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STABILITY ANALYSIS OF MUTUALISTIC INTERACTIONS AMONG THREE SPECIES WITH LIMITED RESOURCES FOR FIRST AND SECOND SPECIES AND UNLIMITED RESOURCES FOR THIRD SPECIES

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ABSTRACT

In this paper, a mathematical model consisting of mutualistic interactions among three species is proposed and analyzed. The local stability analysis of the system is carried out in each of the following three cases: (1) The death rate of any one (say third) species is greater than its birth rate. (2) The death rate of any two (say second and third) species are greater than their birth rate. (3) The death rate of all the species are greater than their birth rate.

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1. INTRODUCTION:

Mutualism is the interaction of two or more species of organisms that benefits each other (Odum [9]). In an ecosystem, mutualisms are found in many diverse communities; such interactions are well documented in the field. Examples include the algal-fungal associations of lichens (Ahmadjian [1]; Hale [5]). the legume-nitrogen-fixing bacteria interactions (Burns and Hardy [2]); plant pollinator interactions (Janzen [6]).

An example of a three way mutualism in Panama rainforests. This involves Virola trees, toucans, and agoutis. Agoutis are small rainforest mammals that eat fruits and seeds. A Virola tree produces fruits high in the top of the tree and they are eaten by toucans. Agoutis cannot climb Virola tree and eat Virola seeds from fruits eaten by toucans that drop the seeds. In times of plenty, when there are too many seeds to eat, agoutis bury some seeds that they plan on digging up when food is in short supply in the dry season. You can imagine that an agouti might forget the location of some of its buried Virola seeds, which are now, in effect, planted under the soil and can germinate and grow when conditions are favorable. So, the toucan helps both the tree and the agouti. The tree supplies the food (seeds) for both animal species, and the agouti helps the tree by planting its seeds so new trees will grow (Robbins [11]}, Simpson K, and Day N. [12]).

A general concept of mathematical modeling can be found in Meyer [8]. A detailed study on ecological species is given in the treatises of the authors such as Cushing [3], Freedman [4] and Paul colinvaux [10]. Recently the authors in [7, 13] investigated problems related to (i) competitor-competitor-mutualist Lotka-Volterra model, (ii) mutualism-competition model. However the volume of work on mutualism is significantly small compared to that of the work dealing with Prey-Predator and competition interactions. This motivates the author to study three species mutualistic interactions in ecosystem. In this paper, we investigate the Stability analysis of mutualistic interactions among three species with limited resources for first and second species and unlimited resources for third species.

2. MATHEMATICAL MODEL:

The mathematical model for three mutually interacting species with limited resources for the first and second species and unlimited resources for third species is given by the following system of equations:

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$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - \alpha_{11}N_1 + \alpha_{12}N_2 + \alpha_{13}N_3) \\ \frac{dN_2}{dt} &= N_2(a_2 + \alpha_{21}N_1 - \alpha_{22}N_2 + \alpha_{23}N_3) \\ \frac{dN_3}{dt} &= N_3(a_3 + \alpha_{31}N_1 + \alpha_{32}N_2) \end{aligned} \tag{2.1}$$

where N_i, $i = 1,2,3$ represent the population density of first, second and third species respectively, a_i represent the intrinsic growth rate of first, second and third species respectively, α_{11} and α_{22} are the rate of decrease of first and second species due to insufficient food, α_{13} is the mutual coefficient of third species to first species, α_{21} is the mutual coefficient of first species to second species, α_{23} is the mutual coefficient of third species to second species, α_{31} is the mutual coefficient of first species to third species, α_{32} is the mutual coefficient of second species to third species. $a_i, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32}$ are nonnegative constants. If the death rate is greater than the birth rate for any species, we continue to use the same notation as intrinsic growth rate with negative sign for the rate of difference.

2.1 Stability analysis:

In this section we shall study the local stability analysis of the nonnegative equilibrium points of the system (2.1). There are at most four possible nonnegative equilibrium points as follows

(i) $E_{21} = (0,0,0)$ always exist.

(ii) $E_{22} = \left(\dfrac{a_1}{\alpha_{11}}, 0, 0\right)$ always exist.

(iii) $E_{23} = \left(0, \dfrac{a_2}{\alpha_{22}}, 0\right)$ always exist.

(iv) $E_{24} = \left(\dfrac{a_1\alpha_{22} + a_2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \dfrac{a_1\alpha_{21} + a_2\alpha_{11}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, 0\right)$ exists if $\alpha_{11}\alpha_{22} > \alpha_{12}\alpha_{21}$.

From the variational matrix about the equilibrium point E_{21}, it is shown that E_{21} is unstable node with locally unstable manifold in the (N_1, N_2, N_3) space. Further, all the three species populations grow indefinitely as $t \to \infty$. The equilibrium point E_{22} [respectively E_{23}] is saddle point with locally stable manifold in the N_1 [respectively N_2] direction and with locally unstable manifold in the (N_2, N_3) [respectively (N_1, N_3)] plane. Furthermore, first species population [respectively second] decline near equilibrium point E_{22} [respectively E_{23}] and second and third species populations [respectively first and third] grow indefinitely as $t \to \infty$. For the local stability analysis of the equilibrium point E_{24}, we consider subsystem of the system (2.1) in the following equations

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - \alpha_{11}N_1 + \alpha_{12}N_2) \\ \frac{dN_2}{dt} &= N_2(a_2 + \alpha_{21}N_1 - \alpha_{22}N_2) \end{aligned} \tag{2.2}$$

We see that $E_{24}^* = \left(\dfrac{a_1\alpha_{22} + a_2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \dfrac{a_1\alpha_{21} + a_2\alpha_{11}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right)$ is one of the nonnegative equilibrium point of subsystem (2.2). Clearly, E_{24} has the same stability behavior as E_{24}^* in the interior of the (N_1, N_2) plane. The variational matrix of subsystem (2.2) at the equilibrium point E_{24}^* is

$$J_1\left(\frac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}},\frac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)=\begin{pmatrix}\dfrac{-\alpha_{11}(a_1\alpha_{22}+a_2\alpha_{12})}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}} & \dfrac{\alpha_{12}(a_1\alpha_{22}+a_2\alpha_{12})}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\\ \dfrac{\alpha_{21}(a_1\alpha_{21}+a_2\alpha_{11})}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}} & \dfrac{-\alpha_{22}(a_1\alpha_{21}+a_2\alpha_{11})}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\end{pmatrix}$$

Thus, the characteristic equation of J_1 is $\lambda^2+\sigma_1\lambda+\sigma_2=0$ where

$$\sigma_1=\left[\alpha_{11}\left(\frac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)+\alpha_{22}\left(\frac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)\right],$$
$$\sigma_2=\left[\left(\frac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)\left(\frac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)\right]\left(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}\right)$$

According to Routh-Hurwitz criterion, the necessary and sufficient conditions for stability are $\sigma_1>0,\ \sigma_2>0$. Since $\alpha_{11}\alpha_{22}>\alpha_{12}\alpha_{21},\ \sigma_1>0,\ \sigma_2>0$ and therefore, by Routh-Hurwitz criterion, the equilibrium point E_{24}^* of subsystem (2.2) is stable node. Obviously, E_{24} is stable node in the (N_1,N_2) plane. However, the roots of the characteristic equation of the variational matrix of system (2.1) at E_{24} satisfy

$$\xi_1+\xi_2=-\left[\alpha_{11}\left(\frac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)+\alpha_{22}\left(\frac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)\right]$$
$$\xi_1\xi_2=\left[\left(\frac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)\left(\frac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)\right]\left(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}\right)$$
$$\xi_3=a_3+\alpha_{31}\left(\frac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)+\alpha_{32}\left(\frac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right)$$

3. THE DEATH RATE OF ANY ONE (SAY THIRD) SPECIES IS GREATER THAN ITS BIRTH RATE:

Under this situation, system (2.1) has the form

$$\begin{aligned}\frac{dN_1}{dt}&=N_1(a_1-\alpha_{11}N_1+\alpha_{12}N_2+\alpha_{13}N_3)\\ \frac{dN_2}{dt}&=N_2(a_2+\alpha_{21}N_1-\alpha_{22}N_2+\alpha_{23}N_3)\\ \frac{dN_3}{dt}&=N_3(-a_3+\alpha_{31}N_1+\alpha_{32}N_2)\end{aligned}\tag{3.1}$$

3.1 Stability analysis:

The present section deals with the existence of the equilibrium points of system (3.1) and local stability analysis of each one are investigated. At most there are seven possible nonnegative equilibrium points for system (3.1), the existence conditions on them are given as the following.

(1) The equilibrium point $E_{31}=(0,0,0)$ always exist.

(2) The equilibrium point $E_{32}=\left(\dfrac{a_1}{\alpha_{11}},0,0\right)$ always exist.

(3) The equilibrium point $E_{33}=\left(0,\dfrac{a_2}{\alpha_{22}},0\right)$ always exist.

(4) The equilibrium point $E_{34}=\left(0,\dfrac{a_3}{\alpha_{32}},\dfrac{a_3\alpha_{22}-a_2\alpha_{32}}{\alpha_{23}\alpha_{32}}\right)$ exists if $a_3\alpha_{22}>a_2\alpha_{32}$.

(5) The equilibrium point $E_{35}=\left(\dfrac{a_3}{\alpha_{31}},0,\dfrac{a_3\alpha_{11}-a_1\alpha_{31}}{\alpha_{13}\alpha_{31}}\right)$ exists if $a_3\alpha_{11}>a_1\alpha_{31}$.

(6) The equilibrium point $E_{36}=\left(\dfrac{a_1\alpha_{22}+a_2\alpha_{12}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}},\dfrac{a_1\alpha_{21}+a_2\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}},0\right)$ exists if $\alpha_{11}\alpha_{22}>\alpha_{12}\alpha_{21}$.

(7) The equilibrium point $E_{37}(N_1^*,N_2^*,N_3^*)$ exists if and only if there is a unique positive solution to the following equations

$$
\begin{aligned}
-\alpha_{11}N_1+\alpha_{12}N_2+\alpha_{13}N_3&=-a_1\\
\alpha_{21}N_1-\alpha_{22}N_2+\alpha_{23}N_3&=-a_2\\
\alpha_{31}N_1+\alpha_{32}N_2&=a_3
\end{aligned}
$$

provided that the three conditions

$$
\begin{aligned}
&(C_1)\ a_1\alpha_{23}\alpha_{32}+a_3(\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{22})>a_2\alpha_{13}\alpha_{32}\\
&(C_2)\ a_2\alpha_{13}\alpha_{31}+a_3(\alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21})>a_1\alpha_{23}\alpha_{31}\\
&(C_3)\ a_1(\alpha_{22}\alpha_{31}+\alpha_{21}\alpha_{32})+a_2(\alpha_{11}\alpha_{32}+\alpha_{12}\alpha_{31})+a_3\alpha_{12}\alpha_{21}<a_3\alpha_{11}\alpha_{22}
\end{aligned}
$$

hold, where

$$
\begin{aligned}
N_1^*&=\frac{a_1\alpha_{23}\alpha_{32}-a_2\alpha_{13}\alpha_{32}+a_3(\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{22})}{\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{22}\alpha_{31}}\\
N_2^*&=\frac{-a_1\alpha_{23}\alpha_{31}+a_2\alpha_{13}\alpha_{31}+a_3(\alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21})}{\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{22}\alpha_{31}}\\
N_3^*&=\frac{-a_1(\alpha_{22}\alpha_{31}+\alpha_{21}\alpha_{32})-a_2(\alpha_{11}\alpha_{32}+\alpha_{12}\alpha_{31})+a_3(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})}{\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{22}\alpha_{31}}
\end{aligned}
$$

Computing the variational matrices corresponding to each equilibrium point and then using the Routh-Hurwitz criteria the following dynamical behaviour are observed.

(1) The equilibrium point E_{31} is saddle point.

(2) If $a_1\alpha_{31}<a_3\alpha_{11}$ [resp. $a_2\alpha_{32}<a_3\alpha_{22}$], the equilibrium point E_{32} [resp. E_{33}] is saddle point with locally stable manifold in the (N_1,N_3) [resp. (N_2,N_3)] plane and locally unstable manifold in the N_2 [resp. N_1] direction and if $a_1\alpha_{31}>a_3\alpha_{11}$ [resp. $a_2\alpha_{32}>a_3\alpha_{22}$], the equilibrium point E_{32} [resp. E_{33}] is saddle point with locally stable manifold in the N_1 [resp. N_2] direction and locally unstable manifold in the (N_2,N_3) [resp. (N_1,N_3)] plane.

(3) For the stability analysis of the equilibrium point E_{34}, we consider subsystem of the system (3.1) in the following equations.

$$
\begin{aligned}
\frac{dN_2}{dt}&=N_2(a_2-\alpha_{22}N_2+\alpha_{23}N_3)\\
\frac{dN_3}{dt}&=N_3(-a_3+\alpha_{32}N_2)
\end{aligned}
\qquad (3.2)
$$

We see that $E_{34}^* = \left(\frac{a_3}{\alpha_{32}}, \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{23}\alpha_{32}} \right)$ is one of the nonnegative equilibrium point of subsystem (3.2). Clearly, E_{34} has the same stability behavior as E_{34}^* in the interior of the (N_2, N_3) plane. The variational matrix of subsystem (3.2) at the equilibrium point E_{34}^* is

$$J_2\left(\frac{a_3}{\alpha_{32}}, \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{23}\alpha_{32}} \right) = \begin{pmatrix} \frac{-a_3\alpha_{22}}{\alpha_{32}} & \frac{a_3\alpha_{23}}{\alpha_{32}} \\ \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{23}} & 0 \end{pmatrix}$$

Thus, the characteristic equation of J_2 is $\lambda^2 + \delta_1\lambda + \delta_2 = 0$ with $\delta_1 = \frac{a_3\alpha_{22}}{\alpha_{32}}$, $\delta_2 = \frac{-a_3(a_3\alpha_{22} - a_2\alpha_{32})}{\alpha_{32}}$. According to Routh-Hurwitz criteria, we conclude that the equilibrium point E_{34}^* of subsystem (3.2) is saddle point. Obviously, E_{34} is saddle point in the (N_2, N_3) plane. However, the roots of the characteristic equation of the variational matrix of system (3.1) at E_{34} satisfy

$$\eta_1 + \eta_2 = -\frac{a_3\alpha_{22}}{\alpha_{32}}$$
$$\eta_1\eta_2 = -\frac{a_3(a_3\alpha_{22} - a_2\alpha_{32})}{\alpha_{32}}$$
$$\eta_3 = \frac{a_1\alpha_{23}\alpha_{32} - a_2\alpha_{13}\alpha_{32} + a_3\alpha_{12}\alpha_{23} + a_3\alpha_{13}\alpha_{22}}{\alpha_{23}\alpha_{32}}$$

(4) The stability behavior of the equilibrium points E_{35} and E_{36} are similar to the equilibrium points E_{34} and E_{24} respectively and hence we omit the details.

Theorem: 3.1 *The positive equilibrium point* $E_{37}(N_1^*, N_2^*, N_3^*)$ *is not stable.*

Proof: Assume $N_1 = N_1^* + u_1$, $N_2 = N_2^* + u_2$, $N_3 = N_3^* + u_3$ where u_1, u_2, u_3 small perturbations are. The variational matrix about equilibrium point E_{37} is given by

$$J^* = \begin{bmatrix} -\alpha_{11}N_1^* & \alpha_{12}N_1^* & \alpha_{13}N_1^* \\ \alpha_{21}N_2^* & -\alpha_{22}N_2^* & \alpha_{23}N_2^* \\ \alpha_{31}N_3^* & \alpha_{32}N_3^* & 0 \end{bmatrix}$$

The characteristic equation of the above variational matrix about equilibrium point E_{37} is $\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 = 0$ where

$$k_1 = \alpha_{11}N_1^* + \alpha_{22}N_2^*$$
$$k_2 = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})N_1^*N_2^* - \alpha_{23}\alpha_{32}N_2^*N_3^* - \alpha_{13}\alpha_{31}N_1^*N_3^*$$
$$k_3 = -(\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{13}\alpha_{22}\alpha_{31})N_1^*N_2^*N_3^*$$

According to Routh-Hurwitz criterion, the necessary and sufficient conditions for stability are

$$k_1 > 0,\ k_3 > 0,\ k_1k_2 > k_3$$

We observe that $k_1 > 0$ but $k_3 < 0$ and therefore, by Routh-Hurwitz criterion, the equilibrium point E_{37} is unstable.

4. THE DEATH RATE OF ANY TWO SPECIES (SAY SECOND AND THIRD) ARE GREATER THAN THEIR BIRTH RATE:

Under this situation, system (2.1) has the form

$$\begin{aligned}\frac{dN_1}{dt} &= N_1(a_1 - \alpha_{11}N_1 + \alpha_{12}N_2 + \alpha_{13}N_3)\\ \frac{dN_2}{dt} &= N_2(-a_2 + \alpha_{21}N_1 - \alpha_{22}N_2 + \alpha_{23}N_3)\\ \frac{dN_3}{dt} &= N_3(-a_3 + \alpha_{31}N_1 + \alpha_{32}N_2)\end{aligned} \tag{4.1}$$

4.1 Stability analysis:

In this section, the existence and the local stability analysis of the nonnegative equilibrium points of the system (4.1) are investigated. Six nonnegative equilibrium points are found. The existence of these equilibrium points shows that:

(1) The equilibrium point $E_{41} = (0,0,0)$ always exist.

(2) The equilibrium point $E_{42} = \left(\frac{a_1}{\alpha_{11}}, 0, 0\right)$ always exist, as the first species survives and second and third species are washed out.

(3) The equilibrium point $E_{43} = \left(0, \frac{a_3}{\alpha_{32}}, \frac{a_3\alpha_{22} + a_2\alpha_{32}}{\alpha_{23}\alpha_{32}}\right)$ always exists, as the second and third species survive and first species is washed out.

(4) The equilibrium point $E_{44} = \left(\frac{a_3}{\alpha_{31}}, 0, \frac{a_3\alpha_{11} - a_1\alpha_{31}}{\alpha_{13}\alpha_{31}}\right)$ exists if $a_3\alpha_{11} > a_1\alpha_{31}$.

(5) The equilibrium point $E_{45} = \left(\frac{a_2\alpha_{12} - a_1\alpha_{22}}{\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}, \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}, 0\right)$ exists if $\frac{a_1}{a_2} > \frac{\alpha_{11}}{\alpha_{21}} > \frac{\alpha_{12}}{\alpha_{22}}$.

(6) The equilibrium point $E_{46}(\overline{N_1}, \overline{N_2}, \overline{N_3})$ exists if and only if there is a unique positive solution to the following equations

$$\begin{aligned}-\alpha_{11}N_1 + \alpha_{12}N_2 + \alpha_{13}N_3 &= -a_1\\ \alpha_{21}N_1 - \alpha_{22}N_2 + \alpha_{23}N_3 &= a_2\\ \alpha_{31}N_1 + \alpha_{32}N_2 &= a_3\end{aligned}$$

provided that the two conditions

$$(C_4)\ \ a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21}) > a_1\alpha_{23}\alpha_{31} + a_2\alpha_{13}\alpha_{31}$$
$$(C_5)\ \ a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3\alpha_{11}\alpha_{22} > a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_3\alpha_{12}\alpha_{21}$$

hold, where

$$\overline{N_1} = \frac{a_1\alpha_{23}\alpha_{32} + a_2\alpha_{13}\alpha_{32} + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22})}{\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31}}$$
$$\overline{N_2} = \frac{-a_1\alpha_{23}\alpha_{31} - a_2\alpha_{13}\alpha_{31} + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21})}{\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31}}$$
$$\overline{N_3} = \frac{-a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})}{\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31}}$$

The local stability analysis of system (4.1) by computing the variational matrices corresponding to each equilibrium point and then using the Routh-Hurwitz criteria shows that:

(1) The equilibrium point E_{41} is saddle point.
(2) The eigenvalues of the variational matrix about E_{42} are given as follows

$$\lambda_1 = -a_1, \quad \lambda_2 = \frac{a_1\alpha_{21} - a_2\alpha_{11}}{\alpha_{11}}, \quad \lambda_3 = \frac{a_1\alpha_{31} - a_3\alpha_{11}}{\alpha_{11}}$$

Thus, the equilibrium point E_{42} is stable node or saddle point, depending on both λ_2 and λ_3 are negative or otherwise. (i.e. both λ_2 and λ_3 are positive or one positive and other negative).

Dynamical behavior of equilibrium points E_{43}, E_{44}, E_{45} and E_{46} are shown in the Theorem 5.1.

5. THE DEATH RATE OF ALL THE SPECIES ARE GREATER THAN THEIR BIRTH RATE:

Under this situation, system (2.1) has the form

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(-a_1 - \alpha_{11}N_1 + \alpha_{12}N_2 + \alpha_{13}N_3) \\ \frac{dN_2}{dt} &= N_2(-a_2 + \alpha_{21}N_1 - \alpha_{22}N_2 + \alpha_{23}N_3) \\ \frac{dN_3}{dt} &= N_3(-a_3 + \alpha_{31}N_1 + \alpha_{32}N_2) \end{aligned} \tag{5.1}$$

5.1 Stability analysis:

This section establishes the local stability analysis of the nonnegative equilibrium points of the system (5.1). There are at most five possible nonnegative equilibrium points as follows.

(1) The equilibrium point $E_{51} = (0,0,0)$ always exist.

(2) The equilibrium point $E_{52} = \left(0, \frac{a_3}{\alpha_{32}}, \frac{a_2\alpha_{32} + a_3\alpha_{22}}{\alpha_{23}\alpha_{32}}\right)$ always exists.

(3) The equilibrium point $E_{53} = \left(\frac{a_3}{\alpha_{31}}, 0, \frac{a_1\alpha_{31} + a_3\alpha_{11}}{\alpha_{13}\alpha_{31}}\right)$ always exists.

(4) The equilibrium point $E_{54} = \left(\frac{a_1\alpha_{22} + a_2\alpha_{12}}{\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}, \frac{a_1\alpha_{21} + a_2\alpha_{11}}{\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}, 0\right)$ exists if $\alpha_{12}\alpha_{21} > \alpha_{11}\alpha_{22}$.

(5) The positive equilibrium point $E_{55}(N_1^{**}, N_2^{**}, N_3^{**})$ exists if and only if there is a unique positive solution to the following equations

$$\begin{aligned} -\alpha_{11}N_1 + \alpha_{12}N_2 + \alpha_{13}N_3 &= a_1 \\ \alpha_{21}N_1 - \alpha_{22}N_2 + \alpha_{23}N_3 &= a_2 \\ \alpha_{31}N_1 + \alpha_{32}N_2 &= a_3 \end{aligned}$$

provided that the two conditions

$$(C_6)\ a_2\alpha_{13}\alpha_{32} + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22}) > a_1\alpha_{23}\alpha_{32}$$
$$(C_7)\ a_1\alpha_{23}\alpha_{31} + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21}) > a_2\alpha_{13}\alpha_{31}$$
$$(C_8)\ a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3\alpha_{11}\alpha_{22} > a_3\alpha_{12}\alpha_{21}$$

hold, where

$$N_1^{**} = \frac{-a_1\alpha_{23}\alpha_{32} + a_2\alpha_{13}\alpha_{32} + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22})}{\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31}}$$

$$N_2^{**} = \frac{a_1\alpha_{23}\alpha_{31} - a_2\alpha_{13}\alpha_{31} + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21})}{\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31}}$$

$$N_3^{**} = \frac{a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})}{\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31}}$$

From the variational matrix about the equilibrium point E_{51}, it is shown that E_{51} is stable point. We now state the local dynamical behavior of the equilibrium points E_{52}, E_{53}, E_{54} and E_{55} in the form of Theorem 5.1. The proof of this Theorem follows directly from the Routh-Hurwitz criteria and hence omitted.

Theorem: 5.1

(i) The equilibrium points E_{43} and E_{52} have the same stability behavior in the N_2N_3 plane.

(ii) The equilibrium points E_{44} and E_{53} have the same stability behavior in the N_1N_3 plane.

(iii) The equilibrium points E_{45} and E_{54} have the same stability behavior in the N_1N_2 plane.

(iv) The equilibrium points E_{46} and E_{55} are saddle point.

6. NUMERICAL SIMULATIONS:

In this section, numerical simulations are carried out to investigate the dynamic behavior of the systems (3.1), (4.1) and (5.1) about the equilibrium points E_{37}, E_{46} and E_{55} respectively. Consider the parameter values

$$\begin{aligned} &a_1 = 0.5,\ a_2 = 0.4,\ a_3 = 0.3,\ \alpha_{11} = 2,\ \alpha_{12} = 0.2,\ \alpha_{13} = 0.5, \\ &\alpha_{21} = 0.1,\ \alpha_{22} = 1.8,\ \alpha_{23} = 0.6,\ \alpha_{31} = 0.3,\ \alpha_{32} = 0.4 \end{aligned} \tag{6.1}$$

For this set of parameter values, the system (3.1) has an equilibrium point at (0.4293, 0.4280, 0.5459). It is saddle point as the eigenvalues of the variational matrix of the system (3.1) about the equilibrium point (0.4293, 0.4280, 0.5459) are approximately -0.8956, -0.8408, 0.1074. Furthermore, third species is extinct out due to death rate is greater than its birth rate while first and second species populations are grow constantly (See Fig. (1)). Similarly, for the same data set (6.1), the systems (4.1) and (5.1) have equilibrium points at (0.6278, 0.2792, 1.3995) and (0.3300, 0.5025, 2.1191) respectively which are saddle points. The dynamical behavior of the systems (4.1) and (5.1) about the equilibrium points (0.6278, 0.2792, 1.3995) and (0.3300, 0.5025, 2.1191) are shown in the Fig. 2 and Fig. 3 respectively.

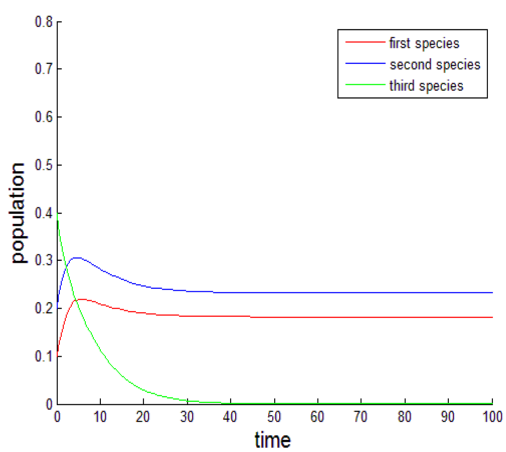

Fig- 1.

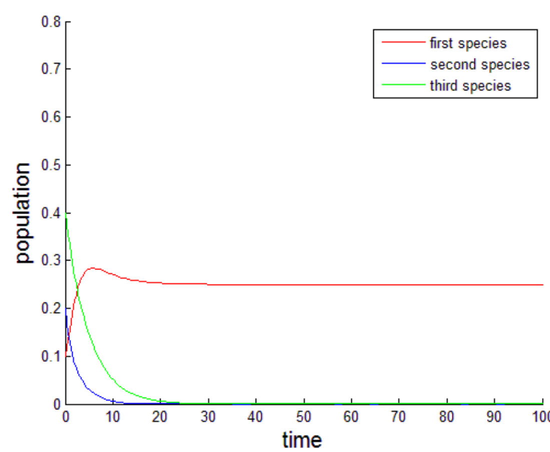

Fig.- 2

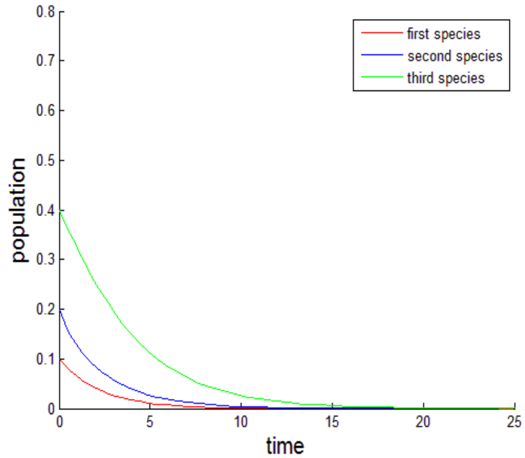

Fig.- 3

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