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COMPOSITION OPERATOR AND WEIGHTED COMPOSITION OPERATOR ON WEIGHTED SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT

In this paper, we characterized invertible composition operator on weighted spaces of analytic functions and obtained dynamical system induced by weighted composition operator on weighted space of vector-valued analytic functions.

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1. INTRODUCTION AND PRELIMINARIES:

The theory of weighted composition operators on spaces of analytic functions are gaining importance as it includes two nice classes of operators and very less information is known about weighted composition operators as compared to the theory of composition operators which has been presented in three monographs(see Cowen and MacCluer [3], Shapiro [8] and Singh and Manhas [9]). In [5] Manhas has made a application of multiplication operators can be found in the theory of dynamical systems. This paper is an continuation of the paper [6]. For more details about weighted composition operator on weighted spaces of continuous and analytic function spaces [1], [2], [4], [5], [6] and [9]. We have organized this paper into three sections. The preliminaries required for proving the results in the remaining sections are presented in section 1. In section two we have characterized the invertible composition operator on weighted spaces of analytic functions. In the last section we endowed with important theorems and obtained dynamical system induced by weighted composition operator on weighted spaces of analytic functions.

Let G be an open connected subset of $C^N\left(N \geq 1\right)$ and $H(G,E)$ be the space of all vector-valued analytic functions from G into the Banach space E. Let V be a set of non-negative upper semi continuous functions on G. Then V is directed upward if for every pair $u_1 u_2 \in V$ and $\lambda u > 0$, there exists $v \in V$ such that $\lambda u_i \leq v$ (point wise on G) for $i = 1, 2$. If V is directed upward and for each $z \in G$, there exists $v \in V$ such that $v(z) > 0$. Then we call V as an arbitrary system of weights on G. If U and V are two arbitrary systems of weights on G such that for each $\mathrm{u} \in U$, there exists $v \in V$ for which $\mathrm{u} \leq \mathrm{v}$, then we write $U \leq V$. If $U \leq V$ and $V \leq U$, then we write $U \cong V$. Let V be an arbitrary system of weights on G.

Then we define $HV_b(G,E) = \{ f \in H(G,E) : vf(G) \, is \, bounded \, in \, E \, for \, each \, v \in V\}$ and

$HV_o(G,E) = \{ f \in H(G,E) : vf \;\; vanishes \, at \, \infty \, on \, G \, for \, each \, v \in V\}$.

For $v \in V$ and $f \in H(G,E)$, we define $\left\| f \right\|_{v,E} = \sup\left\{ v(z) \left\| f(z) \right\| : z \in G \right\}$. Clearly, the family $\left\| . \right\|_{v,E} : v \in V$ of semi norms defines a Hausdroff locally convex topology on each of these spaces $HV_b(G,E)$ and $HV_0(G,E)$. With this topology on each of these spaces $HV_b(G,E)$ and $HV_0(G,E)$ are called weighted locally convex spaces

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of vector-valued analytic functions. These spaces have a basis of closed absolutely convex neighborhoods of the form

$$B_{v,E} = \left\{ f \in HV_b(G,E) : \|f\|_{v,E} \le 1 \right\}.$$

Definitions of reasonable and essential system of weights: [5]:

An arbitrary system of weights V is called reasonable if it satisfies the following properties: for each $v \in V$, there exists $\tilde{v} \in \tilde{V}$ *such that for each* $v \in V, \|f\|_v \le 1$ iff $\|f\|_{\tilde{v}} \le 1$ such that for every $f \in H(G)$; if $v \in V$, then for every $z \in G \; \exists f_z \in B_v$ such that $|f_z(z)| = \dfrac{1}{\tilde{v}(z)}$. A weight $v \in V$ is called essential if there exists a constant $c > 0$ such that $v(z) \le \tilde{v}(z) \le cv(z), \textit{for each}\, z \in G$.

A reasonable system of weights V is called an essential system if each $v \in V$ is an essential weight.

Let G be a topological group with e as identity, let X be a topological space and $\pi : G \times X \to X$ be a continuous map such that

(i) $\pi(e,x) = x \textit{ for every } x \in X$
(ii) $\pi(st,x) = \pi(s,\pi(t,x)) \textit{ for every } t,s \in G \;\, x \in X$.

Then the triple (G, X, π) is called a transformation group, X is a state space. If $G = (R,+)$ the corresponding transformation group is called a dynamical system. The transformation group $(\mathbb{R}, X, \pi)$ is known as continuous dynamical system. If X is a Banach space and

$$\pi(t, \alpha x + \beta y) = \alpha\pi(t,x) + \beta\pi(t,y) \textit{ for } t \in \mathbb{R},\, \alpha,\beta \in \mathbb{C},\, x,y \in X$$

then $(\mathbb{R}^+, X, \pi)$ is called a linear dynamical system.

2. INVERTIBLE COMPOSITION OPERATOR ON ANALYTIC FUNCTION SPACES:

Theorem: 2.1 Let V be an arbitrary system of weights on G. Let $\phi : G \to G$ be analytic map such that C_ϕ is a composition operator on $HV_b(G)$. Then C_ϕ is invertible if (i) for each $v \in V$, there exist $u \in V$ such that $v(\phi(z)) \le u(z)$ for every $z \in G$. (ii) for each $z \in G$, ϕ is a conformal mapping of G.

Proof: It is enough to show that C_ϕ is bounded below and onto. For let $v \in V$ and B_v be a neighborhood of the origin in $HV_b(G)$. Then by condition (i), there exist $u \in V$ such that $v(\phi(z)) \le u(z)$ for every $z \in G$.
We claim that $C_\phi(B_v^c) \subseteq B_u^c$. Let $f \in B_v^c$.

Then
$$1 < \|f\|_v = \sup\{v(z)\|f(\phi(z))\| : z \in G\} \le \sup\{u(z)\|f(\phi(z))\| : z \in G\} \le \|C_\phi f\|_u .$$

This shows that $C_\phi f \in B_u^c$ and hence C_ϕ is bounded below. Now we shall show that C_ϕ is onto. For let $g \in HV_b(G)$. Fix $z_0 \in G$ and let $v \in V$ be such that $v(\phi(z_0)) > 0$. Then by condition (i), there exists $u \in V$ such that $v(\phi(z_0)) \le u(z_0)$ for every $z_0 \in G$. By condition (ii) define an analytic map $\phi^{-1} : G \to G$. Also define $h : G \to \mathbb{C}$ as $h = g \circ \phi^{-1}$.

Then we show that $h \in HV_b(G)$. For, let $v \in V$. Then by condition (i), there exists $v \in V$ such that $v(\phi(z)) \le u(z)$ for every $z \in G$. Now we have

$$\begin{aligned} \|h\|_v &= \sup\{v(\phi(z))\|h(\phi(z))\| : z \in G\} \\ &\le \sup\{u(z)\|g \circ \phi^{-1}\| : z \in G\} \\ &= \sup\{v(z)\|g(z)\| : z \in G\} \le \|g\|_{u,E} < \infty . \end{aligned}$$

This proves that $h \in HV_b(G)$ and clearly $C_\phi(h) = g$. Hence C_ϕ is invertible.

Corollary: 2.2 Let V be an arbitrary system of weights on G. Let $\phi : G \to G$ be analytic map. Then C_ϕ is invertible composition operator if ϕ is a conformal mapping of G onto itself such that $V \circ \phi \cong V$.

Proof: Since, $\phi : G \to G$ is a conformal mapping such that $V \le V \circ \phi$ [5]. Implies that C_ϕ is a composition operator on $HV_b(G)$.

Let $v \in V$. Then, since $V \circ \phi \le V$ such that $u \in V$ such that $v(\phi(z)) \le u(z)$ for every $z \in G$. Further, there exists $w \in V$ such that $\dfrac{1}{mv} \le w$ and $v(\phi(z)) \le w(z)$ for every $z \in G$. Thus according to the given condition and theorem: 3.1, it follows that C_ϕ is invertible composition operator on $HV_b(G)$.

Corollary: 2.3 Let G be a simply connected open subset of $\mathbb{C}$ and let $V = \left\{\lambda_{\chi_k} : \lambda > 0, K \subseteq G, K\, is\, compact\right\}$. Let $\phi : G \to G$ be analytic map. Then C_ϕ is invertible composition operator on $HV_b(G)$ iff ϕ is a conformal mapping of G onto itself.

Theorem: 2.4 Let G be a simply connected open subset of $\mathbb{C}$ and let V be a reasonable system of weights on G. Let $\phi : G \to G$ be analytic map such that C_ϕ is composition operator on $\widetilde{HV_b(G)}$. Then C_ϕ is invertible iff

(i) ϕ is a conformal mapping of G onto itself.

(ii) for each $v \in V$ there exists $u \in V \ni \tilde{v}(\phi(z)) \le \tilde{u}(z)$ for every $z \in G$.

Proof: Suppose, conditions (i) and (ii) hold. Let $z \in G$ and let $v \in V$ be such that $v(\phi(z_0)) > 0$. Then $\tilde{v}(\phi(z_0)) > 0$ and by condition (ii), there exists $u \in V$ such that $\tilde{v}(\phi(z_0)) \le \tilde{u}(z_0)$. Further by condition (i), we have that $\tilde{v}(z) \le \tilde{u}(\phi^{-1}(z))$ for every $z \in G$.

(i.e) $\tilde{v}(z)\|(\phi^{-1}(z))\| \le \tilde{u}(\phi^{-1}(z))$ for every $z \in G$.

Therefore $C_{\phi^{-1}}$ is a composition operator on $\widetilde{HV_b(G)}$ such that $C_\phi C_{\phi^{-1}} = C_{\phi^{-1}} C_\phi = I$, the identity operator.

Hence C_ϕ is invertible. Conversely, suppose that C_ϕ is invertible on $\widetilde{HV_b(G)}$. First to show that ϕ is a conformal mapping. We need to prove that ϕ is bijective. Let $z_1, z_2 \in G$ be such that $z_1 \ne z_2$. Then by Riemann mapping theorem, there exists one-one analytic function $g \in H^\infty(G)$ such that $g(z_1) \ne g(z_2)$.

Define $h:G\to\mathbb{C}$ as $h(z)=g(z)-g(z_1)$ for every $z\in G$. Let $g_{z_2}\in\widetilde{HV_b(G)}$ be such that $g_{z_2}(z_2)\neq 0$ and $h_0=h_{g_{z_2}}$. Since $\widetilde{HV_b(G)}$ is module over $H^\infty(G)$, it follows that $h_0\in\widetilde{HV_b(G)}$ such that $h_0(z_1)=0$ and $h_0(z_2)\neq 0$. Let $\varepsilon=\varepsilon_1+\varepsilon_2$ and let $v\in V$ be such that $v(z_1)\geq 1$ and $v(z_2)\geq 1$. There exists $\tilde{v}\in\tilde{V}$ such that $\tilde{v}(z_1)\geq 1$ and $\tilde{v}(z_2)\geq 1$.

Since $C_\phi(\widetilde{HV_b(G)})$ is dense in $\widetilde{HV_b(G)}$ there exists $F\in\widetilde{HV_b(G)}$ such that $\left\|C_\phi F\right\|<\dfrac{1}{2\varepsilon}$.

(i.e) $\tilde{v}_{z_1}\left\|F(\phi(z_1)\right\|<\dfrac{1}{2\varepsilon}$ and $\tilde{v}_{z_2}\left\|F(\phi(z_2)\right\|-\dfrac{1}{\varepsilon_2}<\dfrac{1}{2\varepsilon}$.

Further it implies that $\left\|F(\phi(z_1)\right\|<\dfrac{\varepsilon_1}{2\varepsilon}<\dfrac{1}{2}$ and $\left|\left\|F(\phi(z_2)\right\|-1\right|<\dfrac{\varepsilon_2}{2\varepsilon}<\dfrac{1}{2}$.

From the last two inequalities, it is easy to conclude that $\phi(z_1)\neq\phi(z_2)$. Thus ϕ is injective and also using the injectivity argument of C_ϕ, we can conclude that ϕ is onto. Now to prove condition (ii), we fix $v\in V$. Then there exists $\tilde{v}\in\tilde{V}$. According to theorem:2.1, C_ϕ is bounded below on $\widetilde{HV_b(G)}$. Thus there exists $\tilde{u}\in\tilde{V}$ with $u\in V$. We claim that $\tilde{v}(\phi(z))\leq\tilde{u}(z)$ for every $z\in G$.

Let $z_0\in G$. For $z=\phi(z_0)$. We have $\tilde{v}\left\|(\phi(z_0))g_0(\phi(z_0))\right\|\leq 1$, where $g_0\in\widetilde{HV_b(G)}$.

That is $\tilde{v}(\phi(z_0))\leq\tilde{u}(z_0)$. This proves our claim.

Corollary: 2.5 Let G be a simply connected open subset of $\mathbb{C}$ and let V be an essential system of weights on G.

Let $\phi:G\to G$ be analytic map such that C_ϕ is composition operator on $HV_b(G)$. Then C_ϕ is invertible iff

(i) ϕ is a conformal mapping of G onto itself for every $z\in G$.

(ii) for each $v\in V$ there exists $u\in V\ni v(\phi(z))\leq u(z)$ for every $z\in G$.

Proof: Follows from the above theorem, since each $v\in V$ is essential.

3. DYNAMICAL SYSTEM AND WEIGHTED COMPOSITION OPERATOR:

Let $g\in H^\infty(G,B(E))$ and $\|g\|_\infty=\sup\{\|g(z)\|:z\in G\}$. Then for each $t\in\mathbb{R}$. We define $\psi_t:G\to B(E)$ as $\psi_t(z)=e^{tg(z)}\,\forall z\in G$ and let $\varphi_t:\mathbb{R}\to\mathbb{R}$ be the analytic map defined by $\varphi_t(\omega)=t+\omega\,\forall t,\omega\in\mathbb{R}$.

Theorem: 3.1 Let $\varphi:X\to X$ and $\psi_t:X\to X$ be continuous functions. Then $W_{\psi_t,\varphi}f$ is bounded $\forall t\in\mathbb{R}, f\in H(X)$.

Proof: We shall show that $W_{\psi_t,\varphi}f$ is continuous at the origin. We claim that $W_{\psi_t,\varphi}(B)\subseteq B$. We have

$$\begin{aligned}\left\|W_{\psi_t,\varphi}f\right\|_v&=\sup\{v(x)\left\|\psi_t(x)f(\varphi(x))\right\|:x\in X\ and\ t\in\mathbb{R}\}\\&\leq\sup\{v(x)\left\|\psi_t(x)\right\|\left\|f(\varphi(x))\right\|:x\in X\ and\ t\in\mathbb{R}\}\\&\leq\sup\{v(x)\left\|e^{tg(x)}\right\|\left\|f(x)\right\|:x\in X\ and\ t\in\mathbb{R}\}\\&\leq e^{|t|M}\sup\{v(x)\left\|f(x)\right\|x\in X\}\\&\leq\|f\|_v\leq 1\text{ as }t\to 0.\end{aligned}$$

Therefore $W_{\psi_t,\varphi}f$ is continuous at the origin. Hence proved.

Theorem: 3.2 Let $H^{\infty}(G,B(E))$ be the space of bounded analytic functions. Let $h_{\alpha}\left(\varphi_{t_{\alpha}}\right)$ converges to $h\left(\varphi_{t}\right)$ in $H^{\infty}(G,B(E))$ and let f_{α} be a sequence converging to f in $HV_{b}(G,E)$ .Then the product of $f_{\alpha}h_{\alpha}\left(\varphi_{t_{\alpha}}\right)$ converges to $fh\left(\varphi_{t}\right)$ in $HV_{b}(G,E)$.

Proof: Let $v\in V$.
Then

$$\begin{aligned}\left\|f_{n}h_{n}\left(\varphi_{t_{n}}\right)-fh\left(\varphi_{t}\right)\right\|_{v,E} &= \sup\left\{v(z)\left\|f_{n}(z)h_{n}\left(\varphi_{t_{n}}(z)\right)-f(z)h\left(\varphi_{t}(z)\right)\right\|:z\in G\right\}\\ &= \sup\left\{v(z)\left\|f_{n}(z)h_{n}\left(t_{n}+z\right)-f(z)h\left(t+z\right)\right\|:z\in G\right\}\\ &\leq \sup\left\{v(z)\left\|f_{n}(z)h_{n}(t_{n})h_{n}(z)-f_{n}(z)h(t)h(z)\right\|:z\in G\right\}\\ &\quad+\sup\left\{v(z)\left\|f_{n}(z)h(t)h(z)-f(z)h(t)h(z)\right\|:z\in G\right\}\\ &\quad+\sup\left\{v(z)\left\|f_{n}(z)-f(z)\right\|\left\|h(t)h(z)\right\|:z\in G\right\}\\ &\leq \sup\left\{v(z)\left\|f_{n}(z)\right\|\left\|h_{n}(t_{n})h_{n}(z)-h(t)h(z)\right\|:z\in G\right\}\\ &\leq \left\|h_{n}\left(\varphi_{t_{n}}\right)-h\left(\varphi_{t}\right)\right\|_{\infty}\left\|f_{n}\right\|_{v}+\left\|h\left(\varphi_{t}\right)\right\|_{\infty}\left\|f_{n}-f\right\|_{v}\to 0\end{aligned}$$

as $\left\|h_{n}\left(\varphi_{t_{n}}\right)-h\left(\varphi_{t}\right)\right\|_{\infty}\to 0$ and $\left\|f_{n}-f\right\|_{v}\to 0$.

Theorem: 3.3 Let V be an arbitrary system of weights on G and let $\pi:\mathbb{R}\times HV_{b}(G,E)\to H(G,E)$ be the function defined by $\pi(t,f)=W_{\psi_{t},\varphi_{t}}f$ for every $t\in\mathbb{R}\, and\, f\in HV_{b}(G,E)$. Then π is linear dynamical system on $HV_{b}(G,E)$.

Proof: Since, for every $t\in\mathbb{R}, W_{\psi_{t},\varphi_{t}}$ is a weighted composition operator on $HV_{b}(G,E)$. Thus it follows that $\pi(t,f)\in HV_{b}(G,E)$ for every $t\in\mathbb{R}$ and $f\in HV_{b}(G,E)$.

Clearly π is linear and

$$\begin{aligned}\pi(0,f)(z) &= W_{\psi_{0},\varphi_{0}}f(z)\ for\, all\ z\in G\\ &= f(z)\ for\, all\ z\in G\\ &= \psi_{0}(z)f(z).\end{aligned}$$

Therefore $\pi(0,f)=f$.

Similarly $\pi(t+s,f)=\pi(t,\pi(s,f))\ for\, all\ t,s\in\mathbb{R}$.

Next to show that π is dynamical system, it sufficient to show that π is jointly continuous [7]. Let the sequence $\left\{(t_{\alpha},f_{\alpha})\right\}$ be a net in $\mathbb{R}\times HV_{b}(G,E)$ such that $(t_{\alpha},f_{\alpha})\to(t,f)$.

Let $v\in V$. Then

$$\begin{aligned}\left\|\pi(t_{\alpha},f_{\alpha})-\pi(t,f)\right\|_{v,E} &= \left\|W_{\psi_{t_{\alpha}},\varphi_{t_{\alpha}}}f_{\alpha}-W_{\psi_{t},\varphi_{t}}f\right\|_{v,E}\\ &= \sup\{v(z)\left\|\psi_{t_{\alpha}}(z)f_{\alpha}(\varphi_{t_{\alpha}}(z))-\psi_{t}(z)f(\varphi_{t}(z))\right\|:z\in G\}\\ &\leq \sup\{v(z)\left\|\psi_{t_{\alpha}}(z)f_{\alpha}(\varphi_{t_{\alpha}}(z))-\psi_{t}(z)f_{\alpha}(\varphi_{t_{\alpha}}(z))\right\|:z\in G\}\end{aligned}$$

$$
\begin{aligned}
&+\sup\{v(z)\left\|\psi_t(z)f_\alpha(\varphi_{t_\alpha}(z))-\psi_t(z)f(\varphi_t(z))\right\|: z\in G\}\\
&\leq \sup\{v(z)\left\|\psi_{t_\alpha}(z)-\psi_t(z)\right\|\left\|f_\alpha(\varphi_{t_\alpha}(z))\right\|: z\in G\}\\
&+\sup\{v(z)\left\|\psi_t(z)\right\|\left\|f_\alpha(\varphi_{t_\alpha}(z))-f(\varphi_t(z))\right\|: z\in G\}\\
&\leq \sup\{\left\|e^{(t_\alpha-t)g(z)}\right\|v(z)\left\|f_\alpha(t_\alpha+z)\right\|: z\in G\}\\
&+\sup\{\left\|e^{tg(z)}\right\|v(z)\left\|f_\alpha(t_\alpha+z)-f(t+z)\right\|: z\in G\}\\
&\leq e^{|t|\|g\|_\infty}(e^{|t_\alpha-t|\|g\|_\infty}-1)v(z)\left\|f_\alpha(\varphi_{t_\alpha})\right\|_{v,E}\\
&+e^{|t|\|g\|_\infty}\left\|f_\alpha(\varphi_{t_\alpha})-f(\varphi_t)\right\|_{v,E}\to 0.
\end{aligned}
$$

Since by theorem: 3.2. This shows that π is jointly continuous and hence π is a linear dynamical system on $HV_b(G,E)$.

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