Coefficient Bounds for Certain Subclasses of Meromorphic p-valent Functions

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**ABSTRACT**

In this present investigation, the authors obtain sharp bounds for \(a_1 - \mu a_2\) and \(|a_2|\) for certain subclasses of meromorphic p-valent functions. As an application of these results, coefficients bounds for classes of functions defined through Ruscheweyh derivatives are also obtained.

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**1. INTRODUCTION:**

Let \(\Sigma_p\) denotes the class of functions \(f(z)\) of the form

\[
f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^{k-p+1} \quad (p \in \mathbb{N} := \{1,2,3,...\}),
\]

which are analytic and p-valent in the punctured unit disk

\[\Delta’ = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \Delta - \{0\}.
\]

For functions \(f(z) \in \Sigma_p\) given by Eqn. (1) and \(g(z) \in \Sigma_p\) given by

\[
g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^{k-p+1} \quad (p \in \mathbb{N} := \{1,2,3,...\}).
\]

we define the Hadamard product (or convolution) of \(f\) and \(g\) by

\[
(f \ast g)(z) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^{k-p+1} = (g \ast f)(z).
\]

In terms of the Hadamard product (or convolution) of two functions, we define an analogue of the family Ruscheweyh derivative [3] by

\[
D_{\lambda}^{k-p} f(z) = \frac{1}{z^p (1-z)^{\lambda-p}} \ast f(z), \quad (\lambda > -p, p \in \mathbb{N}, f \in \Sigma_p)
\]

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or equivalently
\[
D^{k+1} f(z) = \frac{1}{z^p} \left( \frac{z^{k+2p} f(z)}{(\lambda + p - 1)!} \right)^{k+1}, \quad (\lambda > -p, p \in \mathbb{N}, f \in \Sigma_p) \tag{4}
\]

It follows readily from Eqn. (1), Eqn. (3), Eqn. (4) that
\[
D^{k+1} f(z) = z^{-p} + \sum_{k=0}^{\infty} \delta(\lambda, k) a_k z^{-k-p+1}, \quad (\lambda > -p, p \in \mathbb{N}, f \in \Sigma_p) \tag{5}
\]

where \( f \in \Sigma_p \) is given by Eqn. (1) and
\[
\partial(\lambda, k) = \binom{\lambda + p + k}{k + 1} = \frac{(\lambda + p + k)!}{(\lambda + p - 1)! k!}.
\]

The function \( f(z) \) is subordinate to the function \( g(z) \), written \( f \prec g \), provided there is an analytic function \( w(z) \) defined on \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \).

**Definition 1.1** Let \( \phi(z) \) be an analytic function with positive real part in the unit disk \( \Delta \) with \( \phi(0) = 1 \) and \( \phi'(0) > 0 \) that maps \( \Delta \) onto a region starlike with respect to \( '1' \) and symmetric with respect to real axis. A function \( f \in \Sigma_p \) is in the class \( \Sigma_{p,\alpha}^*(\phi) \) if
\[
-\frac{z^{1-p} f'(z)}{p[f(z)]^{1+p}} < \phi(z), \quad (z \in \Delta, \alpha > -1/p, p \in \mathbb{N}).
\]

In the case of \( \alpha = 0 \) let us see the basic class as follows.

**Definition 1.2** Let \( \phi(z) \) be an analytic function with positive real part in the unit disk \( \Delta \) with \( \phi(0) = 1 \) and \( \phi'(0) > 0 \) that maps \( \Delta \) onto a region starlike with respect to \( '1' \) and symmetric with respect to real axis. A function \( f \in \Sigma_p \) is in the class \( \Sigma_{p}^*(\phi) \) if
\[
-\frac{z f'(z)}{p f(z)} < \phi(z), \quad (z \in \Delta, p \in \mathbb{N}).
\]

The classes \( \Sigma_{p,\alpha}^*(\phi), \Sigma_{p}^*(\phi) \) were studied by Srutha Keerthi et al. [7].

**Lemma 1.1** If \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part, then, for any complex number \( \mu \)
\[
|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}
\]

and the result is sharp for the functions given by
\[
p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.
\]

**Lemma 1.2** If \( p_{1}(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is an analytic function with positive real part in \( \Delta \), then for \( r_1, r_2 \) real,
\[
|c_3 + r_1 c_1 c_2 + r_2 c_1^3| \leq 2H(s_1, s_2),
\]

where \( s_1 = 2(r_1 + 1), s_2 = 2r_1 + 4r_2 + 1 \) and
\[
H(s_1, s_2) = \begin{cases} 
1 & \text{for } (s_1, s_2) \in D_1 \cup D_2 \\
|s_2| & \text{for } (s_1, s_2) \in \cup_{k=3}^{8} D_k \\
\frac{1}{2} \left( |s_1| + 1 \right) \left( \frac{|s_1| + 1}{|s_2|} \right)^{\frac{1}{2}} & \text{for } (s_1, s_2) \in D_8 \cup D_9 \\
\frac{1}{2} s_2 \left( \frac{s_1^2 - 4}{(s_2^2 - 4 s_1)} \right)^{\frac{1}{2}} & \text{for } (s_1, s_2) \in D_{10} \cup D_{11} - \{\pm 2, 1\} \\
\frac{1}{2} \left( |s_1| - 1 \right) \left( \frac{|s_1| - 1}{|s_2|} \right)^{\frac{1}{2}} & \text{for } (s_1, s_2) \in D_{12}.
\end{cases}
\]

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The extremal functions up to the rotations are of the form

\[ p_1(z) = \frac{1 + z^3}{1 - z}, \quad p_1(z) = \frac{1 + z}{1 - z}, \]

\[ p_1(z) = p_{1,0}(z) = \frac{1 + (1 - 2\lambda)(\varepsilon_1 - \varepsilon_2)z - \varepsilon_1\varepsilon_2 z^2}{1 - (\varepsilon_1 + \varepsilon_2)z + \varepsilon_1\varepsilon_2 z^2}, \]

\[ p_1(z) = p_{1,1}(z) = \frac{1 - z^2}{1 - 2t_1z + z^2}, \]

\[ p_1(z) = p_{1,1}(z) = \frac{1 + 2t_1z + z^2}{1 - z^2}, \]

\[ \varepsilon_1 = t_0 - e^{-\frac{\theta_0}{2}}(a \pm b), \]

\[ \varepsilon_2 = -e^{-\frac{\theta_0}{2}}(ia \pm b), \]

\[ a = t_0 \cos \left( \frac{\theta_0}{2} \right), \quad b = \sqrt{1 - t_0^2 \sin^2 \left( \frac{\theta_0}{2} \right)}, \quad \lambda = \frac{b \pm a}{2b}, \]

\[ t_0 = \frac{2s_2(s_1^2 + 2) - 3s_1^2}{3(s_2 - 1)(s_1^2 - 4s_2)}, \]

\[ t_1 = \left( \frac{|s_1| + 1}{3|s_1| + 1 + s_2} \right)^{\frac{1}{\lambda}}, \]

\[ t_2 = \left( \frac{|s_1| - 1}{3|s_1| - 1 - s_2} \right)^{\frac{1}{\lambda}}, \]

\[ \cos \left( \frac{\theta_0}{2} \right) = \frac{s_1}{2} \left[ \frac{s_2(s_1^2 + 8) - 2(s_1^2 + 2)}{2s_2(s_1^2 + 2) - 3s_1^2} \right]. \]

The sets \( D_k, k = 1, 2\ldots 12 \) are defined as follows.

\[ D_1 = \left\{ (s_1, s_2) : |s_1| \leq \frac{1}{2}, |s_2| \leq 1 \right\}, \]

\[ D_2 = \left\{ (s_1, s_2) : \frac{1}{2} \leq |s_1| \leq 2, \frac{4}{27}(|s_1| + 1)^3 - (|s_1| + 1) \leq |s_2| \leq 1 \right\}, \]

\[ D_3 = \left\{ (s_1, s_2) : |s_1| \leq \frac{1}{2}, |s_2| \leq -1 \right\}, \]

\[ D_4 = \left\{ (s_1, s_2) : |s_1| \geq \frac{1}{2}, |s_2| \leq -\frac{2}{3}(|s_1| + 1) \right\}, \]

\[ D_5 = \left\{ (s_1, s_2) : |s_1| \leq 2, |s_2| \geq 1 \right\}, \]

\[ D_6 = \left\{ (s_1, s_2) : 2 \leq |s_1| \leq 4, \frac{1}{12}(s_1^2 + 8) \right\}, \]

\[ D_7 = \left\{ (s_1, s_2) : |s_1| \geq 4, |s_2| \geq \frac{2}{3}(|s_1| - 1) \right\}, \]

\[ D_8 = \left\{ (s_1, s_2) : \frac{1}{2} \leq |s_1| \leq 2, -\frac{2}{3}(|s_1| + 1) \leq |s_2| \leq \frac{4}{27}(|s_1| + 1)^3 - (|s_1| + 1) \right\}. \]
Lemma 1.2 is a reformulation of a Lemma given by Prokhorov and Szynal [6]. For earlier works refer also [1, 2, 5].

2. COEFFICIENT BOUNDS:

Theorem: 2.1 Let \( \phi(z) = 1 + B_1z + B_2z^2 + \cdots \). If \( f(z) \) given by (1) belongs to \( \Sigma_{p,a}^+(\phi) \), then for any complex number \( \mu \),
\[
|a_1 - \mu a_0| \leq \frac{B_1p}{(2 + \alpha)p} \max \left\{ 1, \left| \frac{B_2}{B_1} - \left( \frac{2 + \alpha}{2(1 + \alpha p)} \right) (1 + \alpha - 2\mu) \right| \right\}. \tag{6}
\]

This result is sharp. Further,
\[
|a_2| \leq \frac{B_1p}{2(3 + \alpha)p} H(s_1, s_2), \tag{7}
\]

where \( s_1, s_2 \) and \( H(s_1, s_2) \) is as defined in lemma 1.2, also
\[
r_1 = \frac{B_2}{B_1} - \frac{(1 + \alpha)(3 + \alpha)pB_1p}{2(2 + \alpha p)(1 + \alpha p)} - 1, \tag{8}
\]
\[
r_2 = \frac{B_2}{4B_1} - \frac{B_3}{2B_1} + \frac{(1 + \alpha)(3 + \alpha)p}{8(2 + \alpha p)(1 + \alpha p)} \left( 2B_1 - 2B_2 \right) p + \frac{(1 + \alpha)(2 + \alpha)(1 - \alpha)pB_1^2p^2}{8(1 + \alpha p)^3} + \frac{1}{4}. \tag{9}
\]

Proof: If \( f(z) \in \Sigma_{p,a}^+(\phi) \) let
\[
p(z) = -\frac{z^\alpha f'(z)}{p[f(z)]^{1+\alpha}} = 1 + b_1z + b_2z^2 + \cdots. \tag{10}
\]
from (10)
\[
a_0 = -\frac{b_1p}{1 + \alpha p} \tag{11}
\]
\[
a_1 = -\frac{b_2p}{(2 + \alpha p)} + \frac{b_3^2(1 + \alpha)p^2}{2(1 + \alpha p)^3} \tag{12}
\]
\[
a_2 = b_3 - 3b_2b_1 \left[ \frac{(1 + \alpha)p}{(1 + \alpha p)(2 + \alpha p)} \right] + b_1^2 \left[ \frac{(1 + \alpha)(1 + 2\alpha)p^2}{2(1 + \alpha p)^2} - \frac{2\alpha(1 + \alpha)(1 + 2\alpha)p^3}{6(1 + \alpha p)^3} \right] \tag{13}
\]
Since \( \phi(z) \) is univalent and \( p(z) \prec \phi(z) \), then the function
\[
p(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1z + c_2z^2 + \cdots, \tag{14}
\]

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is analytic and has positive real part in $\Delta$. Also we have

$$p(z) = \frac{p_1(z) - 1}{p_1(z) + 1}$$

(15)

and from (15), we obtain

$$b_1 = \frac{B_1c_1}{2}$$

(16)

$$b_2 = \frac{1}{2} \left[ B_1 \left( c_2 - \frac{c_1^2}{2} \right) \right] + \frac{1}{4} B_2c_1^2,$$

(17)

$$b_3 = \frac{B_1c_3}{2} + (B_2 - B_1) \frac{c_3c_1}{2} + (B_1 + B_3 - 2B_2) \frac{c_1^3}{8}$$

(18)

Substituting (16), (17), (18) in (11), (12), (13) we get

$$a_0 = -\frac{(B_1c_1)p}{2(1 + \alpha p)},$$

(19)

$$a_1 = \frac{B_1c_1^2 (1 + \alpha)p^2}{8(1 + \alpha p)^2} + \frac{[B_1c_1^2 - B_2c_1^2 - 2B_1c_3]p}{4(2 + \alpha p)}$$

(20)

$$a_2 = \frac{-B_1p}{2(3 + \alpha p)} \left[ c_3 + \frac{B_2}{B_1} - \frac{(1 + \alpha)(3 + \alpha p)B_1p}{2(2 + \alpha p)(1 + \alpha p)} - 1 \right] c_1c_2$$

$$+ c_3 \left[ \frac{(1 + \alpha)(1 + 2\alpha)(1 - \alpha p)B_1^2p^2}{8(1 + \alpha p)^3} + \frac{2(1 + \alpha)(3 + \alpha p)(B_1 - B_2)p}{8(1 + \alpha p)(2 + \alpha p)} + \frac{B_3}{4B_1} - \frac{B_2}{2B_1} + \frac{1}{4} \right]$$

(21)

Therefore we have

$$a_1 - \mu a_0^2 = \frac{-B_1p}{2(2 + \alpha p)} (c_2 - v c_1^2)$$

(22)

where

$$v = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{B_1p(2 + \alpha p)(1 + \alpha - 2\mu)}{2(1 + \alpha p)^2} \right]$$

Now we obtain inequality (6) by applying Lemma 1.1 in (22) and inequality (7) by applying Lemma 1.2 in (21). The result (6) is sharp for the function defined by

$$\frac{-z^{1-\alpha}f'(z)}{p[f(z)]^{1-\alpha}} = \phi(z^2)$$

and

$$\frac{-z^{1-\alpha}f'(z)}{p[f(z)]^{1-\alpha}} = \phi(z).$$

**Corollary 2.1** Let $\phi(z) = 1 + B_1z + B_2z^2 + \ldots$. If $f(z)$ given by (1) belongs to $\Sigma_p^{\phi}$, then for any complex number $\mu$,

$$\left| a_1 - \mu a_0^2 \right| \leq \frac{B_1p}{2} \max \left\{ 1, \left| \frac{B_2}{B_1} - (1 - 2\mu)B_1p \right| \right\}.$$  

(23)

Further,

$$\left| a_2 \right| \leq \frac{B_1p}{2} H(s_1, s_2)$$

(26)

where $s_1, s_2$ and $H(s_1, s_2)$ is as defined in lemma 1.2, also
The result (23) is sharp for the function defined by

\[
-z^f(z) = \phi(z^2)
\]

and

\[
-z^f(z) = \phi(z).
\]

These are obtained by taking \(\alpha = 0\) in equations (6) and (7).

3. APPLICATIONS TO FUNCTIONS DEFINED THROUGH RUSCHEWEYH DERIVATIVES:

The following classes \(\Sigma_{p,a}^\lambda (\phi)\) and \(\Sigma_p^\lambda (\phi)\) were defined by Srutha Keerthi et al. [7].

Definition: 3.1 Let \(\phi(z)\) be an analytic function with positive real part in the unit disk \(\Delta\) with \(\phi(0) = 1\) and \(\phi'(0) > 0\) that maps \(\Delta\) onto a region starlike with respect to 1 and symmetric with respect to the real axis. A function \(f(z) \in \Sigma_p\) is in the class \(\Sigma_{p,a}^\lambda (\phi)\) if

\[
\frac{-z^f(z)}{p[D^f(z)]^{1/a}} < \phi(z), \quad (z \in \Delta, \alpha > -1, \lambda > -p, p \in \mathbb{N}).
\]

By taking \(\alpha = 0\), we set the class \(\Sigma_p^\lambda (\phi)\).

Definition: 3.2 Let \(\phi(z)\) be an analytic function with positive real part in the unit disk \(\Delta\) with \(\phi(0) = 1\) and \(\phi'(0) > 0\) that maps \(\Delta\) onto a region starlike with respect to 1 and symmetric with respect to the real axis. A function \(f(z) \in \Sigma_p\) is in the class \(\Sigma_p^\lambda (\phi)\) if

\[
\frac{-z[D^f(z)]}{p[D^f(z)]} < \phi(z), \quad (z \in \Delta, \lambda > -p, p \in \mathbb{N}).
\]

Theorem: 3.1 Let the function \(\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots\). If \(f(z)\) given by (1) belongs to \(\Sigma_{p,a}^\lambda (\phi)\) then

\[
|a_1 - \mu a_0^2| = \frac{2B_1 p}{(\lambda + p + 1)(\lambda + p)(2 + \alpha p)} \max \left\{ 1, \frac{B_2}{B_1} - \left[ \frac{(2 + \alpha p)}{2(1 + \alpha p)^2} \left( 1 + \alpha - \frac{(\lambda + p + 1)\mu}{(\lambda + p)} \right) \right] B_1 p \right\}.
\]

This result is sharp.

Further

\[
|a_2| \leq \frac{3B_1 p}{(\lambda + p + 2)(\lambda + p + 1)(\lambda + p)(3 + \alpha p)} H(s_1, s_2),
\]

where \(s_1, s_2\) and \(H(s_1, s_2)\) is as defined in lemma 1.2, also \(r_1, r_2\) are given by equations (8), (9). The proof is similar to theorem 2.1, so the details are omitted.

Corollary: 3.1 Let the function \(\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots\). If \(f(z)\) given by (1) belongs to \(\Sigma_p^\lambda (\phi)\) then

\[
|a_1 - \mu a_0^2| = \frac{B_1 p}{(\lambda + p + 1)(\lambda + p)} \max \left\{ 1, \frac{B_2}{B_1} - \left[ \left( 1 - \frac{(\lambda + p + 1)\mu}{(\lambda + p)} \right) \right] B_1 p \right\}.
\]

This result is sharp.
Further

\[ |a_{n}| \leq \frac{B_p}{(\lambda + p + 2)(\lambda + p + 1)(\lambda + p)} H(s_1, s_2), \]

where \( s_1, s_2 \) and \( H(s_1, s_2) \) is as defined in lemma 1.2, also \( r_1, r_2 \) are given by equations (25), (26).

Corollary 3.1 is obtained by taking \( \alpha = 0 \) in theorem 3.1.

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