab}\delta_j^h - R_{ij,ab}\delta_k^h\right) \tag{2.7}$$

Above equation in view of (2.5) and (2.6) reduces to

$$W^h_{ijk,ab} = \lambda_{ab}R^h_{ijk} + \frac{(R_{,ab}-\lambda_{ab}R)}{n(n-1)}\left(g_{ik}\delta^h_j - g_{ij}\delta^h_k\right) + \frac{1}{(n-1)}\left[\lambda_{ab}R_{ik}\delta^h_j - \frac{(R_{,ab}-\lambda_{ab}R)}{n}g_{ik}\right] - \delta^h_k\left[\lambda_{ab}R_{ij} - \frac{(R_{,ab}-\lambda_{ab}R)}{n}g_{ij}\right] \tag{2.8}$$

$$= \lambda_{ab}\left[R^h_{ijk} + \frac{1}{(n-1)}\left(R_{ik}\delta^h_j - R_{ij}\delta^h_k\right)\right] \tag{2.9}$$

$$= \lambda_{ab}W^h_{ijk}$$

therefore the space is $W^* - GW_n$.

Conversely let the space be $W^* - GW_n$ then (2.4) in view of (1.23) gives

$$R^h_{ijk,ab} + \frac{1}{(n-1)}\left(R_{ik,ab}\delta^h_j - R_{ij,ab}\delta^h_k\right) = \lambda_{ab}[R^h_{ijk} + \frac{1}{(n-1)}\left(R_{ik}\delta^h_j - R_{ij}\delta^h_k\right)] \tag{2.10}$$

transecting (2.10) by g^{ij} we obtain

$$R_{ij,ab} = \lambda_{ab}R_{ij} - \frac{1}{n}\left(\lambda_{ab}R - R_{,ab}\right)g_{ij} \tag{2.11}$$

Therefore we have

$$\delta^h_k R_{ij,ab} - \delta^h_j R_{ik,ab} = \lambda_{ab}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) - \frac{R}{n}\lambda_{ab}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{R_{,ab}}{n}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) \tag{2.12}$$

Substituting (2.12) in (2.10) we have

$$R^h_{ijk,ab} - \frac{R_{,ab}}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) = \lambda_{ab}\left[R^h_{ijk} - \frac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right)\right]$$

which reduces to

$$Z^h_{i\ jk,ab} = \lambda_{ab}Z^h_{ijk}$$

Theorem: 2.2 The necessary and sufficient condition for a Generalized Weyl space GW_n to be $C^* - GW_n$ is that it should be $Z^* - GW_n$.

Proof: Let GW_n satisfies the relation (2.3) then (2.3) in view of (1.24) reduces to (2.5) and (2.6).

Now differentiating (1.24), (1.25) and (1.26) covariantly with respect to a and then again differentiating the result thus obtained with respect to b we have respectively

$$Z^h_{ijk,ab} = R^h_{ijk,ab} + \frac{R_{,ab}}{n(n-1)}\left(g_{ik}\delta^h_j - g_{ij}\delta^h_k\right), \tag{2.13}$$

$$C^h_{ijk,ab} = Z^h_{ijk,ab} + \frac{2}{n(n-2)}[\delta^h_k Z_{[ij],ab} - \delta^h_j Z_{[ik],ab} - g_{ik}g^{hm}Z_{[mj],ab} + g_{ij}g^{hm}Z_{[mk],ab} - (n-2)\delta^h_i Z_{[kj],ab}] - \frac{1}{(n-2)}\left[\delta^h_k Z_{ij,ab} - \delta^h_j Z_{ik,ab} - g_{ik}g^{hm}Z_{mj,ab} + g_{ij}g^{hm}Z_{mk,ab}\right] \tag{2.14}$$

and

$$Z_{jk,ab} = R_{jk,ab} - \frac{R_{,ab}}{n}g_{jk} \tag{2.15}$$

Equation (1.14) in view of (2.13) and (2.15) reduces to

$$C^h_{ijk,ab} = R^h_{ijk,ab} + \frac{R_{,ab}}{n(n-1)}\left(g_{ik}\delta^h_j - g_{ij}\delta^h_k\right) + \frac{2}{n(n-2)}[\delta^h_k\left(R_{ij,ab} - \frac{R_{,ab}}{n}g_{ij}\right) - \delta^h_j\left(R_{ik,ab} - \frac{R_{,ab}}{n}g_{ik}\right) - g_{ik}g^{hm}\left(R_{mj,ab} - \frac{R_{,ab}}{n}g_{mj}\right) + g_{ij}g^{hm}\left(R_{mk,ab} - \frac{R_{,ab}}{n}g_{mk}\right) - (n-2)\delta^h_i\left(R_{kj,ab} - \frac{R_{,ab}}{n}g_{kj}\right) - \frac{1}{(n-2)}[\delta^h_k\left(R_{ij,ab} - \frac{R_{,ab}}{n}g_{ij}\right) - \delta^h_j\left(R_{ik,ab} - \frac{R_{,ab}}{n}g_{ik}\right) - g_{ik}g^{hm}\left(R_{mj,ab} - \frac{R_{,ab}}{n}g_{mj}\right) + g_{ij}g^{hm}\left(R_{mk,ab} - \frac{R_{,ab}}{n}g_{mk}\right). \tag{2.16}$$

Equation (2.16) in view of (2.5), (1.24) and (2.6) reduces to

$$C^h_{ijk,ab} = \lambda_{ab}\{Z^h_{ijk} + \tfrac{2}{n(n-2)}\left[\delta^h_k Z_{[ij]} - \delta^h_j Z_{[ik]} - g_{ik}g^{hm}Z_{[mj]} + g_{ij}g^{hm}Z_{[mk]} - (n-2)\delta^h_i Z_{kj}\right] - \tfrac{1}{(n-2)}\left[\delta^h_k Z_{ij} - \delta^h_j Z_{ik} - g_{ik}g^{hm}Z_{mj} + g_{ij}g^{hm}Z_{mk}\right]\}$$

which reduces to

$$C^h_{ijk,ab} = \lambda_{ab}C^h_{ijk}$$

Conversely, let the space be $C^* - GW_n$ then (2.2) in view of (1.25) reduces to

$$Z^h_{ijk,ab} + \tfrac{2}{n(n-2)}\left[\delta^h_k Z_{[ij],ab} - \delta^h_j Z_{[ik],ab} - g_{ik}g^{hm}Z_{[mj],ab} + g_{ij}g^{hm}Z_{[mk],ab} - (n-2)\delta^h_i Z_{kj,ab}\right] - \tfrac{1}{(n-2)}\left[\delta^h_k Z_{ij,ab} - \delta^h_j Z_{ik,ab} - g_{ik}g^{hm}Z_{mj,ab} + g_{ij}g^{hm}Z_{mk,ab}\right] = \lambda_{ab}\{Z^h_{ijk} + \tfrac{2}{n(n-2)}[\delta^h_k Z_{[ij]} - \delta^h_j Z_{[ik]} - g_{ik}g^{hm}Z_{[mj]} + g_{ij}g^{hm}Z_{[mk]} - (n-2)\delta^h_i Z_{kj}] - \tfrac{1}{(n-2)}\left[\delta^h_k Z_{ij} - \delta^h_j Z_{ik} - g_{ik}g^{hm}Z_{mj} + g_{ij}g^{hm}Z_{mk}\right]\}$$

which reduces to

$$\begin{aligned} Z^h_{ijk,ab} - \lambda_{ab}Z^h_{ijk} = \frac{2}{n(n-2)}[&\delta^h_k(Z_{[ij],ab} - \lambda_{ab}Z_{[ij]}) - \delta^h_j(Z_{[ik],ab} - \lambda_{ab}Z_{[ik],ab}) \\ &-g_{ik}g^{hm}(Z_{[mj],ab} - \lambda_{ab}Z_{[mj]}) + g_{ij}g^{hm}(Z_{[mk],ab} - \lambda_{ab}Z_{[mk]}) \\ &-(n-2)\delta^h_i(Z_{kj,ab} - \lambda_{ab}Z_{kj})] - \tfrac{1}{(n-2)}[\delta^h_k(Z_{ij,ab} - \lambda_{ab}Z_{ij}) - \delta^h_j(Z_{ik,ab} - \lambda_{ab}Z_{ik}) \\ &-g_{ik}g^{hm}(Z_{[mj]} - \lambda_{ab}Z_{mj}) + g_{ij}g^{hm}(Z_{mk,ab} - \lambda_{ab}Z_{mk})] \end{aligned} \tag{2.17}$$

Equation (2.17) in view of (1.26)(2.6) and (2.15) reduces to

$$Z^h_{ijk,ab} - \lambda_{ab}Z^h_{ijk} = 0$$

therefore the space is $Z^* - GW_n$.

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