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Coefficient Bounds for Certain Subclasses of Meromorphic p-valent Functions

B. A. Frasin[1], *B. Srutha Keerthi[2], B. Adolf Stephen[3], A. Gangadharan[4]

[1]Department of Mathematics, Al al-Bayt University, P.O.Box: 130095, Mafraq, Jordan
E-mail: bafrasin@yahoo.com

[2]Department of Applied Mathematics, Sri Venkateswara College of Engineering, Sriperumbudur, Chennai – 602105, India
E-mail : sruthilaya06@yahoo.co.in, laya@svce.ac.in

[3]Department of Mathematics, Madras Christian College, Chennai – 600059, India
E-mail : adolfmcc2003@yahoo.co.in

[4]Department of Applied Mathematics, Sri Venkateswara College of Engineering, Sriperumbudur, Chennai - 602105, India
E-mail : ganga@svce.ac.in

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ABSTRACT

In this present investigation, the authors obtain sharp bounds for $\left|a_1 - \mu a_0^2\right|$ *and* $|a_2|$ *for certain subclasses of meromorphic p-valent functions. As an application of these results, coefficients bounds for classes of functions defined through Ruscheweyh derivatives are also obtained.*

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1. INTRODUCTION:

Let Σ_p denotes the class of functions f(z) of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^{k-p+1} \qquad (p \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disk

$$\Delta^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \Delta - \{0\}.$$

For functions $f(z) \in \Sigma_p$ given by Eqn. (1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^{k-p+1} \qquad (p \in \mathbb{N} := \{1,2,3,\dots\}).$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^{k-p+1} = (g * f)(z). \tag{2}$$

In terms of the Hadamard product (or convolution) of two functions, we define an analogue of the family Ruscheweyh derivative [3] by

$$D^{\lambda+p-1} f(z) = \frac{1}{z^p (1-z)^{\lambda+p}} * f(z), \qquad (\lambda > -p, p \in \mathbb{N}, f \in \Sigma_p) \tag{3}$$

***Corresponding author: *B. Srutha Keerthi[2], *E-mail: sruthilaya06@yahoo.co.in**

or equivalently

$$D^{\lambda+p-1}f(z)=\frac{1}{z^p}\left(\frac{z^{\lambda+2p-1}f(z)}{(\lambda+p-1)!}\right)^{\lambda+p-1}, \qquad (\lambda>-p, p\in\mathbb{N}, f\in\Sigma_p) \tag{4}$$

It follows readily from Eqn. (1), Eqn. (3), Eqn. (4) that

$$D^{\lambda+p-1}f(z)=z^{-p}+\sum_{k=0}^{\infty}\delta(\lambda,k)a_k z^{k-p+1}, \qquad (\lambda>-p, p\in\mathbb{N}, f\in\Sigma_p) \tag{5}$$

where $f\in\Sigma_p$ is given by Eqn. (1) and

$$\partial(\lambda,k)=\binom{\lambda+p+k}{k+1}=\binom{\lambda+p+k}{\lambda+p-1}.$$

The function f(z) is subordinate to the function g(z), written $f\prec g$, provided there is an analytic function w(z) defined on Δ with w(0) = 0 and |w(z)| < 1 such that f(z) = g(w(z)).

Definition: 1.1 Let $\phi(z)$ be an analytic function with positive real part in the unit disk Δ with $\phi(0)=1$ and $\phi'(0)>0$ that maps Δ onto a region starlike with respect to '1' and symmetric with respect to real axis. A functions $f\in\Sigma_p$ is in the class $\Sigma^*_{p,\alpha}(\phi)$ if

$$-\frac{z^{1-\alpha p}f'(z)}{p[f(z)]^{1+\alpha}}\prec\phi(z), \quad (z\in\Delta, \alpha>-\frac{1}{p}, p\in\mathbb{N}).$$

In the case of $\alpha=0$ let us see the basic class as follows.

Definition: 1.2 Let $\phi(z)$ be an analytic function with positive real part in the unit disk Δ with $\phi(0)=1$ and $\phi'(0)>0$ that maps Δ onto a region starlike with respect to '1' and symmetric with respect to real axis. A function $f\in\Sigma_p$ is in the class $\Sigma^*_p(\phi)$ if

$$-\frac{zf'(z)}{pf(z)}\prec\phi(z), \quad (z\in\Delta, p\in\mathbb{N}).$$

The classes $\Sigma^*_{p,\alpha}(\phi)$, $\Sigma^*_p(\phi)$ were studied by Srutha Keerthi et al. [7].

Lemma: 1.1 If $p(z)=1+c_1z+c_2z^2+\cdots$ is a function with positive real part, then, for any complex number μ

$$\left|c_2-\mu c_1^2\right|\le 2\max\{1,|2\mu-1|\}$$

and the result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2}, \qquad p(z)=\frac{1+z}{1-z}.$$

Lemma: 1.2 If $p_1(z)=1+c_1z+c_2z^2+\cdots$ is an analytic function with positive real part in Δ, then for r_1, r_2 real,

$$\left|c_3+r_1c_1c_2+r_2c_1^3\right|\le 2H(s_1,s_2),$$

where $s_1=2(r_1+1)$, $s_2=2r_1+4r_2+1$ and

$$H(s_1,s_2)=\begin{cases}1 & \text{for } (s_1,s_2)\in D_1\cup D_2\\ |s_2| & \text{for } (s_1,s_2)\in\cup_{k=3}^{7}D_k\\ \frac{2}{3}(|s_1|+1)\left(\frac{|s_1|+1}{3|s_1|+s_2}\right)^{\frac{1}{2}} & \text{for } (s_1,s_2)\in D_8\cup D_9\\ \frac{1}{3}s_2\left(\frac{s_1^2-4}{s_1^2-4s_2}\right)\left(\frac{s_1^2-4}{3(s_2-1)}\right)^{\frac{1}{2}} & \text{for } (s_1,s_2)\in D_{10}\cup D_{11}-\{\pm 2,1\}\\ \frac{2}{3}(|s_1|-1)\left(\frac{|s_1|-1}{3(|s_1|-1-s_2)}\right)^{\frac{1}{2}} & \text{for } (s_1,s_2)\in D_{12}\end{cases}$$

the extremal functions up to the rotations are of the form

$$p_1(z) = \frac{1+z^3}{1-z^3}, \qquad p_1(z) = \frac{1+z}{1-z},$$

$$p_1(z) = p_{1,0}(z) = \frac{1+(1-2\lambda)(\varepsilon_1-\varepsilon_2)z-\varepsilon_1\varepsilon_2 z^2}{1-(\varepsilon_1+\varepsilon_2)z+\varepsilon_1\varepsilon_2 z^2},$$

$$p_1(z) = p_{1,1}(z) = \frac{1-z^2}{1-2t_1 z+z^2},$$

$$p_1(z) = p_{1,1}(z) = \frac{1+2t_1 z+z^2}{1-z^2},$$

$$\varepsilon_1 = t_0 - e^{-\frac{i\theta_0}{2}}(a \pm b),$$

$$\varepsilon_2 = -e^{-\frac{i\theta_0}{2}}(ia \pm b),$$

$$a = t_0\cos\left(\frac{\theta_0}{2}\right), \qquad b = \sqrt{1-t_0^2\sin^2\left(\frac{\theta_0}{2}\right)}, \qquad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left(\frac{2s_2(s_1^2+2)-3s_1^2}{3(s_2-1)(s_1^2-4s_2)}\right)^{\frac{1}{2}},$$

$$t_1 = \left(\frac{|s_1|+1}{3(|s_1|+1+s_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|s_1|-1}{3(|s_1|-1-s_2)}\right)^{\frac{1}{2}},$$

$$\cos\left(\frac{\theta_0}{2}\right) = \frac{s_1}{2}\left[\frac{s_2(s_1^2+8)-2(s_1^2+2)}{2s_2(s_1^2+2)-3s_1^2}\right].$$

The sets D_k, k = 1, 2… 12 are defined as follows.

$$D_1 = \left\{(s_1,s_2) : |s_1| \le \frac{1}{2}, |s_2| \le 1\right\},$$

$$D_2 = \left\{(s_1,s_2) : \frac{1}{2}|s_{1\le}| \le 2, \frac{4}{27}(|s_1|+1)^3-(|s_1|+1) \le |s_2| \le 1\right\},$$

$$D_3 = \left\{(s_1,s_2) : |s_1| \le \frac{1}{2}, |s_2| \le -1\right\},$$

$$D_4 = \left\{(s_1,s_2) : |s_1| \ge \frac{1}{2}, |s_2| \le -\frac{2}{3}(|s_1|+1)\right\},$$

$$D_5 = \{(s_1,s_2) : |s_1| \le 2, |s_2| \ge 1\},$$

$$D_6 = \left\{(s_1,s_2) : 2 \le |s_1| \le 4, |s_2| \ge \frac{1}{12}(s_1^2+8)\right\},$$

$$D_7 = \left\{(s_1,s_2) : |s_1| \ge 4, |s_2| \ge \frac{2}{3}(|s_1|-1)\right\},$$

$$D_8 = \left\{(s_1,s_2) : \frac{1}{2} \le |s_1| \le 2, -\frac{2}{3}(|s_1|+1) \le |s_2| \le \frac{4}{27}(|s_1|+1)^3-(|s_1|+1)\right\},$$

$$D_9 = \left\{ (s_1, s_2) : |s_1| \geq 2, -\frac{2}{3}(|s_1|+1) \leq |s_2| \leq \frac{2|s_1|(|s_1|+1)}{s_1^2 + 2|s_1| + 4} \right\},$$

$$D_{10} = \left\{ (s_1, s_2) : 2 \leq |s_1| \leq 4, \frac{2|s_1|(|s_1|+1)}{s_1^2 + 2|s_1| + 4} \leq |s_2| \leq \frac{1}{12}(s_1^2 + 8) \right\},$$

$$D_{11} = \left\{ (s_1, s_2) : |s_1| \geq 4, \frac{2|s_1|(|s_1|+1)}{s_1^2 + 2|s_1| + 4} \leq |s_2| \leq \frac{2|s_1|(|s_1|-1)}{s_1^2 - 2|s_1| + 4} \right\},$$

$$D_{12} = \left\{ (s_1, s_2) : |s_1| \geq 4, \frac{2|s_1|(|s_1|-1)}{s_1^2 - 2|s_1| + 4} \leq |s_2| \leq \frac{2}{3}(|s_1|-1) \right\}.$$

Lemma 1.2 is a reformulation of a Lemma given by Prokhorov and Szynal [6]. For earlier works refer also [1, 2, 5].

2. COEFFICIENT BOUNDS:

Theorem: 2.1 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If f(z) given by (1) belongs to $\Sigma^*_{p,\alpha}(\phi)$, then for any complex number μ,

$$|a_1 - \mu a_0^2| \leq \frac{B_1 p}{(2+\alpha p)} \max\left\{ 1, \left| \frac{B_2}{B_1} - \left(\frac{(2+\alpha p)}{2(1+\alpha p)^2}(1+\alpha-2\mu) \right) B_1 p \right| \right\}. \tag{6}$$

This result is sharp. Further,

$$|a_2| \leq \frac{B_1 p}{2(3+\alpha p)} H(s_1, s_2), \tag{7}$$

where s_1, s_2 and $H(s_1, s_2)$ is as defined in lemma 1.2, also

$$r_1 = \frac{B_2}{B_1} - \frac{(1+\alpha)(3+\alpha p)B_1 p}{2(2+\alpha p)(1+\alpha p)} - 1, \tag{8}$$

$$r_2 = \frac{B_3}{4B_1} - \frac{B_2}{2B_1} + \frac{(1+\alpha)(3+\alpha p)}{8(2+\alpha p)(1+\alpha p)}(2B_1 - 2B_2)p + \frac{(1+\alpha)(2+\alpha)(1-\alpha p)B_1^2 p^2}{8(1+\alpha p)^3} + \frac{1}{4}. \tag{9}$$

Proof: If $f(z) \in \Sigma^*_{p,\alpha}(\phi)$ let

$$p(z) = \frac{-z^{1-\alpha p} f'(z)}{p[f(z)]^{1+\alpha}} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{10}$$

from (10)

$$a_0 = \frac{-b_1 p}{1+\alpha p} \tag{11}$$

$$a_1 = -\frac{b_2 p}{(2+\alpha p)} + \frac{b_1^2 (1+\alpha) p^2}{2(1+\alpha p)^2} \tag{12}$$

$$a_2 = b_3 - 3b_2 b_1 \left[\frac{(1+\alpha)p}{(1+\alpha p)(2+\alpha p)} \right] + b_1^3 \left[\frac{(1+\alpha)(1+2\alpha)p^2}{2(1+\alpha p)^2} - \frac{2\alpha(1+\alpha)(1+2\alpha)p^3}{6(1+\alpha p)^3} \right] \tag{13}$$

Since $\phi(z)$ is univalent and $p(z) \prec \phi(z)$, then the function

$$p(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots, \tag{14}$$

is analytic and has positive real part in Δ. Also we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{15}$$

and from (15), we obtain

$$b_1 = \frac{B_1 c_1}{2} \tag{16}$$

$$b_2 = \frac{1}{2}\left[B_1\left(c_2 - \frac{c_1^2}{2}\right)\right] + \frac{1}{4}B_2 c_1^2, \tag{17}$$

$$b_3 = \frac{B_1 c_3}{2} + (B_2 - B_1)\frac{c_2 c_1}{2} + (B_1 + B_3 - 2B_2)\frac{c_1^3}{8} \tag{18}$$

Substituting (16), (17), (18) in (11), (12), (13) we get

$$a_0 = \frac{-(B_1 c_1)p}{2(1+\alpha p)}, \tag{19}$$

$$a_1 = \frac{B_1^2 c_1^2 (1+\alpha)p^2}{8(1+\alpha p)^2} + \frac{[B_1 c_1^2 - B_2 c_1^2 - 2B_1 c_2]p}{4(2+\alpha p)} \tag{20}$$

$$\begin{aligned} a_2 = \frac{-B_1 p}{2(3+\alpha p)}\Bigg\{ & c_3 + \left[\frac{B_2}{B_1} - \frac{(1+\alpha)(3+\alpha p)B_1 p}{2(2+\alpha p)(1+\alpha p)} - 1\right]c_1 c_2 \\ & + c_1^3\left[\frac{(1+\alpha)(1+2\alpha)(1-\alpha p)B_1^2 p^2}{8(1+\alpha p)^3} + \frac{2(1+\alpha)(3+\alpha p)(B_1 - B_2)p}{8(1+\alpha p)(2+\alpha p)} + \frac{B_3}{4B_1} - \frac{B_2}{2B_1} + \frac{1}{4}\right]\Bigg\} \end{aligned} \tag{21}$$

Therefore we have

$$a_1 - \mu a_0^2 = \frac{-B_1 p}{2(2+\alpha p)}(c_2 - v c_1^2) \tag{22}$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{B_1 p(2+\alpha p)(1+\alpha-2\mu)}{2(1+\alpha p)^2}\right]$$

Now we obtain inequality (6) by applying Lemma 1.1 in (22) and inequality (7) by applying Lemma 1.2 in (21). The result (6) is sharp for the function defined by

$$\frac{-z^{1-\alpha p} f'(z)}{p[f(z)]^{1+\alpha}} = \phi(z^2)$$

and

$$\frac{-z^{1-\alpha p} f'(z)}{p[f(z)]^{1+\alpha}} = \phi(z).$$

Corollary: 2.1 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If f(z) given by (1) belongs to $\Sigma_p^*(\phi)$, then for any complex number μ,

$$\left|a_1 - \mu a_0^2\right| \le \frac{B_1 p}{2}\max\left\{1, \left|\frac{B_2}{B_1} - (1-2\mu)B_1 p\right|\right\}. \tag{23}$$

Further,

$$\left|a_2\right| \le \frac{B_1 p}{2} H(s_1, s_2) \tag{26}$$

where s_1, s_2 and $H(s_1, s_2)$ is as defined in lemma 1.2, also

$$r_1 = \frac{B_2}{B_1} - \frac{3B_1 p}{4} - 1, \tag{25}$$

$$r_2 = \frac{B_3}{4B_1} - \frac{B_2}{2B_1} + \frac{3(B_1 - B_2)p}{8} + \frac{B_1^2 p^2}{8} + \frac{1}{4}. \tag{26}$$

The result (23) is sharp for the function defined by

$$-\frac{zf'(z)}{pf(z)} = \phi(z^2)$$

and

$$-\frac{zf'(z)}{pf(z)} = \phi(z).$$

These are obtained by taking $\alpha = 0$ in equations (6) and (7).

3. APPLICATIONS TO FUNCTIONS DEFINED THROUGH RUSCHEWEYH DERIVATIVES:

The following classes $\Sigma_{p,\alpha}^{\lambda}(\phi)$ and $\Sigma_{p}^{\lambda}(\phi)$ were defined by Srutha Keerthi et al. [7].

Definition: 3.1 Let $\phi(z)$ be an analytic function with positive real part in the unit disk Δ with $\phi(0) = 1$, and $\phi'(0) > 0$ that maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. A function $f(z) \in \Sigma_p$ is in the class $\Sigma_{p,\alpha}^{\lambda}(\phi)$ if

$$\frac{-z^{1-\alpha p}[D^{\lambda}f(z)]'}{p[D^{\lambda}f(z)]^{1+\alpha}} \prec \phi(z), \quad \left(z \in \Delta, \alpha > \frac{-1}{p}, \lambda > -p, p \in \mathbb{N}\right).$$

By taking $\alpha = 0$, we set the class $\Sigma_{p}^{\lambda}(\phi)$.

Definition: 3.2 Let $\phi(z)$ be an analytic function with positive real part in the unit disk Δ with $\phi(0) = 1$ and $\phi'(0) > 0$ that maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. A function $f(z) \in \Sigma_p$ is in the class $\Sigma_{p}^{\lambda}(\phi)$ if

$$\frac{-z[D^{\lambda}f(z)]'}{p[D^{\lambda}f(z)]} \prec \phi(z), \quad (z \in \Delta, \lambda > -p, p \in \mathbb{N}).$$

Theorem: 3.1 Let the function $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If f(z) given by (1) belongs to $\Sigma_{p,\alpha}^{\lambda}(\phi)$ then

$$\left|a_1 - \mu a_0^2\right| = \frac{2B_1 p}{(\lambda+p+1)(\lambda+p)(2+\alpha p)} \max\left\{1, \left|\frac{B_2}{B_1} - \left[\frac{(2+\alpha p)}{2(1+\alpha p)^2}\left(1+\alpha - \frac{(\lambda+p+1)\mu}{(\lambda+p)}\right)\right]B_1 p\right|\right\}. \tag{27}$$

This result is sharp.
Further

$$\left|a_2\right| \le \frac{3B_1 p}{(\lambda+p+2)(\lambda+p+1)(\lambda+p)(3+\alpha p)} H(s_1, s_2),$$

where s_1, s_2 and $H(s_1, s_2)$ is as defined in lemma 1.2, also r_1, r_2 are given by equations (8), (9). The proof is similar to theorem 2.1, so the details are omitted.

Corollary: 3.1 Let the function $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If f(z) given by (1) belongs to $\Sigma_{p}^{\lambda}(\phi)$ then

$$\left|a_1 - \mu a_0^2\right| = \frac{B_1 p}{(\lambda+p+1)(\lambda+p)} \max\left\{1, \left|\frac{B_2}{B_1} - \left[\left(1 - \frac{(\lambda+p+1)\mu}{(\lambda+p)}\right)\right]B_1 p\right|\right\}.$$

This result is sharp.

Further

$$|a_2| \leq \frac{B_1 p}{(\lambda+p+2)(\lambda+p+1)(\lambda+p)} H(s_1, s_2),$$

where s_1, s_2 and $H(s_1, s_2)$ is as defined in lemma 1.2, also r_1, r_2 are given by equations (25), (26).

Corollary 3.1 is obtained by taking $\alpha = 0$ in theorem 3.1.

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