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# COUPLED FIXED POINT THOEREMS IN PARTIALLY ORDERED 2-METRIC SPACE

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*ABSTRACT*

*In this paper, some existence theorems of coupled fixed points for mixed monotone operators are proved. We derive new coupled fixed point theorems for contractive mappings on 2-metric space.*

***Keywords:*** *Coupled fixed point, mixed monotone property, partially ordered set, 2-metric space.*

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## 1. INTRODUCTION:

Fixed point theory plays a major role in many applications, including variational and linear inequalities, optimization and applications in the field of approximation theory and minimum norm problem. S. Banach [1] proved the famous and well known Banach contraction principle concerning the fixed point of contraction mappings defined on a complete metric space. This theorem has been generalized and extended by many authors [2],[3],[4]. The concept of 2-metric space was initially given by Gahler [10] whose abstract properties were suggested by the area of function in Euclidean space. Iseki [11] set out the tradition of proving fixed point theorem in 2-metric spaces employing various contractive conditions. Ran and Reurings [5], Bhaskar and Lakshmikantham [6], Lakshmikantham and Ciric [7], Nguyen Van[8] presented some new results for contractions in partially ordered metric spaces .In [9] W. Shatanawi proved coupled fixed point theorem in Generalized Metric space. In the present paper, we prove a coupled fixed point theorem in the setting of 2-metric space

## 2. PRELIMINARIES:

**Definition: 2.1** Let X be a non-empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on $X$ if given distinct elements x, y of X, there exists an element z of $X$ such that

(i) $d(x,y,z) \neq 0$

(ii) $d(x,y,z) = 0$ when at least two of $x, y, z$ are equal,

(iii) $d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all $x, y, z$ in $X$ , and

(iv) $d(x,y,z) \leq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x, y, z, w$ in $X$ .

When d is a 2-metric on $X$ , then the ordered pair $(X,d)$ is called a 2-metric space.

**Definition: 2.2** A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence if for each $a \in X$ , $\lim d(x_n, x_m, a) = 0$ as $n, m \to \infty$ .

**Definition: 2.3** A sequence $\{x_n\}$ in $X$ is convergent to an element $x \in X$ if for each $a \in X$ $\lim d(x_n, x, a) = 0$ as $n \to \infty$ .

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**Definition: 2.4** A complete 2-metric space is one in which every Cauchy sequence in $X$ converges to an element of $X$.

**Example: 1.1** Let $R^2$ be the Euclidean space. Let $d(x, y, z)$ denote the area of the triangle formed by joining the three points $x, y, z \in R^2$.Then $(R^2, d)$ is a 2-metric space and $d(x, y, z) = 0$ for any three distinct points $x, y, z \in R^2$ lying on the same straight line.

**Definition: 2.5** An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

**3. MAIN RESULT:**

**Theorem: 3.1** Let $(X, \leq)$ be a partially ordered set and suppose there is a 2-metric d on X such that $(X, d)$ is a complete 2-metric space. Let $\Phi : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \leq \Phi(x_0, y_0)$ and $y_0 \geq \Phi(y_0, x_0)$

Suppose there exist non-negative real numbers $a_1, a_2 and a_3$ with $a_1 + a_2 < 1$ such that

$$d(\Phi(x, y), \Phi(l, m), t) \leq a_1 d(x, l, t) + a_2 d(y, m, t) + a_3 \min\{d(\Phi(x, y), l, t), d(\Phi(l, m), x, t) \\ d(\Phi(x, y), x, t), d(\Phi(l, m), l, t)\} \tag{3.1}$$

For all $x, y, l, m, t \in X$ with $x \geq l$ and $y \leq m$.Suppose either

(a) $\Phi$ is continuous or

(b) X has the following property:

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ for all n.

(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y_n \geq y$ for all n.

then $\Phi$ has a coupled fixed point in X that is there exist $x, y \in X$ such that $x = \Phi(x, y)$ and $y = \Phi(y, x)$.

**Proof:** Let $x_0, y_0 \in X$ be such that $x_0 \leq \Phi(x_0, y_0)$ and $y_0 \geq \Phi(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows,

$$x_{n+1} = \Phi(x_n, y_n) \text{ and } y_{n+1} = \Phi(y_n, x_n) \text{ for all } n \geq 0 \tag{3.2}$$

We shall show that $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for all $n \geq 0$ (3.3)

We shall use the mathematical induction

Let $n = 0$, since $x_0 \leq \Phi(x_0, y_0)$ and $y_0 \geq \Phi(y_0, x_0)$ And as $x_1 = \Phi(x_0, y_0)$ and $y_1 = \Phi(y_0, x_0)$, we have $x_0 \leq x_1$ and $y_0 \geq y_1$.

Thus (3.3) holds for $n = 0$.Now suppose that (3.3) holds for some fixed $n \geq 0$ .then since $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, and by the mixed monotone property of $\Phi$, we have

$$x_{n+2} = \Phi(x_{n+1}, y_{n+1}) \geq \Phi(x_n, y_{n+1}) \geq \Phi(x_n, y_n) = x_{n+1} \tag{3.4}$$

$$y_{n+2} = \Phi(y_{n+1}, x_{n+1}) \leq \Phi(y_n, x_{n+1}) \leq \Phi(y_n, x_n) = y_{n+1} \tag{3.5}$$

Thus by mathematical induction we conclude that (3.3) holds for all $n \geq 0$

Therefore $x_0 \le x_1 \le x_2 \le \ldots\ldots\ldots \le x_n \le x_{n+1}$ and $y_0 \le y_1 \le y_2 \le \ldots\ldots\ldots \le y_n \le y_{n+1}$ (3.6)

Since $x_n \ge x_{n-1}$ and $y_n \le y_{n-1}$

From (3.1) and (3.2), we have

$$d\left(\Phi\left(x_n, y_n\right), \Phi\left(x_{n-1}, y_{n-1}\right), t\right) \le a_1 d\left(x_n, x_{n-1}, t\right) + a_2 d\left(y_n, y_{n-1}, t\right) + a_3 \min\left\{d\left(\Phi\left(x_n, y_n\right), x_{n-1}, t\right), d\left(\Phi\left(x_{n-1}, y_{n-1}\right), x_n, t\right), d\left(\Phi\left(x_n, y_n\right), x_n, t\right), d\left(\Phi\left(x_{n-1}, y_{n-1}\right), x_{n-1}, t\right)\right\}$$

Or

$$d\left(x_{n+1}, x_n, t\right) \le a_1 d\left(x_n, x_{n-1}, t\right) + a_2 d\left(y_n, y_{n-1}, t\right) \tag{3.7}$$

Similarly since $y_{n-1} \ge y_n$ and $x_{n-1} \le x_n$, we have

$$d\left(\Phi\left(y_{n-1}, x_{n-1}\right), \Phi\left(y_n, x_n\right), t\right) \le a_1 d\left(y_{n-1}, y_n, t\right) + a_2 d\left(x_{n-1}, x_n, t\right) + a_3 \min\left\{d\left(\Phi\left(y_{n-1}, x_{n-1}\right), y_n, t\right), d\left(\Phi\left(y_n, x_n\right), y_{n-1}, t\right), d\left(\Phi\left(y_{n-1}, x_{n-1}\right), y_{n-1}, t\right), d\left(\Phi\left(y_n, x_n\right), y_n, t\right)\right\}$$

$$d\left(y_n, y_{n+1}, t\right) \le a_1 d\left(y_{n-1}, y_n, t\right) + a_2 d\left(x_{n-1}, x_n, t\right) \tag{3.8}$$

Adding (3.7) and (3.8) we get

$$d\left(x_{n+1}, x_n, t\right) + d\left(y_{n+1}, y_n, t\right) \le \left(a_1 + a_2\right)\left[d\left(x_n, x_{n-1}, t\right) + d\left(y_n, y_{n-1}, t\right)\right] \tag{3.9}$$

Set $d_n = d\left(x_{n+1}, x_n, t\right) + d\left(y_{n+1}, y_n, t\right)$ and $w = a_1 + a_2 < 1$

We have $0 \le d_n \le w d_{n-1} \le w^2 d_{n-2} \ldots\ldots\ldots \le w^n d_0$

This implies

$$\lim_{n\to\infty}\left[d\left(x_{n+1}, x_n, t\right) + d\left(y_{n+1}, y_n, t\right)\right] = \lim_{n\to\infty} d_n = 0$$

Thus, $\lim_{n\to\infty} d\left(x_{n+1}, x_n, t\right) = d\left(y_{n+1}, y_n, t\right) = 0$

For each $m > n$, we have

$$d\left(x_n, x_m, t\right) \le d\left(x_n, x_{n+1}, t\right) + d\left(x_{n+1}, x_{n+2}, t\right) \ldots\ldots. + d\left(x_{m-1}, x_m, t\right) \quad \text{and}$$

$$d\left(y_n, y_m, t\right) \le d\left(y_n, y_{n+1}, t\right) + d\left(y_{n+1}, y_{n+2}, t\right) \ldots\ldots. + d\left(y_{m-1}, y_m, t\right)$$

On adding we get

$$\begin{aligned} d\left(x_n, x_m, t\right) + d\left(y_n, y_m, t\right) &\le \left[d\left(x_n, x_{n+1}, t\right) + d\left(y_n, y_{n+1}, t\right)\right] + \left[d\left(x_{n+1}, x_{n+2}, t\right) + d\left(y_{n+1}, y_{n+2}, t\right)\right] + \\ &\quad \ldots.. + \left[d\left(x_{m-1}, x_m, t\right) + d\left(y_{m-1}, y_m, t\right)\right] \\ &= d_n + d_{n+1} + \ldots\ldots\ldots. + d_{m-1} \\ &\le \left(w^n + w^{n+1} + \ldots\ldots\ldots. w^{m-1}\right) d_0 \\ &\le \frac{w^n}{1-w} d_0 \end{aligned} \tag{3.10}$$

This implies that

$$\lim_{n\to\infty}\left[d\left(x_m, x_n, t\right) + d\left(y_m, y_n, t\right)\right] = 0$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$ .since $X$ is a complete 2-metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y . \tag{3.11}$$

Now suppose that the assumption (a) holds,taking the limit as $n \to \infty$ in (3.2) and by (3.11), we get

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \Phi(x_{n-1}, y_{n-1}) = \Phi\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}\right) = \Phi(x, y) \text{ and}$$

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \Phi(y_{n-1}, x_{n-1}) = \Phi\left(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}\right) = \Phi(y, x)$$

Thus we proved that $x = \Phi(x, y)$ and $y = \Phi(y, x)$.

Finally suppose that (b) holds. since $\{x_n\}$ is non-decreasing sequence and $x_n \to x$ and $\{y_n\}$ is non-increasing sequence and $\{y_n\} \to y$ .by assumption (b) we have $x_n \le x$ and $y_n \ge y$ , and

$$d(\Phi(x, y), \Phi(x_n, y_n), t) \le a_1 d(x, x_n, t) + a_2 d(y, y_n, t)$$
$$+a_3\{d(\Phi(x, y), x_n, t), d(\Phi(x_n, y_n), x, t), d(\Phi(x, y), x, t), d(\Phi(x_n, y_n), x_n, t)\}$$

Taking $n \to \infty$, we get $d(\Phi(x, y), x, t) \le 0$ This implies $\Phi(x, y) = x$.

Similarly we can show that $\Phi(y, x) = y$ .therefore we have proved that $\Phi$ has a coupled fixed point.

**Corollary: 3.2**: Let $(X, \le)$ be a partially ordered set and suppose there is a 2-metric d in X such that $(X, d)$ is a complete 2-metric space. Let $\Phi : X \times X \to X$ be a mapping having tha mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with

$x_0 \le \Phi(x_0, y_0)$ and $y_0 \ge \Phi(y_0, x_0)$

Suppose there exist non-negative real numbers $a_1, a_2$ with $a_1 + a_2 < 1$ such that

$$d(\Phi(x, y), \Phi(l, m), t) \le a_1 d(x, l, t) + a_2 d(y, m, t)$$

For all $x, y, l, m \in X$ with $x \ge u$ and $y \le v$ . Suppose either

(a) $\Phi$ is continous or
(b) $X$ has the following property:

(i) If a non-decreasing sequence $\{x_n\} \to x$ ,then $x_n \le x$ ,for all n,

(ii) If a non-increasing sequence $\{y_n\} \to y$ , then $y \le y_n$ ,for all n.

Then $\Phi$ has a coupled fixed point in $X$ .

**Proof:** Taking $a_3 = 0$ in Theorem 1, we obtain corollary 1.

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