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A NEW CLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH GENERALIZED-DERIVATIVE OPERATOR

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ABSTRACT

In this paper, we study a new class of harmonic univalent functions associated with modified generalized – derivative operator in the open unit disk. We obtain numerous sharp results including coefficient conditions, extreme points, distortion bounds, convolution properties and convex combinations for the above class of harmonic univalent functions. The results obtained for the class reduce to the corresponding results for various well-known classes in the literature.

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1. INTRODUCTION:

A continuous complex valued function $f = u + iv$ defined in a simply connected domain *D* is said to be harmonic in *D* if both u and v are harmonic in *D*. In any simply connected domain *D* we can write $f = h + \overline{g}$, where h and g are analytic in *D*. A necessary and sufficient condition for f to be locally univalent and sense preserving in *D* is that $|h'(z)| > |g'(z)|, z \in D$, see [4].

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$, we may express the analytic function h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1. \tag{1.1}$$

Note that the class S_H reduces to the class *S* of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero.

In 1984, Clunie and Sheil-Small [4] investigated the class S_H and studied some coefficient bounds. Since then, there have been published several related papers on S_H and its subclasses.

In fact by introducing new subclasses Sheil-Small [14] Silverman [15], Silverman and Silvia [16] and Jahangiri [9] presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Ahuja [1], Duren [7] Ponnusamy and Rasila [11] and references therein for basic results on the subjects.

To study our work systematically first we give the class $S_H^n(\alpha,\beta)$ introduced by Gencel and Yalcin [8]

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Let $S_H^n(\alpha,\beta)$ denote the family of harmonic functions f of the form (1.1) such that

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)-D^n f(z)}{D^{n+1}f(z)+(1-2\alpha)D^n f(z)}\right\}<\beta,$$

for $0\le\alpha<1, 0<\beta\le 1, n\epsilon N_0$ *and* $z\epsilon U$. It should be noted that the differential operator D^n was introduced by Salagean and

$$D^n f(z)=D^n h(z)+(-1)^n\overline{D^n g(z)}\ .$$

For $n,m\epsilon N_0$, $N_0=N\cup\{0\}$, $\lambda>0$, and $f=h+\overline{g}$ given by (1.1), we define the modified generalized-derivative operator of f as

$$D_{m,\lambda}^n f(z)=D_{m,\lambda}^n h(z)+(-1)^n\overline{D_{m,\lambda}^n g(z)}\ , \tag{1.2}$$

where $$D_{m,\lambda}^n h(z)=z+\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n C(m,k)a_k z^k,$$

$$D_{m,\lambda}^n g(z)=\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n C(m,k)b_k z^k$$

and $C(m,k)=\binom{k+m-1}{m}=\dfrac{\Gamma(k+m)}{\Gamma(k)\Gamma(m+1)}$.

Remark: 1.1 It is worthy to note that the generalized derivative operator $D_{m,\lambda}^n$ was introduced by Darus and Shaqsi [5]

Remark: 1.2 Some special cases of this operator includes the following derivative operator $D_{(m,1)}^n\equiv D_m^n$ ([3],[10]), the Salagean derivative operator $D_{0,1}^n\equiv D^n$ ([6],[13]), the generalized Salagean derivative operator $D_{0,\lambda}^n\equiv D_\lambda^n$ [2], the Ruscheweyh derivative operator $D_{m,1}^0\equiv D_m$ [12], and the generalized Ruscheweyh derivative operator $D_{m,\lambda}^1\equiv D_{m,\lambda}$ [17].

Darus and Shaqsi [5], gave the following inclusion relations:

$$D_{m,\lambda}^{n+1}f(z)=(1-\lambda)D_{m,\lambda}^n f(z)+\lambda z(D_{m,\lambda}^n f(z))'$$
$$z(D_{m,\lambda}^n f(z))'=(1+\lambda)D_{m+1,\lambda}^n f(z)-\lambda D_{m,\lambda}^n f(z)$$
$$D_{1,0}^0 f(z)=f(z).$$

For $a_1=1, n,m\in N_0, \lambda>0$,$0\le\alpha<1$, $0<\beta\le 1$ and $z\in U$, we let $S_H^n(\alpha,\beta,\lambda,m,k)$ denote the family of harmonic functions f of the form (1.1) such that

$$\mathrm{Re}\left\{\frac{D_{m,\lambda}^{n+1}f(z)-D_{m,\lambda}^n f(z)}{D_{m,\lambda}^{n+1}f(z)+(1-2\alpha)D_{m,\lambda}^n f(z)}\right\}<\beta, \tag{1.3}$$

$D_{m,\lambda}^n f$ is defined by (1.2).

We let the subclass $\overline{S}_H^n(\alpha,\beta,\lambda,m,k)$ consist of harmonic function $f_n=h+\overline{g}_n$ in $\overline{S}_H^n(\alpha,\beta,\lambda,m,k)$ so that h and g_n are of the form

$$h(z)=z-\sum_{k=2}^{\infty}a_k z^k,\quad g_n(z)=(-1)^n\sum_{k=1}^{\infty}b_k z^k.\ \ a_k,b_k\ge 0 \text{ and } |b_1|<1. \tag{1.4}$$

In this paper, the coefficient condition given in [8] for the class $S_H^n(\alpha,\beta)$ is extended to the class $S_H^n(\alpha,\beta,\lambda,m,k)$ of the form (1.3). Furthermore, we obtain extreme points, distortion theorem, convolution conditions and convex combinations for the functions in $\overline{S}_H^n(\alpha,\beta,\lambda,m,k)$.

2. MAIN RESULT:

In our first theorem, we introduce a sufficient bound for harmonic functions in $S_H^n(\alpha,\beta,\lambda,m,k)$.

Theorem: 1 Let $f(z)=h(z)+\overline{g}(z)$ be given by (1.1). Furthermore, let

$$\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n\{[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)|a_k|+(2+(k-1)\lambda+\beta[(k-1)\lambda+2\alpha])C(m,k)|b_k|\}$$
$$\le 4\beta(1-\alpha) \tag{2.1}$$

where $a_1=1,\ \ n,m\in N_0,\ \ \lambda>0,\ \ 0\le\alpha<1,\ \ 0<\beta\le 1.$

Then $f(z)$ is harmonic univalent, sense preserving in *U* and $f\in S_H^n(\alpha,\beta,\lambda,m,k)$.

Proof: For $|z_1|<|z_2|<1$, we have

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$
$$\ge 1-\left|\frac{\sum_{k=1}^{\infty}b_k\left(z_1^k-z_2^k\right)}{(z_1-z_2)-\sum_{k=2}^{\infty}a_k\left(z_1^k-z_2^k\right)}\right|$$
$$>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|}$$
$$>1-\frac{\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n(2+(k-1)\lambda+\beta[(k-1)\lambda+2\alpha])C(m,k)|b_k|}{1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)|a_k|}$$
$$\ge 0$$

Hence, $f(z)$ is univalent in *U*. Note that $f(z)$ is sense preserving in *U*.

This is because $|h'(z)|\ge 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}$

$$>1-\sum_{k=2}^{\infty}k|a_k|$$
$$>1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)|a_k|$$
$$>\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n[2+(k-1)\lambda+\beta((k-1)\lambda+2\alpha)]C(m,k)|b_k||z|^{k-1}$$
$$\ge\sum_{k=1}^{\infty}k|b_k||z|^{k-1}$$
$$\ge|g'(z)|$$

It remains to show that $f(z) \in S_H^n(\alpha,\beta,\lambda,m,k)$.Suppose that the inequality (2.1) holds true and let $z \in \partial U = \left\{ z : z \in C \ \text{ and } \ |z| = 1 \right\}$. Then we find from definition (1.2) that

$$\left| \frac{D_{m,\lambda}^{n+1} f(z) - D_{m,\lambda}^{n} f(z)}{D_{m,\lambda}^{n+1} f(z) + (1-2\alpha) D_{m,\lambda}^{n} f(z)} \right|$$

$$= \left| \frac{\sum_{k=2}^{\infty} [1+(k-1)\lambda]^n C(m,k)(k-1)\lambda a_k z^k - (-1)^n \sum_{k=1}^{\infty} [1+(k-1)\lambda]^n C(m,k)[2+(k-1)\lambda] b_k \bar{z}^k}{2(1-\alpha)z + \sum_{k=2}^{\infty} [1+(k-1)\lambda]^n C(m,k)[2+(k-1)\lambda-2\alpha] a_k z^k - (-1)^n \sum_{k=1}^{\infty} [1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda+2\alpha] b_k \bar{z}^k} \right|$$

$$\le \frac{\sum_{k=2}^{\infty} \left[1+(k-1)\lambda\right]^n C(m,k)(k-1)\lambda |a_k| |z|^k + \sum_{k=1}^{\infty} \left[1+(k-1)\lambda\right]^n C(m,k)\left[2+(k-1)\lambda\right] |b_k| |z|^k}{2(1-\alpha)|z| - \sum_{k=2}^{\infty} \left[1+(k-1)\lambda\right]^n C(m,k)\left[2+(k-1)\lambda-2\alpha\right] |a_k| |z|^k - \sum_{k=1}^{\infty} \left[1+(k-1)\lambda\right]^n C(m,k)\left[(k-1)\lambda+2\alpha\right] |b_k| |z|^k}$$

$$\le \beta$$

provided that the inequality (2.1) is satisfied . Hence, by the maximum modulus theorem, we have

$f \in \bar{S}_H^n(\alpha,\beta,\lambda,m,k)$.

The harmonic mappings

$$f(z) = z + \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda + \beta(2+(k-1)\lambda-2\alpha)]} x_k z^k + \sum_{k=1}^{\infty} \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n C(m,k)(2+(k-1)\lambda + \beta[(k-1)\lambda+2\alpha])} \overline{y_k z^k} \tag{2.2}$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.2) are in $\bar{S}_H^n(\alpha,\beta,\lambda,m,k)$ because

$$\sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda + \beta(2+(k-1)\lambda-2\alpha)]}{2\beta(1-\alpha)} |a_k| + \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n C(m,k)(2+(k-1)\lambda + \beta[(k-1)\lambda+2\alpha])}{2\beta(1-\alpha)} |b_k|$$

$$= \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| + 1 = 2$$

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f_n = h + \bar{g}_n$ where h and g_n are of the form (1.4).

Theorem: 2 Let $f_n = h + \overline{g}_n$ be given by(1.4). Then $f_n \in \overline{S}_H^n(\alpha,\beta,\lambda,m,k)$ if and only if

$$\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^n\left\{\left[(k-1)\lambda+\beta\left(2+(k-1)\lambda-2\alpha\right)\right]C(m,k)|a_k|+\left(2+(k-1)\lambda+\beta\left[(k-1)\lambda+2\alpha\right]\right)C(m,k)|b_k|\right\}$$

$$\leq 4\beta(1-\alpha) \tag{2.3}$$

Proof: Since $\overline{S}_H^n(\alpha,\beta,\lambda,m,k) \subset S_H^n(\alpha,\beta,\lambda,m,k)$, we only need to prove the "only if" part of the theorem . To this end, for functions f_n of the form (1.4), we notice that the condition

$$Re\left\{\frac{D_{m,\lambda}^{n+1}f(z)-D_{m,\lambda}^{n}f(z)}{D_{m,\lambda}^{n+1}f(z)+(1-2\alpha)D_{m,\lambda}^{n}f(z)}\right\}<\beta$$

is equivalent to

$$\mathrm{Re}\left\{\frac{-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n C(m,k)(k-1)\lambda a_k z^k-(-1)^{2n}\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^n C(m,k)\left[2+(k-1)\lambda\right]b_k\overline{z}^k}{2(1-\alpha)z-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n C(m,k)\left[2+(k-1)\lambda-2\alpha\right]a_k z^k-(-1)^{2n}\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^n C(m,k)\left[(k-1)\lambda+2\alpha\right]b_k\overline{z}^k}\right\}$$

$$>-\beta.$$

If we choose z on the real axis and $z \longrightarrow 1^-$ we get

$$\frac{\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n C(m,k)(k-1)\lambda a_k+\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n C(m,k)[2+(k-1)\lambda]b_k}{2(1-\alpha)-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n C(m,k)[2+(k-1)\lambda-2\alpha]a_k-\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda+2\alpha]b_k}<\beta$$

whence

$$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n C(m,k)(k-1)\lambda a_k+\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n C(m,k)[2+(k-1)\lambda]b_k$$

$$<2\beta(1-\alpha)-\beta\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n C(m,k)[2+(k-1)\lambda-2\alpha]a_k-\beta\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda+2\alpha]b_k$$

and so

$$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)a_k+\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n[2+(k-1)\lambda+\beta((k-1)\lambda+2\alpha)]C(m,k)b_k$$

$$<2\beta(1-\alpha)$$

which is equivalent to (2.3).

Next, we determine the extreme points of the closed convex hulls of $\overline{S}_H^n(\alpha,\beta,\lambda,m,k)$ denoted by $clco\overline{S}_H^n(\alpha,\beta,\lambda,m,k)$.

Theorem: 3 Let $f_n = h + \overline{g}_n$ be given by (1.4). Then $f_n \in clco\overline{S}_H^n(\alpha,\beta,\lambda,m,k)$ if and only if

$$f_n(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_{nk}(z)\right),$$

where

$$h_1(z)=z,$$

$$h_k(z) = z - \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)}z^k, \qquad (\mathrm{k}=2,3,\ldots\ldots)$$

and

$$g_{nk}(z) = z + (-1)^n \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n[2+(k-1)\lambda+\beta((k-1)\lambda+2\alpha)]C(m,k)}\bar{z}^k, \ (\mathrm{k}=1,2,3,\ldots)$$

$$\sum_{k=1}^{\infty}(x_k + y_k) = 1, \ x_k \geq 0 \ \text{ and } \ y_k \geq 0.$$

In particular, the extreme points of $\bar{S}_H^n(\alpha,\beta,\lambda,m,k)$ are $\{h_k\}$ and $\{g_{nk}\}$.

Proof: For the functions $f(z)$ of the form (2.6),we have

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k + y_k g_k)$$

$$= \sum_{k=1}^{\infty}\left(x_k + y_k\right)z - \sum_{k=2}^{\infty}\frac{2\beta(1-\alpha)}{\left[1+(k-1)\lambda\right]^n\left[(k-1)\lambda+\beta\left(2+(k-1)\lambda-2\alpha\right)\right]C(m,k)}x_k z^k$$

$$+(-1)^n\sum_{k=1}^{\infty}\frac{2\beta(1-\alpha)}{\left[1+(k-1)\lambda\right]^n\left(2+(k-1)\lambda+\beta\left[(k-1)\lambda+2\alpha\right]\right)C(m,k)}y_k\bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)}{2\beta(1-\alpha)}\left(\frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)}x_k\right)$$

$$+\sum_{k=1}^{\infty}\frac{[1+(k-1)\lambda]^n(2+(k-1)\lambda+\beta[(k-1)\lambda+2\alpha])C(m,k)}{2\beta(1-\alpha)}\left(\frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n(2+(k-1)\lambda+\beta[(k-1)\lambda+2\alpha])C(m,k)}y_k\right)$$

$$= \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k = 1 - x_1 \leq 1,$$

and so $f_n(z) \in \bar{S}_H^n(\alpha,\beta,\lambda,m,k)$, conversely, if $f_n(z) \in clco \ \ \bar{S}_H^n(\alpha,\beta,\lambda,m,k)$, then

$$a_k \leq \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)},$$

and $$b_k \leq \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n[2+(k-1)\lambda+\beta((k-1)\lambda+2\alpha)]C(m,k)}.$$

Set $$x_k = \frac{[1+(k-1)\lambda]^n[(k-1)\lambda+\beta(2+(k-1)\lambda-2\alpha)]C(m,k)}{2\beta(1-\alpha)}a_k, \quad (\mathrm{k}=2, 3\ldots)$$

$$y_k = \frac{[1+(k-1)\lambda]^n[2+(k-1)\lambda+\beta((k-1)\lambda+2\alpha)]C(m,k)}{2\beta(1-\alpha)}b_k. \quad (\mathrm{k}=1, 2\ldots)$$

Then by Theorem 2, $0 \leq x_k \leq 1, (k = 2,3,\ldots\ldots)$ and $0 \leq y_k \leq 1, (k = 1,2,3,\ldots\ldots)$.

We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$ and note that by Theorem 2, $x_1 \geq 0$.

Consequently, we obtain

$$f_n(z) = \sum_{k=1}^{\infty} \left(x_k h_k(z) + y_k g_{nk}(z) \right)$$

as required.

The following theorem gives the distortion bounds for functions in $\overline{S}_H^n(\alpha, \beta, \lambda, m, k)$ which yields a covering result for this class.

Theorem: 4. Let $f_n(z) \in \overline{S}_H^n(\alpha, \beta, \lambda, m, k)$. Then for $|z| = r < 1$ we have

$$|f_n(z)| \leq (1 + |b_1|) r + \frac{2}{[1+\lambda]^n (m+1)} \left(\frac{\beta(1-\alpha) - (1+\beta\alpha)}{[\lambda + \beta(2+\lambda-2\alpha)]} C(m,k)|b_1| \right) r^2, \qquad |z| = r < 1$$

and

$$|f_n(z)| \geq (1 - |b_1|) r - \frac{2}{[1+\lambda]^n (m+1)} \left(\frac{\beta(1-\alpha) - (1+\beta\alpha)}{[\lambda + \beta(2+\lambda-2\alpha)]} C(m,k)|b_1| \right) r^2, \qquad |z| = r < 1.$$

Proof: We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted.

Let $f_n \in \overline{S}_H^n(\alpha, \beta, \lambda, m, k)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq (1+|b_1|) r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \\
&\leq (1+|b_1|) r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2 \\
&= (1+|b_1|) r + \frac{2\beta(1-\alpha)}{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1)} \left(\sum_{k=2}^{\infty} \frac{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1)}{2\beta(1-\alpha)} (|a_k| + |b_k|) r^2 \right) \\
&\leq (1+|b_1|) r + \frac{2\beta(1-\alpha)}{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1)} \left(1 - \frac{2+2\beta\alpha}{2\beta(1-\alpha)} C(m,k)|b_1| \right) r^2 \\
&= (1+|b_1|) r + \frac{2}{[1+\lambda]^n (m+1)} \left(\frac{\beta(1-\alpha) - (1+\beta\alpha)}{[\lambda + \beta(2+\lambda-2\alpha)]} C(m,k)|b_1| \right) r^2
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary: 1 If $f_n(z) \in \overline{S}_H(\alpha, \beta, \lambda, m, k)$, then

$$\left\{ w : |w| < \frac{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1) - 2\beta(1-\alpha)}{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1)} - \frac{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1) - 2(1+\beta\alpha)}{[1+\lambda]^n [\lambda + \beta(2+\lambda-2\alpha)](m+1)} |b_1| \right\}$$
$$\subset f_n(U).$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \bar{z}^k,$$

and

$$F_n(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| \bar{z}^k,$$

we define the convolution of two harmonic functions $f(z)$ and $F(z)$ as

$$f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k||B_k| \bar{z}^k. \tag{2.4}$$

Using this definition, we show that the class $\bar{S}_H^n(\alpha, \beta, \lambda, m, k)$ is closed under convolution.

Theorem: 5 For $0 \le \alpha_1 \le \alpha_2 < 1$, let $f_n(z) \in \bar{S}_H^n(\alpha_2, \beta, \lambda, m, k)$ and $F_n(z) \in \bar{S}_H^n(\alpha_1, \beta, \lambda, m, k)$.

Then $f_n * F_n \in \bar{S}_H^n(\alpha_2, \beta, \lambda, m, k) \subset \bar{S}_H^n(\alpha_1, \beta, \lambda, m, k)$.

Proof: Let $f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \bar{z}^k$ be in $\bar{S}_H^n(\alpha_2, \beta, \lambda, m, k)$, and

$$F_n(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| \bar{z}^k \text{ be in } \bar{S}_H^n(\alpha_1, \beta, \lambda, m, k).$$

Then the convolution $(f_n * F_n)$ is given by (2.4). We wish to show that the coefficient of $(f_n * F_n)$ satisfies the required condition given in Theorem 2.

For $F_n(z) \in \bar{S}_H^n(\alpha_1, \beta, \lambda, m, k)$, we note that $|A_k| < 1$ and $|B_k| < 1$.

Now, for the convolution function $(f_n * F_n)$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n [(k-1)\lambda + \beta(2+(k-1)\lambda - 2\alpha_1)] C(m,k)}{2\beta(1-\alpha_1)} |a_k||A_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n [2+(k-1)\lambda + \beta((k-1)\lambda + 2\alpha_1)] C(m,k)}{2\beta(1-\alpha_1)} |b_k||B_k| \\
&\le \sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n [(k-1)\lambda + \beta(2+(k-1)\lambda - 2\alpha_1)] C(m,k)}{2\beta(1-\alpha_1)} |a_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n [2+(k-1)\lambda + \beta((k-1)\lambda + 2\alpha_1)] C(m,k)}{2\beta(1-\alpha_1)} |b_k| \\
&\le \sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n [(k-1)\lambda + \beta(2+(k-1)\lambda - 2\alpha_2)] C(m,k)}{2\beta(1-\alpha_2)} |a_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n [2+(k-1)\lambda + \beta((k-1)\lambda + 2\alpha_2)] C(m,k)}{2\beta(1-\alpha_2)} |b_k| \\
&\le 1
\end{aligned}$$

Since $0 \le \alpha_1 \le \alpha_2 < 1$, and $f_n(z) \in \bar{S}_H^n(\alpha_2, \beta, \lambda, m, k)$.

Therefore $f_n * F_n \in \bar{S}_H^n(\alpha_2, \beta, \lambda, m, k) \subset \bar{S}_H^n(\alpha_1, \beta, \lambda, m, k)$.

Now we show that $\bar{S}_H^n(\alpha, \beta, \lambda, m, k)$ is closed under convex combination of its members.

Theorem: 6 The class $\bar{S}_H^n(\alpha, \beta, \lambda, m, k)$ is closed under convex combination.

Proof: For $i = 1, 2\ldots$ let $f_{n_i}(z) \in \bar{S}_H^n(\alpha, \beta, \lambda, m, k)$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k_i}| \bar{z}^k,$$

Then by (2.3),

$$\sum_{k=1}^{\infty} \left(\frac{[1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda + \beta(2+(k-1)\lambda - 2\alpha)]}{\beta(1-\alpha)} a_{k_i} + \frac{[1+(k-1)\lambda]^n C(m,k)(2+(k-1)\lambda + \beta[(k-1)\lambda + 2\alpha])}{\beta(1-\alpha)} b_{k_i} \right) \le 4. \tag{2.5}$$

For $\sum_{i=1}^{\infty} t_i = 1, \quad 0 \le t_i \le 1,$ the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) \bar{z}^k,$$

Then by (2.5),

$$\sum_{k=1}^{\infty} \left(\frac{[1+(k-1)\lambda]^n [(k-1)\lambda + \beta(2+(k-1)\lambda - 2\alpha)]}{\beta(1-\alpha)} \sum_{i=1}^{\infty} t_i a_{k_i} + \frac{[1+(k-1)\lambda]^n (2+(k-1)\lambda + \beta[(k-1)\lambda + 2\alpha])}{\beta(1-\alpha)} \sum_{i=1}^{\infty} t_i b_{k_i} \right)$$

$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=1}^{\infty} \left(\frac{[1+(k-1)\lambda]^n C(m,k)[(k-1)\lambda + \beta(2+(k-1)\lambda - 2\alpha)]}{\beta(1-\alpha)} a_{k_i} + \frac{[1+(k-1)\lambda]^n C(m,k)(2+(k-1)\lambda + \beta[(k-1)\lambda + 2\alpha])}{\beta(1-\alpha)} b_{k_i} \right) \right\} \le 4 \sum_{i=1}^{\infty} t_i = 4.$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \bar{S}_H^n(\alpha, \beta, \lambda, m, k)$.

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