International Journal of Mathematical Archive-2(12), 2011, Page: 2549-2560
IJMA Available online through www.ijma.info ISSN 2229 – 5046

FUZZY RANKING METHOD TO ASSIGNMENT PROBLEM WITH FUZZY COSTS

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(Received on: 19-11-11; Accepted on: 10-12-11)

ABSTRACT

This paper presents solution methodology for assignment problem with fuzzy cost. The fuzzy costs are considered as trapezoidal fuzzy numbers. Ranking method (introduced by S. Abbasbandy and T. Hajjary, 2009) has been used for ranking the trapezoidal fuzzy numbers. Hungarian method is extended to solve this type of fuzzy assignment problem. Two Numerical examples are solved to check the validity of the proposed method.

Keywords: *Assignment problem, Fuzzy assignment problem, Hungarian method, Trapezoidal fuzzy number.*

AMS subject Classification: *90B06.*

1. INTRODUCTION:

Assignment problem (AP) is a particular type of transportation problem where n jobs are to be assigned to an equal number of n workers in one to one basis such that the assignment cost (or profit) is minimum (or maximum). The Hungarian method [1] is one of the classical methods, which is used to solve different types of APs. In this paper, we consider more realistic AP with fuzzy cost. Hungarian method is extended to deal with fuzzy assignment problem (FAP). We consider the fuzzy cost $\tilde{C}_{ij}$ as trapezoidal fuzzy number (TrFN).

Kuhn [1] introduced an algorithm so called Hungarian method for solving an AP in crisp environment. However, decision-making unit always comes in close contact with environment where fuzziness is common in realistic decision-making situation. Bellman and Zadeh [2] discussed decision making in fuzzy environment in 1970. Lin and Wen [3] solved AP with fuzzy interval cost by a labelling algorithm. They showed that AP can usually be simplified into either a linear fractional programming problem or a bottleneck assignment problem. Wang [4] used graph theory to solve FAP. Chen [5] proposed a FAP by considering that all the individuals involved have same skills. Sakwa et al. [6] presented interactive fuzzy programming approach to solve two levels AP.

Mukherjee and Kajla [7] used Yager's ranking method [8] for solving FAP. They transform FAP into a crisp AP and solve it by simplex method. The same procedure namely robust ranking method of Yager [8] is used to solve FAP by Nagarajan and Solairaju [9].

Ranking of fuzzy numbers are introduced in different forms in the literature [8], [10], [11]. In this paper, we proposed a extended Hungarian method for FAP. Here, fuzzy numbers are ranked by ranking method introduced by Abbasbandy and Hajjari [12]. Alternative solution of FAP is also discussed. To demonstrate the efficiency of the proposed method, two numerical examples are solved where all costs are considered as trapezoidal fuzzy numbers (TrFNs).

Rest of the paper is organized in the following way: Section 2 describes the preliminaries of fuzzy sets required for the paper. Subsections 2.4- 2.13 present the fuzzy arithmetic. Section 3 represents the formulation of FAP. Section 4 is devoted to present the proposed extended Hungarian method to FAP. In Section 5, two problems on fuzzy assignment problems are solved to demonstrate the efficiency of the proposed method. Finally, Section 6 presents the concluding remarks.

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2. PRELIMINARIES OF FUZZY SETS:

In 1965, Zadeh [13] introduced the concept of fuzzy set as a mathematical way of representing impreciseness in real world problems.

2.1 Definition: Fuzzy set: A fuzzy set $\tilde{A}$ in a universe of discourse X is defined by $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\langle \mid x\in X\}$, where $\mu_{\tilde{A}}(x): X \to [0, 1]$ is called the membership function of $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is the degree of membership to which x∈ $\tilde{A}$.

2.2 Definition: Normal fuzzy set**:** A fuzzy set $\tilde{A}$ is said to normal if there exists a point x in X such that $\mu_{\tilde{A}}(x) = 1$.

Otherwise $\tilde{A}$ is said to be the subnormal fuzzy set.

2.3 Definition: Trapezoidal fuzzy number: A trapezoidal fuzzy number $\tilde{a}(x)$ is denoted by (a_1, a_2, a_3, a_4) where a_1, a_2, a_3, a_4 are real numbers and its membership function

$\mu_{\tilde{a}}(x)$ is given by $$\mu_{\tilde{a}}(x) = \begin{cases} 0, & x \le a_1, \\ \mu_L(x) = \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2, \\ 1, & a_2 \le x \le a_3, \\ \mu_U(x) = \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4, \\ 0, & x \ge a_4 \end{cases}$$

$\mu_{\tilde{a}}(x)$ (see Figure1) satisfies the following conditions.

(i) $\mu_{\tilde{a}}(x)$ is a continuous mapping from R to closed interval [0, 1]

(ii) $\mu_{\tilde{a}}(x) = 0$ for every $x \in (-\infty, a_1]$

(iii) $\mu_{\tilde{a}}(x)$ is strictly increasing and continuous on $[a_1, a_2]$

(iv) $\mu_{\tilde{a}}(x) = 1$ for every $x \in [a_2, a_3]$

(v) $\mu_{\tilde{a}}(x)$ is strictly decreasing and continuous on $[a_3, a_4]$

(vi) $\mu_{\tilde{a}}(x) = 0$ for every $x \in [a_4, \infty)$

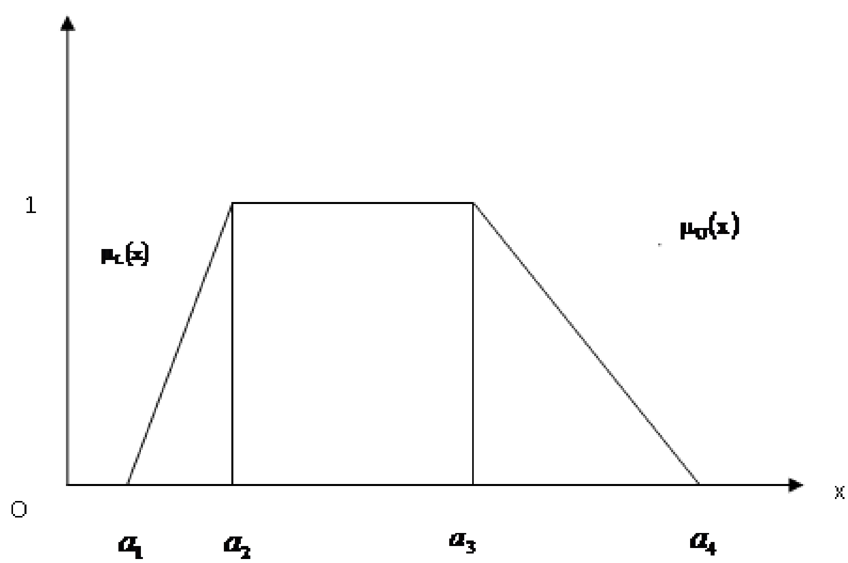

Figure-1: Trapezoidal fuzzy number $\tilde{a}$

2.4 Arithmetic Operation:

Fuzzy arithmetic is based on the extension principle introduced by Zadeh in 1975 [13]. The arithmetic operations of addition, subtraction, multiplication, and division are discussed in [12].

2.4.1 Addition: Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be trapezoidal fuzzy numbers, then

$$\tilde{A} + \tilde{B} = (a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4)$$
$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

2.4.2 Subtraction: $\tilde{A} - \tilde{B} = (a_1, a_2, a_3, a_4) - (b_1, b_2, b_3, b_4)$

$$= (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$

2.4.3 Multiplication:

$$\tilde{A}_1 \otimes \tilde{A}_2 = (a_1, a_2, a_3, a_4) \otimes (b_1, b_2, b_3, b_4)$$
$$=\{\min (a_1b_1, a_1b_4, a_4b_1, a_4b_4), \min(a_2b_2, a_2b_3, a_3b_2, a_3b_3), \max(a_2b_2, a_2b_3, a_3b_2, a_3b_3), \max(a_1b_1, a_1b_4, a_4b_1, a_4b_4)\}$$

2.5 Definition : Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number, then the defuzzified value or the crisp value of $\tilde{a}$, is denoted by

$$\text{Defuzz}(\tilde{a}) = \left(\frac{a_1 + 2a_2 + 2a_3 + a_4}{6}\right) \tag{1}$$

2.6 Definition: If $\tilde{a} = (a_1, a_2, a_3, a_4)$ is a trapezoidal fuzzy number, then $\tilde{U} = (x_0 - \beta, x_0, y_0, y_0 + \beta)$ with parametric form $\tilde{U} = (\underline{U}(r), \overline{U}(r))$ where $\underline{U}(r) = (x_0 - \beta + \beta r)$ and $\overline{U}(r) = (y_0 + \gamma - \gamma r)$ are defined as

$Mag(\tilde{U}) = \frac{1}{2}\left(\int_0^1 \left(\underline{U}(r) + \overline{U}(r) + x_0 + y_0\right) r dr\right)$ where, $r \in [0,1]$, The magnitude of the trapezoidal fuzzy number

$\tilde{U} = (a_1, a_2, a_3, a_4)$ is given by Mag $(\tilde{U}) = \dfrac{a_1 + 5a_2 + 5a_3 + a_4}{12}$ (2)

2.7 Definition: Let $\tilde{U}$ and $\tilde{V}$ be two trapezoidal fuzzy numbers. The ranking of $\tilde{U}$ and $\tilde{V}$ by the Mag (.) on E, the set of trapezoidal fuzzy numbers is defined as follows:

(i) Mag ($\tilde{U}$) > Mag ($\tilde{V}$) iff $\tilde{U} \succ \tilde{V}$
(ii) Mag ($\tilde{U}$) < Mag ($\tilde{V}$) iff $\tilde{U} \prec \tilde{V}$
(iii) Mag ($\tilde{U}$) = Mag ($\tilde{V}$) iff $\tilde{U} \approx \tilde{V}$

2.8 Definition: The ordering $\geq$ and $\leq$ between any two trapezoidal fuzzy numbers $\tilde{U}$ and $\tilde{V}$ are defined as follows:

(i) $\tilde{U} \succcurlyeq \tilde{V}$ iff $\tilde{U} \succ \tilde{V}$ or $\tilde{U} \approx \tilde{V}$
(ii) $\tilde{U} \preccurlyeq \tilde{V}$ iff $\tilde{U} \prec \tilde{V}$ or $\tilde{U} \approx \tilde{V}$

2.9 Definition:

(i) $\tilde{U} = (a_1, a_2, a_3, a_4) \approx \tilde{0}$ iff Mag ($\tilde{U}$) = 0
(ii) $\tilde{U} = (a_1, a_2, a_3, a_4) \succcurlyeq \tilde{0}$ iff Mag ($\tilde{U}$) ≥ 0
(iii) $\tilde{U} = (a, b, c, d) \preccurlyeq \tilde{0}$ iff Mag ($\tilde{U}$) ≤ 0

2.10 Definition: A trapezoidal fuzzy number $\tilde{U} = (a_1, a_2, a_3, a_4)$ is said to be non-negative if Mag ($\tilde{U}$) ≥ 0.

2.11 Definition: Let $\{\tilde{a}_i, i = 1, 2, 3, ..., n\}$ be a set of trapezoidal fuzzy numbers. If Mag ($\tilde{a}_k$) $\leq$ Mag ($\tilde{a}_i$) for all i, then the trapezoidal fuzzy number $\tilde{a}_k$ is the minimal of $\tilde{a}_i$ (i = 1, 2,…, n).

2.12 Definition: Let $\tilde{a}_i$ (i = 1, 2… n) be a set of trapezoidal fuzzy numbers.

If Mag ($\tilde{a}_k$) $\geq$ Mag ($\tilde{a}_i$) for all i, then the trapezoidal fuzzy number $\tilde{a}_k$ is the maximal of $\tilde{a}_i$ (i = 1, 2… n)

2.13 Definition: If $\tilde{X} = a\tilde{W} + b\tilde{Z}$ and $\tilde{P} = g\tilde{Q} - h\tilde{R}$ where $\tilde{X}, \tilde{W}, \tilde{Z}, \tilde{P}, \tilde{Q}$ and $\tilde{R}$ fuzzy numbers and a, b, g, h are constants. Then we have, Mag $(\tilde{X}) = a\mathrm{Mag}(\tilde{W}) + b\mathrm{Mag}(\tilde{Z})$ and $\mathrm{Mag}(\tilde{P}) = g\mathrm{Mag}(\tilde{Q}) - h\mathrm{Mag}(\tilde{R})$. Therefore Mag (.) satisfies the linearity and additive properties.

3. FORMULATION OF THE ASSIGNMENT PROBLEM:

Suppose for n jobs, n persons are available with different skills. Let the cost of doing j-th work by i-th person be c_{ij}. The cost matrix is given in Table 1.

Table-1: Cost matrix of assignment problem

job / Worker	job-1	job-2	job-3	... job-j…	job-n
worker-1	c_{11}	c_{12}	c_{13}	…c_{1j}…	c_{1n}
worker-2	c_{21}	c_{22}	c_{23}	…c_{2j}…	c_{2n}
. . . worker-i . .	. . . c_{i1} .	. . . c_{i2} . .	. . . c_{i3} . .	. . . …c_{ij}… . .	. . . c_{in} . .
. . . worker-n	. . . c_{n1}	. . . c_{n2}	. . . c_{n3}	. . . …c_{ni}…	. . . c_{nn}

Now the problem is which work is to be assigned to whom so that the cost of completion of work will be minimal. Mathematically, the problem can be expressed as:

$$\text{minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \ [\ i = 1,2, \dots, n\ ; j = 1, 2, \dots, n\ ] \tag{3}$$

where $x_{ij} = 1$; if i-th person is assigned to j-th job.
= 0; if otherwise.
subject to

(i) $\sum_{j=1}^{n} x_{ij} = 1$(one job is done by the i-th person)

(ii) $\sum_{i=1}^{n} x_{ij} = 1$(only one person should be assigned to the j-th job)

where x_{ij} denotes that the j-th job is to be assigned to the i-th person.

Now, if we consider the costs c_{ij} (i = 1, 2,..., n; j = 1,2,…, n) as TrFNs (see Table 2), then the total cost becomes a TrFN represented by $\tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij}$.

Then, FAP can be stated as follows:

$$\min \tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij}\ , i = 1, 2,..., n \text{ and } j = 1,2,..., n. \tag{4}$$

subject to $\sum_{i=1}^{n} x_{ij} = 1$, i = 1 , 2, ..., n

$\sum_{i=1}^{n} x_{ij} = 1$, j = 1, 2, …,n

$x_{ij} \in [0, 1]$, where where $x_{ij} = 1$; if i-th person is assigned to j-th job.
= 0; if otherwise.

Classical Hungarian method cannot be used directly to solve FAP. Therefore, we need to extend the Hungarian method to deal with FAP. The cost matrix is given in Table 2.

Table-2: Cost matrix of fuzzy assignment problem

Worker \ job	job-1	job-2	job-3	... job-j…	job-n
Worker-1	$\tilde{c}_{11}$	$\tilde{c}_{12}$	$\tilde{c}_{13}$	$\ldots\tilde{c}_{1j}\ldots$	$\tilde{c}_{1n}$
Worker-2	$\tilde{c}_{21}$	$\tilde{c}_{22}$	$\tilde{c}_{23}$	$\ldots\tilde{c}_{2j}\ldots$	$\tilde{c}_{2n}$
. . . Worker-i . .	. . . $\tilde{c}_{i1}$ . .	. . . $\tilde{c}_{i2}$ . .	. . . $\tilde{c}_{i3}$ . .	. . . $\ldots\tilde{c}_{ij}\ldots$ . .	. . . $\tilde{c}_{in}$ . .
Worker-n	$\tilde{c}_{n1}$	$\tilde{c}_{n2}$	$\tilde{c}_{n2}$	$\ldots\tilde{c}_{nj}\ldots$	$\tilde{c}_{nn}$

Theorem: 3.1 (Reduction Theorem): In a fuzzy assignment problem, we assume that the fuzzy cost matrix consists of only TrFNs. If we add or subtract a TrFN to every element of any row (or column) of the fuzzy cost or profit matrix $\tilde{c} = [\,\tilde{c}_{ij}\,]_{n\times n}$, then a fuzzy assignment that minimizes the total fuzzy cost on one fuzzy cost matrix also minimizes the total fuzzy cost on the other fuzzy cost matrix, where $\tilde{c}_{ij}$ are TrFNs. In other words, if $x_{ij} = x^*_{ij}$ minimizes

$\tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij}$ $\tilde{z}^* =$, $\sum_{i=1}^{n} x_{ij} = 1$ and $\sum_{j=1}^{n} x_{ij} = 1$, $x_{ij} = 0$, or 1; then x^*_{ij} also minimizes $\tilde{z}^* = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}{}^* x_{ij}$. where

$\tilde{c}_{ij}{}^* = \tilde{c}_{ij} - \tilde{u}_i - \tilde{v}_j$, for all i, j = 1, 2, …, n and $\tilde{u}_{ij}$ and $\tilde{v}_{ij}$ are some TrFNs.

Proof: We write $\tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij}$ and $\tilde{z}^* = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}{}^* x_{ij}$

$$\tilde{z}^* = \sum_{i=1}^{n}\sum_{j=1}^{n} (\tilde{c}_{ij} - \tilde{u}_i - \tilde{v}_j)\, x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{u}_i x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{v}_j x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij} - \sum_{i=1}^{n}\tilde{u}_i \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{n} x_{ij} \sum_{j=1}^{n} \tilde{v}_j$$

$$= \tilde{Z} - \sum_{i=1}^{n}\tilde{u}_i - \sum_{j=1}^{n}\tilde{v}_j, \text{ since } \sum_{i=1}^{n} x_{ij} = 1 \text{ and } \sum_{j=1}^{n} x_{ij} = 1.$$

As $\sum_{i=1}^{n}\tilde{u}_i$ and $\sum_{j=1}^{n}\tilde{v}_j$ are TrFNs, $\tilde{Z}$ and $\tilde{Z}^*$ differ only by a TrFN.

The difference between $\tilde{Z}$ and $\tilde{Z}^*$ is independent of x_{ij}. Hence, an assignment x_{ij} that minimizes $\tilde{Z}$ will also minimize $\tilde{Z}^*$. Therefore, the optimal assignment of the original problem must be the optimal assignment of the new problem.

Theorem: 3.2 If for a fuzzy assignment problem, all $\tilde{c}_{ij} \geq \tilde{0}$ and we can find a set $x_{ij} = x^*_{ij}$ such that $\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x^*_{ij} = \tilde{0}$, then the solution is optimal.

Proof: Since we have $\tilde{c}_{ij} \geq \tilde{0}$ and also $x_{ij} \geq 0$, so $\tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}x_{ij}$, is non-negative. Hence, its minimum value is fuzzy zero, which is obtained by using $x_{ij} = x^*_{ij}$. Therefore, the present solution is an optimal solution.

4. THE COMPUTATIONAL PROCEDURE OF EXTENDED HUNGARIAN METHOD FOR SOLVING FUZZY ASSIGNMENT PROBLEM

Here we present systematic procedure for solving fuzzy assignment problem that we call extended Hungarian method. Computational procedure for obtaining a fuzzy optimal solution is summarized below:

Step: 1 Determine the fuzzy cost matrix from given problem.

(i) If the number of sources is equal to the number of destinations, go to Step 3.
(ii) If the number of sources is not equal to the number of destinations, go to Step 2.

Step: 2 Add a dummy source, or dummy destination, so that the fuzzy cost matrix reduces to a square fuzzy cost matrix. The cost entries of dummy sources/destinations are always fuzzy zeros.

Step: 3 Locate the minimum fuzzy cost element in each row of the given cost matrix with the help of magnitude defined by the equation (2). Subtract this minimum fuzzy cost element from each fuzzy element of that row. Therefore, there will be at least one fuzzy zero in each row of this new matrix which is called the first reduced cost matrix.

Step: 4 In the reduced fuzzy cost matrix obtained in Step 3, locate the smallest fuzzy cost element of each column and then subtract the same from each fuzzy cost element of that column. As a result, each column and row of the second reduced fuzzy cost matrix now have at least one fuzzy zero. It is also noticed that the rest of the fuzzy cost elements whose magnitude defined by the equation (2) are zeros, they are converted to fuzzy zero cost elements.

Step: 5 In the modified fuzzy cost matrix obtained in Step 4, search for an optimal assignment as follows:
(a) Examine the rows successively until a row with single fuzzy zero is found. Mark this element with asterisk (*) and cross off $(\times)$all other fuzzy zeros in its column. Continue this process until all the rows have been considered.
(b) Repeat the procedure for each column of the reduced fuzzy cost matrix.
(c) If a row or a column has two or more fuzzy zeros and one cannot be chosen by inspection then assign arbitrarily any one of these fuzzy zeros and cross of all other fuzzy zeros of that column/row.
(d) Repeat (a) through (c) above successively until the chain of assigning asterisk (*) or cross off $(\times)$ ends.

Step: 6 If the number of assignments (*) is equal to n (the order of fuzzy cost matrix), optimal solution is reached. If the number of assignments (*) is not equal to n (the order of fuzzy cost matrix), then it is required to go the Step 7.

Step: 7 To make zero assignment:
Draw the minimum number of horizontal and / or vertical lines to cover all the fuzzy zeros of the reduced fuzzy cost matrix. This can be done by using a simple procedure:
(a) Mark ($\longleftarrow$) rows that do not have any assigned fuzzy zero.
(b) Mark ($\downarrow$) columns that have fuzzy zeros in the marked rows.
(c) Mark ($\longleftarrow$) rows that have assigned fuzzy zeros in the marked columns.
(d) Repeat (b) and (c) of Step 7 until the chain of marking is completed.
(e) Draw lines through all the unmarked rows and marked columns. This provides us the desired minimum number of lines.

Step: 8 If the number of drawn lines are less than the number of rows or columns, then we cannot make assignment.

Hence, the following procedure is to be followed:

The cells covered by the lines are known as covered cells. The cells, which are not covered by lines, are known as uncovered cells. The cell at the intersection of horizontal line and vertical line are known as crossed cell.
(a) Identify the smallest element in the uncovered cells.
(b) Subtract this element from the elements of all other uncovered cells.
(c) Add this element to the elements of the crossed cells.
(d) Do not alter the elements of covered cells.

Step: 9 Go to Step 6 and repeat the procedure until an optimal solution is reached.

Note: Sometimes we may find more than one fuzzy zero in a row or column. It indicates that the problem has not unique optimal solution i.e.it has alternative optimal solutions. To get another alternative optimal solution, we mark asterisk (*) on the fuzzy zeros in such a way that each row and column consists of one and only asterisk (*) fuzzy zero except previous case. Then we will obtain another alternative optimal solution.

5. SOLUTION OF FUZZY ASSIGNMNET PROBLEMS:

Example 5.1 A departmental head has four subordinates namely A, B, C, D and four jobs namely Job-1, Job-2, Job-3, and Job-4 to be performed. The subordinates differ in efficiency and the jobs differ in their intrinsic difficulty. The fuzzy cost matrix ($\tilde{C}_{ij}$) is given below (see Table 3) whose elements are trapezoidal fuzzy numbers. Determine the optimal assignment schedule.

Table-3: Fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	(4,6,7,9)	(6,8,12,14)	(10,11,13,15)	(6,8,10,12)
B	(7,8,11,13)	(4,6,7,8)	(6,8,9,11)	(5,8,9,11)
C	(2,4,5,7)	(5,7,10,12)	(8,10,12,14)	(4,6,7,9)
D	(6,8,10,12)	(3,5,6,7)	(5,7,10,12)	(3,4,6,8)

Solution:

Step: 1 It is completed since number of rows and columns are equal.

Step: 2 It is not required due to equality of rows and column.

Step: 3 We calculate the magnitude of each element of the fuzzy cost matrix by using the equation (2).

Here, Mag ($\tilde{c}_{11}$) = $\dfrac{4+5\times6+5\times7+9}{12} = \dfrac{78}{12}$,

Similarly, we get Mag ($\tilde{c}_{1i}$) = $\dfrac{120}{12}, \dfrac{145}{12}, \dfrac{108}{12}$ for i = 2, 3, 4 respectively.

Mag ($\tilde{c}_{2i}$) = $\dfrac{115}{12}, \dfrac{77}{12}, \dfrac{102}{12}, \dfrac{101}{12}$, for i =1, 2, 3, 4

Mag ($\tilde{c}_{3i}$) = $\dfrac{54}{12}, \dfrac{102}{12}, \dfrac{132}{12}, \dfrac{78}{12}$, for i = 1, 2, 3, 4

Mag ($\tilde{c}_{4i}$) = $\dfrac{108}{12}, \dfrac{65}{12}, \dfrac{102}{12}, \dfrac{61}{12}$, for i = 1, 2, 3, 4

Now, using the Step 3, we have the following fuzzy cost matrix:

Table-4: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	$(\tilde{0})$	(-3, 1, 6, 10)	(1, 4, 7, 11)	(-3, 1, 4, 8)
B	(-1, 1, 5, 9)	$(\tilde{0})$	(-2, 1, 3, 7)	(-3, 1, 3, 7)
C	$(\tilde{0})$	(-2, 2, 6, 10)	(1, 5, 8, 12)	(-3, 1, 3, 7)
D	(-2, 2, 6, 9)	(-5,-1, 2, 4)	(-3, 1, 6, 9)	$(\tilde{0})$

Similarly, computing we get min {Mag($\tilde{c}_{i3}$)} = min { $\dfrac{67}{12}, \dfrac{25}{12}, \dfrac{78}{12}, \dfrac{41}{12}$ },for i =1, 2, 3, 4

$$= \frac{25}{12} = \{\text{Mag}(\tilde{c}_{23})\}.$$

Step 4: Here, in the reduced fuzzy cost matrix obtained in Step 3, we locate the smallest fuzzy cost element of each column and then subtract the smallest fuzzy cost element from each fuzzy cost element of that column. We obtain the following fuzzy cost matrix (see Table 5).

Table- 5: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	$(\tilde{0})$	(-3, 1, 6, 10)	(-6, 1, 6, 13)	(-3, 1, 4, 8)
B	(-1, 1, 5, 9)	$(\tilde{0})$	$(\tilde{0})$	(-3, 1, 3, 7)
C	$(\tilde{0})$	(-2, 2, 6, 10)	(-6, 2, 7, 14)	(-3, 1, 3, 7)
D	(-2, 2, 6, 9)	(-5, -1, 2, 4)	(-10, -2, 5, 11)	$(\tilde{0})$

We observe that each row and column of above reduced cost matrix, consists of at least one fuzzy zero. We also see that there is no fuzzy cost element with magnitude zero.

Step 5: Starting with the first row, we mark (*) a single fuzzy zero if it comprises of any fuzzy zero and cross $(\times)$ all other fuzzy zeros in the column so marked. Similarly, we do the same for the remaining rows. We also observe that row 2 comprises of two fuzzy zeros. Thus, we arbitrarily mark (*) on a fuzzy zero in column 2 and cross on a fuzzy zero in column 3. Here, row 3 and column 3 do not have any assignment. Then, we have the following Table 6.

Table- 6: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	$(\tilde{0}^*)$	(-3, 1, 6, 10)	(-6, 1, 6, 13)	(-3, 1, 4, 8)
B	(-1, 1, 5, 9)	$(\tilde{0}^*)$	$(\tilde{0}^\times)$	(-3, 1, 3, 7)
C	$(\tilde{0}^\times)$	(-2, 2, 6, 10)	(-6, 2, 7, 14)	(-3, 1, 3, 7)
D	(-2, 2, 6, 9)	(-5, -1, 2, 4)	(-10, -2, 5, 11)	$(\tilde{0}^*)$

Step: 6 Here, the number of assignment (= 3) $\neq$ order (n = 4) of the fuzzy cost matrix, therefore assignment cannot be made. So we move to next step.

Step: 7

(i) Since row 3 does not have any assigned fuzzy zero, we mark this row with this sign ($\leftarrow$).

(ii) Now there is a fuzzy zero in the first column of the marked row, so we mark first column with ($\downarrow$).

(iii) There is an assignment of the first row of the ticked column. So we mark first row with ($\leftarrow$).

(iv) We draw straight lines through second and fourth row and first column. Then, we get the following table 7.

Table- 7: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
← A	$(\tilde{0}^*)$	--	--	--
B	--	$(\tilde{0}^*)$	$(\tilde{0}^\times)$	--
← C	$(\tilde{0}^\times)$	--	--	--
D		--	--	$(\tilde{0}^*)$
	↓			

Step: 8 Here, the minimum number of drawn lines (N = 3) $\neq$ order (n = 4) of the fuzzy cost matrix, therefore assignment cannot be made. It indicates that the current assignment is not optimum. So, we determine the magnitude of all uncovered fuzzy cost elements by these three lines and find the minimum magnitude.

Here min {Mag ($\tilde{c}_{12}$), Mag ($\tilde{c}_{13}$), Mag ($\tilde{c}_{32}$), Mag ($\tilde{c}_{33}$), Mag ($\tilde{c}_{42}$), Mag ($\tilde{c}_{43}$)}

= Mag ($\tilde{c}_{42}$) = 4/12

Subtract minimum fuzzy cost element from all uncovered elements and add this to the element of the crossed cells.

Then, we have another modified fuzzy cost matrix (8).

Table- 8: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	$(\tilde{0})$	(-7,-1, 7, 15)	(-10,-1, 7, 18)	(-3, 1, 4, 8)
B	(-6, 0, 7, 13)	$(\tilde{0})$	$(\tilde{0})$	(-8, 0, 5, 11)
C	$(\tilde{0})$	(-6, 0, 7, 15)	(-10, 0, 8, 19)	(-3, 1, 3, 7)
D	(-2, 2, 6, 9)	$(\tilde{0})$	(-14, -4, 6, 16)	$(\tilde{0})$

Once again, we cover all the fuzzy zeros by a minimum number of horizontal and vertical lines by using Step 5 and Step 7. Then, we obtain another modified Table 9.

Table 9: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
← A	$(\tilde{0}^{*})$	--	--	--
B	--	$(\tilde{0}^{\times})$	$(\tilde{0}^{*})$	--
C	$(\tilde{0}^{\times})$	--	--	--
D	--	$(\tilde{0}^{*})$	--	$(\tilde{0}^{\times})$

In this situation, the minimum number of drawn lines is not equal to the order of the fuzzy cost matrix (= 4). Therefore, we need to go again Step 8 of section 4.

In order to obtain an optimal assignment, we proceed as Step 8 and get another modified Table 10,

$\min\{\text{Mag}(\tilde{c}_{12}), \text{Mag}(\tilde{c}_{13}), \text{Mag}(\tilde{c}_{14}), \text{Mag}(\tilde{c}_{32}), \text{Mag}(\tilde{c}_{33}), \text{Mag}(\tilde{c}_{34})\} = \text{Mag}(\tilde{c}_{34}) = \frac{24}{12}$.

Table 10: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	$(\tilde{0})$	(-14, -4, 6, 18)	(-17, -4, 6, 21)	(-10, -2, 3, 11)
B	(-9, 1, 10, 20)	$(\tilde{0})$	$(\tilde{0})$	(-8, 0, 5, 11)
C	$(\tilde{0})$	(-13, -3, 6, 18)	(-17, 3, 7, 22)	$(\tilde{0})$
D	(-5, 3, 9, 16)	$(\tilde{0})$	(-14, -4, 6, 16)	$(\tilde{0})$

Now using Step 5 and Step 7, we have

Table- 11: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
← A	$(\tilde{0}^{*})$	--	--	--
B	--	$(\tilde{0}^{\times})$	$(\tilde{0}^{*})$	--
← C	$(\tilde{0}^{\times})$	--	--	$(\tilde{0}^{*})$
D	--	$(\tilde{0}^{*})$	--	$(\tilde{0}^{\times})$

Finally, we notice that the number of drawn lines (N = 4) equals to the order of cost matrix and each row and each column consists of one and only one fuzzy zero. Then, we write the table 11 as Table 12.

Table-12: Reduced fuzzy cost matrix.

	Job-1	Job-2	Job-3	Job-4
A	$(\tilde{0}^{*})$	--	--	--
B	--	$(\tilde{0}^{\times})$	$(\tilde{0}^{*})$	--
C	$(\tilde{0}^{\times})$	--	--	$(\tilde{0}^{*})$
D	--	$(\tilde{0}^{*})$	--	$(\tilde{0}^{\times})$

Therefore, the assignment corresponding to the asterisk (*) offers the optimal assignment as:

A → Job-1; B → Job-3; C → Job-4; D→ Job-2. It yields the fuzzy optimal total cost:

$$\tilde{C}_{13} + \tilde{C}_{22} + \tilde{C}_{31} + \tilde{C}_{44} = (4, 6, 7, 9) + (6, 8, 9, 11) + (4, 6, 7, 9) + (3, 5, 6, 7)$$

$$= (17, 25, 29, 36).$$

The defuzzified value of Z = $\dfrac{17+2\times(25+29)+36}{6}$ = 26.8333 units.

Example: 5.2 A Company has three tasks to be performed and 3 apprentices to do these tasks. The apprentices differ in efficiency and the tasks differ in their intrinsic difficulty. The cost matrix ($\tilde{C}_{ij}$) is given below (see Table 13) whose elements are trapezoidal fuzzy numbers. Determine the optimal assignment schedule.

Table: 13 Fuzzy cost matrix.

	task-1	task-2	task-3
A	(-2, 0, 2, 8)	(-1, 0, 6, 8)	(-2, 0, 2, 8)
B	(4, 8, 12, 16)	(2, 3, 4, 5)	(2, 3, 4, 5)
C	(0, 6, 8, 10)	(0, 6, 8, 10)	(6, 7, 9, 13)

Solution:

Step: 1 It is already completed since $\tilde{C}_{ij}$ is a square fuzzy cost matrix of order three.

Step: 2 It is not required as the number of rows and columns are equal.

Step: 3 We determine the magnitude of each fuzzy cost matrix by the equation (2).

Mag ($\tilde{c}_{1i}$) = $\frac{16}{12}, \frac{37}{12}, \frac{16}{12}$ for i =1, 2, 3, respectively.

So min (Mag ($\tilde{c}_{1i}$)) = Mag ($\tilde{c}_{11}$) = Mag ($\tilde{c}_{13}$) = $\frac{16}{12}$,

Similarly, min (Mag ($\tilde{c}_{2i}$)) = $\min\left(\frac{120}{12}, \frac{42}{12}, \frac{42}{12}\right) = \frac{42}{12}$ = Mag($\tilde{C}_{23}$) = Mag ($\tilde{c}_{22}$) for i = 1, 2, 3, and min (Mag ($\tilde{c}_{3i}$))

$= \min\left(\frac{80}{12}, \frac{80}{12}, \frac{89}{12}\right) = \frac{80}{12}$ = Mag($\tilde{C}_{31}$) = Mag ($\tilde{c}_{32}$).

Now, using the Step 3**,** we get the following fuzzy cost matrix (see Table 14):

Table-14: Reduced fuzzy cost matrix.

	task-1	task-2	task-3
A	$(\tilde{0})$	(-7, -2, 6, 10)	$(\tilde{0})$
B	(-4, 2, 8, 14)	$(\tilde{0})$	$(\tilde{0})$
C	$(\tilde{0})$	$(\tilde{0})$	(-4, -1, 3, 13)

Step 4: Since each column consists of fuzzy zero element, subtracting of smallest element i.e. fuzzy zero from other elements of the respective column of the fuzzy cost matrix, we obtain the same cost matrix (14) obtained in Step 3.

Step 5: Starting with first row, we mark (*) a single fuzzy zero if any and cross $(\times)$ all other fuzzy zeros in the column so marked. Similarly, we do the same for the remaining two rows. We also observe that row 2 consists of two fuzzy zeros. We arbitrarily mark (*) a fuzzy zero in column 3 and cross a fuzzy zero in column 2. Then, we have the following Table 15.

Table-15: Reduced fuzzy cost matrix.

	task-1	task-2	task-3
A	$(\tilde{0}^{*})$	(-7, -2, 6, 10)	$(\tilde{0}^{\times})$
B	(-4, 2, 8, 14)	$(\tilde{0}^{\times})$	$(\tilde{0}^{*})$
C	$(\tilde{0}^{\times})$	$(\tilde{0}^{*})$	(-4, -1, 3, 13)

Step 6: Here, the number of assignment (= 3) = order (n = 3) of the fuzzy cost matrix, therefore assignment can be made. Therefore, we have the optimum assignment schedule.

Therefore, we have the optimal assignment corresponding to the asterisk marked fuzzy zero as A→ task-1; B → task-3; C→task-2. It gives the fuzzy optimal total cost:

$\tilde{C}_{11} + \tilde{C}_{23} + \tilde{C}_{32}$ = (-2, 0, 2, 8) + (2, 3, 4, 5) + (0, 6, 8, 10) = (0, 9, 14, 23).

The defuzzified value of the objective function $Z = \dfrac{0+2\times(9+14)+23}{6}$ = 11.5 units.

Here, we see that there exist more than one fuzzy zeros in a row or column. It reflects that the solution is not unique.

Therefore, we can find another alternative solution (see table 16) as follows:

Table 16: Reduced fuzzy cost matrix.

	task-1	task-2	task-3
A	$(\tilde{0}^{\times})$	--	$(\tilde{0})^{*}$
B	--	$(\tilde{0})^{*}$	$(\tilde{0}^{\times})$
C	$(\tilde{0})^{*}$	$(\tilde{0}^{\times})$	--

Therefore, the other optimal assignment corresponding to the asterisk marked fuzzy zero as:

A → task-3; B → task-2; C → task-1. It offers the fuzzy optimal total cost:

$\tilde{C}_{13} + \tilde{C}_{22} + \tilde{C}_{31}$ = (-2, 0, 2, 8) + (2, 3, 4, 5) + (0, 6, 8, 10) = (0, 9, 14, 23).

Here, we notice that the defuzzified value of the objective function Z is 11.5 units.

6. CONCLUSIONS:

In this paper, extended Hungarian method is introduced to solve fuzzy assignment problems. The proposed method is simple and fruitful. Two fuzzy assignment problems are solved to demonstrate the efficiency of the proposed method. This method is a systematic procedure, which is easy to understand and can be easily applied. In the proposed method, it is not required to convert fuzzy costs to a crisp form to solve the problem. We hope that the proposed method presented here will contribute to future study in restricted assignment problem, travelling salesperson problems in fuzzy environment.

7. REFERENCE:

[1] Kuhn, H. W., The Hungarian method for assignment problem, Naval Research Logistics Quarterly. 2 (1955), 83-97.

[2] Bellman, R. R., and Zadeh, L.A., Decision making in a fuzzy environment, Management Science. 17(B) (1970), 141-164.

[3] Lin, C.J. and Wen, U.P., A labelling algorithm for the fuzzy Assignment problem, Fuzzy Sets and Systems.142 (2004), 373–391.

[4] Wang, X., Fuzzy optimal assignment problem, Fuzzy Math. 3(1987), 101–108.

[5] Chen., M. S., On a fuzzy assignment problem, Tamkang J. 22 (1985), 407– 411.

[6] Sakawa, M., Nishizaki, I., and Uemura, Y., Interactive fuzzy programming for two level linear and linear fractional production and assignment problems: a case study, European Journal of Operation Research. 135(2001), 142–157.

[7] Mukherjee, S., and K. Basu., Application of fuzzy ranking method for solving assignment problem with fuzzy costs, International Journal of Computational and Applied Mathematics. 5(3) (2010), 359-368.

[8] Yager, R. R., A procedure for ordering fuzzy subsets of the unit interval, Information Sciences. 24 (1981), 143-161.

[9] Nagarajan, R., and Solairaju, A., Computing improved fuzzy optimal Hungarian assignment problems with fuzzy costs under robust ranking techniques, International Journal of Computer Applications. 6(4) (2010), 6-13.

[10] Cheng, C. H., A new approach for ranking fuzzy number by distance method, Fuzzy Sets and Systems. 95 (1998), 307-317.

[11] Fortemps, P., and Roubens, M., Ranking and defuzzification methods based on area compensation, Fuzzy Sets and Systems. 82 (1996), 319-330.

[12] Abbasbandy, S., and T. Hajjari., A new approach for ranking of trapezoidal fuzzy numbers, Computers and Mathematics with Applications. 57 (2009), 413-419.

[13] Zadeh, L.A., Fuzzy sets, Information and Control. 8 (1965), 338–353.

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