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A NOTE ON RELATIONS BETWEEN BARNETTE AND SPARSE GRAPHS

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ABSTRACT

Graph theory is one of the oldest concept which is used to solve the problems in real life engineering and many areas of science and technology and also in social sciences. The main aim of this paper is to study the problems on planar graphs, sparse graphs, Barnette graphs and obtained some relations between them.

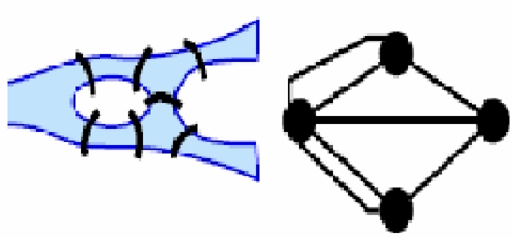
Key words: *Graph, Hamiltonian graph, Sparse graph, Barnette graph, Planar graph.*

1. INTRODUCTION:

Graph theory is one of the oldest subject in Mathematics which is started with the problem of Koinsber bridge, in 1735. This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. This lead to the invention of enumerative graph theory. Later the term "Graph" was introduced by Sylvester in 1878 where he drew an analogy between "Quantic invariants" and covariants of algebra and molecular diagrams. It took quite some time for the Hamiltonian Cycle problem to be formally discovered. It is named for the famous Irish Mathematician Sir William Rowan Hamilton. Barnette's Conjecture, published in [4], states that all planar, cubic, 3-connected, bipartite graphs are Hamiltonian. Sparse graphs are useful most in practical applications such as computer or road networks are and with respect to Hamiltonicity, sparse graphs tend to be more difficult to solve than dense graphs.

In this paper we study the Hamiltonian graphs, Hamiltonian paths Sparse graph and some results between planar and sparse graphs and Barnette graphs.

Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?



K¨onigsberg problem

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1.1 Definition: A graph is a triple G = (V, E,ϕ) where

• V is a finite set, called the vertices of G,
• E is a finite set, called the edges of G, and
• ϕ is a function with domain E and codomain P2 (V), where P_2 (V) stands for the set of all 2-element subsets of the set V.

1.2 Example: The graph depicted in Figure 1 has vertex set V= {a, b, c, d, e. f} and edge set E = {(a, b), (b, c), (c, d), (c, e), (d, e), (e, f)}.

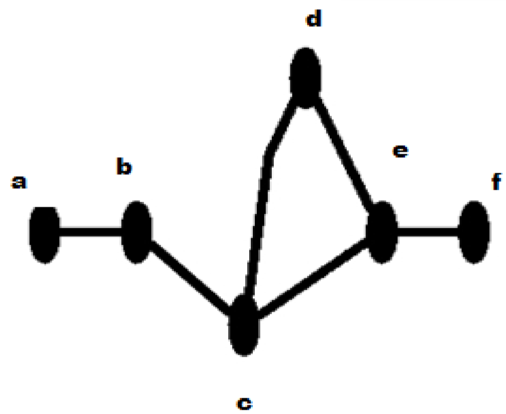


Figure- 1:

1.3 Note: Vertices are also known as nodes, points and (in social networks) as actors, agents or players. Edges are also known as lines and (in social networks) as ties or links. An edge e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Definition: A *component* of a graph is defined as a maximal subgraph in which a path exists from every node to every other (i.e., they are mutually reachable). The size of a component is defined as the number of nodes it contains. A connected graph has only one component.

1.5 Definition: A *cycle* can be defined as a closed path in which *n* >= 3. The sequence *c, e, d* in Figure 3 is a cycle.

1.6 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.7 Definition: A bipartite graph (or **bigraph**) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite **graph is a graph that does not contain any odd-length cycles.**

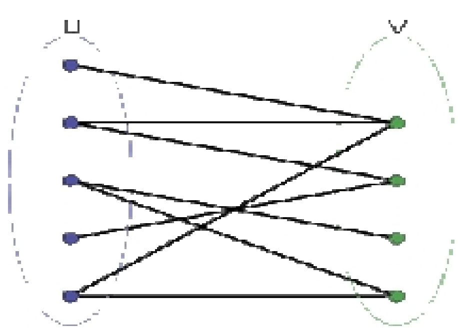


Figure- 3: Example of a bipartite graph.

1.8 Definition: An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

2. HAMILTONIAN, SPARSE AND PLANAR GRAPHS:

In this section we have given some existing literature related to Hamiltonian graphs.

2.1 Definition: Another closely related problem is finding a **Hamilton path** in the graph (named after an Irish mathematician, Sir William Rowan Hamilton). Whereas an Euler path is a path that visits every edge exactly once, a Hamilton path is a path that visits every vertex in the graph 4 exactly once.

2.2 Definition: A **Hamilton circuit** is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit.

2.3 Example:

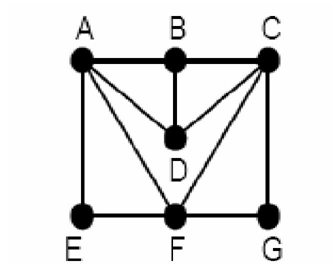


Figure 4: Hamilton Circuit would be AEFGCDBA

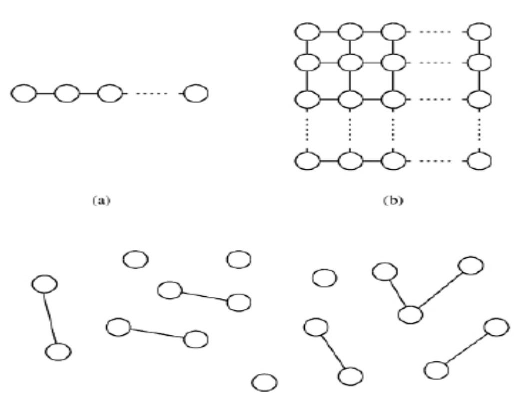
2.4 Travelling Salesman Problem:

Traveling Salesman Problem (TSP) is a very well-known problem which is based on Hamilton cycle. The problem statement is: Given a number of cities and the cost of traveling from any city to any other city, find the cheapest round-trip route that visits every city exactly once and return to the starting city.

In graph terminology, where the vertices of the graph represent cities and the edges represent the cost of traveling between the connected cities (adjacent vertices), traveling salesman problem is just about trying to find the Hamilton cycle with the minimum weight. This problem has been shown to be NP-Hard. Even though the problem is computationally difficult, a large number of heuristics and exact methods are known, so that some instances with tens of thousands of cities have been solved. The most direct solution would be to try all permutations and see which one is cheapest (using brute force search). The running time for this approach is $O(V\,!)$, the factorial of the number of cities, so this solution becomes impractical even for only 20 cities. A dynamic programming solution solves the problem with a runtime complexity of $O(V\ 22V\ )$ by considering dp[end][state] which means the minimum cost to travel from start vertex to end vertex using the vertrices stated in the state (start vertex can be any vertex chosen at the start). As there are $V\ 2V$ subproblems and the time complexity to solve each sub-problems is $O(V\ )$, the overall runtime complexity $O(V\ 22V\ )$.

2.5 Definition: A graph with 'n' vertices and 'm' edges is said to be **sparse graph** if **m < n (n-1).**

2.6 Examples of sparse graphs:
(a) a linear graph, in which each vertex has two incident edges;
(b) a grid graph, in which each vertex has four incident vertices;
(c) a random sparse graph.



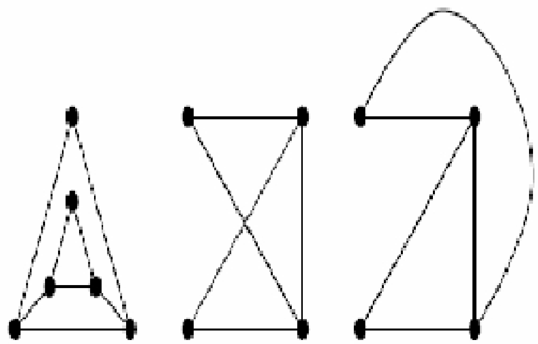
2.7 Problem: The input to the *Sparse- Cut* problem is

• A weighted graph G = (V,E) with positive edge weights (or costs or capacities, as they are called in this context) ce for every edge e € E. As is usual, n = |V |.

• A set of pairs of vertices $\{(s_1, t_1), (s_2, t_2) . . . (s_k, t_k)\}$, with associated demands Di between si and ti.

2.8 Definition: A **planar graph** is one that can be drawn on a plane in such a way that there are no "edge crossings," i.e. edges intersect only at their common vertices.

2.9 Examples for Planar Graphs:



2. 10 Theorem: A plane graph is bipartite if and only if all of its faces have even degree.

Proof: Almost every modern graph theory textbook contains a proof of the statement that a graph is Sparse if and only if it contains no odd cycles. All that remains to be shown is that a plane graph contains no odd cycles if and only if all of its faces have even degree.

3. BARNETTE GRAPHS AND SOME RESULTS ON BARNETTE AND SPARSE GRAPHS:

In this section we studied about Barnette graphs, and obtained some results between Barnette and Sparse graphs.

3.1 Definition: A k-connected graph is one that cannot be disconnected by removing fewer than k vertices.

3.2 Definition: The Graphs of 3-connected, simple, bipartite, 3-dimenstional convex polytops are called Barnette Graphs.

3.3 Barnette Conjecture:

Every graph that is 3-connected, 3-regular, bipartite and planar has a Hamiltonian Cycle.

3.4 Theorem: Barnette's Conjecture holds if and only if every Barnette graph is Sparse cut.

To Prove this result we need the following Lemmas:

Lemma: 1 If some edge in some Barnette graph is part of no Sparse Cut then Barnette's Conjecture does not hold.

Lemma: 2 If Barnette's conjecture holds, then every pair of adjacent vertices in any Barnette graph has a Hamiltonian Path between them.

Proof: Suppose Barnette's Conjecture holds. In a contrary way suppose there exists some Barnette graph, say G, such that G contains an edge between two vertices, call them x and a, which is part of no sparse cut.

Since every Barnette graph is Hamiltonian so G is planar, it is possible to embed it such that x and a are on the outside face.

Let x's other neighbors be called b and c as indicated in Figure 1. Suppose we create a new graph G * by applying the first 3 steps of the mirroring procedure to G.

Since there exists no Hamiltonian Path between a and x, if we were to remove x from G, the resulting graph, call it G_1, would neither contain a Hamiltonian Path from a to b, nor would it contain a Hamiltonian path from a to c.

Similarly, if we were to remove x* from G*, the resulting graph, would neither contain a Hamiltonian Path from a* to b*, nor would it contain a Hamiltonian Path from a* to c*.

We now complete the Mirroring Procedure to create G^1, which by lemma 2 is also a Barnette graph. Consider the cut across the three newly-created edges labeled e1, e2, and e3 in G^1 as shown in following figure Any Sparse Cut in G^1 would have to cross exactly two of those three edges. Such a Sparse Cut cannot cross e1 and e2 because there is no Hamiltonian Path from a to b in G^1. It cannot cross e1 and e3 because neither is there a Hamiltonian Path from a to c in G^1, nor is there a Hamiltonian Path from c* to a*.

Finally, it cannot cross e2 and e3 since there is no Hamiltonian Path from a* to b*.

Therefore, G^1 is a non-Hamiltonian. Hence G is not a Barnette graph, which is a contradiction. Therefore every edge in any Barnette graph must be part of some Sparse Cut, as required.

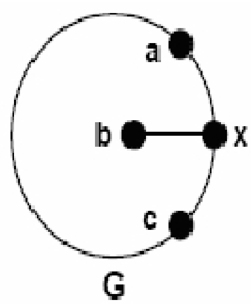


Figure 1: The Mirroring Procedure, step-1

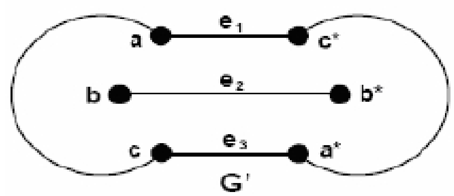


Figure 1: The Mirroring Procedure-2

Converse: Suppose G is Barnette graph which is having Sparse cut. Since G is Barnette graph it follows the Barnette conjecture obviously. Converse is true obviously.

REFERENCES:

[1] Alexander Hertel "Hamiltonian cycles in Sparse Graphs" Ph. D thesis, 2004.

[2] A.M. Frieze. "Finding Hamilton Cycles in Sparse Random Graphs" Journal of Combinatorial Theory, Series B, 44:230{250, 1988. Bibliography 83

[3] J.A. Bondy and U.S.R Murty. "Graph Theory With Applications". Macmillan, London, 1976. 81Bibliography 82

[4] M.N. Ellingham and J.D. Horton. Non-Hamiltonian 3-Connected Cubic Bipartite Graphs. Journal of Combinatorial Theory, Series B, 34 pp : 350 - 353, 1983.

[5] M.R. Garey, D.S. Johnson, and R.E. Tarjan. The Planar Hamiltonian Circuit Problem is NP-Complete. SIAM Journal of Computing, Vol. 5 No. 4, 704 - 714, 1976.

[6] T. Feder and C. Subi. On Barnette's Conjecture. Published On Internet: http://theory.stanford.edu/ tomas/bar.ps. Submitted to Journal of Graph Theory.

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