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ON SOME NEW INEQUALITIES FOR s-CONVEX FUNCTIONS

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ABSTRACT

In this paper some new Hadamard-type inequalities for s-convex functions are estalished by using fairly elementary analysis.

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1. INTRODUCTION:

The following definition for convex functions is well known in the mathematical literature:

A function $f: I \to \mathbb{R}$, $\varnothing \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamar's inequality, due to its rich geometrical significance and applications, which is stated as follows:

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard's inequality [3] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for mean can be derived from (1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see [1]-[9] and the references therein.

In the paper [4], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined as follows:

A function $f:[0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(tx+(1-t)y) \le t^s f(x)+(1-t)^s f(y),$$

holds for all x, $y \in [0,\infty)$, $t \in [0,1]$ and $s \in (0,1]$.

The class of s-convex functions in the second sense is usually denoted by K_s^2. It is easy to observe that for $s = 1$, the class of s-convex functions in the second sense is merely the class of convex functions defined on $[0,\infty)$. It was also proved in [4] that the functions from K_s^2, $s \in (0,1)$ are non-negative.

In [2], Dragomir and Fitzpatrick proved a vartiant of Hermite-Hadamard's inequality which holds for s-convex functions in the second sense:

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Theorem: 1 *[2] Suppose* $f:[0,\infty)\to[0,\infty)$ *is an* s *-convex function in the second sense, where* $s\in(0,1)$*, and let* a, $b\in[0,\infty)$, $a<b$. *If* $f\in L^1([a,b])$*, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}. \tag{2}$$

The constant $k=\dfrac{1}{s+1}$ is the best possible in the second inequality in (2).

For the new inequalities for convex and s -convex functions in the second sense we refer the interested readers to [8] and [9], the recent work of M. Tunç. The main purpose of this paper is to establish new integral inequalities like those established in [8] but for the class of s -convex functions in the second sense by using the same techniques as used in [8] and we believe that inequalities proved in the present paper are of independent interest.

2. MAIN RESULTS:

We begin this section with the following result:

Theorem: 2 *Let* $f,g:[a,b]\to\mathbb{R}$, a, $b\in[0,\infty)$, $a<b$, *be functions such that such that* f, g, $fg\in L^1([a,b])$. *If* f *is* s_1*-convex function in the second sense and* g *is* s_2*-convex function in the second sense for some fixed* s_1, $s_2\in(0,1]$*, then we have the following inequality:*

$$\begin{aligned}&\frac{g(a)}{(b-a)^{s_2+1}}\int_a^b(b-x)^{s_2}f(x)dx+\frac{f(a)}{(b-a)^{s_1+1}}\int_a^b(b-x)^{s_1}g(x)dx\\&\qquad+\frac{g(b)}{(b-a)^{s_2+1}}\int_a^b(a-x)^{s_2}f(x)dx+\frac{f(b)}{(b-a)^{s_1+1}}\int_a^b(a-x)^{s_1}g(x)dx\\&\le\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{M(a,b)}{s_1+s_2+1}+N(a,b)\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)},\end{aligned} \tag{3}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof: Since f is s_1-convex function in the second sense and g is s_2-convex function in the second sense for some fixed s_1, $s_2\in(0,1]$, we have that

$$f(ta+(1-t)b)\le t^{s_1}f(a)+(1-t)^{s_1}f(b)$$

and

$$g(ta+(1-t)b)\le t^{s_2}g(a)+(1-t)^{s_2}g(b),$$

for $t\in[a,b]$.

By using the elementary inequality $e\le f$ and $p\le r$ then $er+fp\le ep+fr$ for $e,f,p,r\in\mathbb{R}$, we get from the above inequalities that

$$\begin{aligned}&f(ta+(1-t)b)\left[t^{s_2}g(a)+(1-t)^{s_2}g(b)\right]+g(ta+(1-t)b)\left[t^{s_1}f(a)+(1-t)^{s_1}f(b)\right]\\&\qquad\le f(ta+(1-t)b)g(ta+(1-t)b)+\left[t^{s_1}f(a)+(1-t)^{s_1}f(b)\right]\left[t^{s_2}g(a)+(1-t)^{s_2}g(b)\right]\end{aligned}$$

which gives the following inequality:

$$\begin{aligned}&t^{s_2}g(a)f(ta+(1-t)b)+(1-t)^{s_2}g(b)f(ta+(1-t)b)\\&\qquad+t^{s_1}f(a)g(ta+(1-t)b)+(1-t)^{s_1}f(b)g(ta+(1-t)b)\\&\le f(ta+(1-t)b)g(ta+(1-t)b)+t^{s_1+s_2}f(a)g(a)+t^{s_1}(1-t)^{s_2}f(a)g(b)\end{aligned}$$

$$+t^{s_2}(1-t)^{s_1} f(b)g(a)+(1-t)^{s_1+s_2} f(b)g(b).$$

Integrating the above inequality over $[0,1]$, we get that

$$g(a)\int_0^1 t^{s_2} f(ta+(1-t)b)dt+g(b)\int_0^1 (1-t)^{s_2} f(ta+(1-t)b)dt$$
$$+f(a)\int_0^1 t^{s_1} g(ta+(1-t)b)dt+f(b)\int_0^1 (1-t)^{s_1} g(ta+(1-t)b)dt$$
$$\le \int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt+f(a)g(a)\int_0^1 t^{s_1+s_2}dt$$
$$+f(a)g(b)\int_0^1 t^{s_1}(1-t)^{s_2}dt+f(b)g(a)\int_0^1 t^{s_2}(1-t)^{s_1}dt+f(b)g(b)\int_0^1 (1-t)^{s_1+s_2}dt.$$

By making use if the substitution $ta+(1-t)b=x$, $(a-b)dt=dx$, we observe that

$$\int_0^1 t^{s_2} f(ta+(1-t)b)dt=\frac{1}{(b-a)^{s_2+1}}\int_a^b (b-x)^{s_2} f(x)dx,$$

$$\int_0^1 t^{s_1} g(ta+(1-t)b)dt=\frac{1}{(b-a)^{s_1+1}}\int_a^b (b-x)^{s_1} g(x)dx,$$

$$\int_0^1 (1-t)^{s_2} f(ta+(1-t)b)dt=\frac{1}{(b-a)^{s_2+1}}\int_a^b (a-x)^{s_2} f(x)dx,$$

$$\int_0^1 (1-t)^{s_1} g(ta+(1-t)b)dt=\frac{1}{(b-a)^{s_1+1}}\int_a^b (a-x)^{s_1} g(x)dx$$

and

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt=\frac{1}{b-a}\int_a^b f(x)g(x)dx.$$

By using Beta function of Euler type

$$\beta(u,v)=\int_0^1 t^{u-1}(1-t)^{1-v}dt=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},u,v>0,$$

we obtain that

$$\int_0^1 t^{s_2}(1-t)^{s_1}dt=\frac{\Gamma(s_2+1)\Gamma(s_1+1)}{\Gamma(s_1+s_2+2)},$$

$$\int_0^1 t^{s_1}(1-t)^{s_2}dt=\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}.$$

Also it can easily be seen that

$$\int_0^1 (1-t)^{s_1+s_2}dt=\int_0^1 t^{s_1+s_2}dt=\frac{1}{s_1+s_2+1}.$$

By taking the above obervations into account we get the desired inequality. This completes the proof of the theorem.

Theorem: 3 *Let* $f,g:[a,b]\to\mathbb{R}$, a, $b\in[0,\infty)$, $a<b$, *be functions such that such that* f, g, $fg\in L^1([a,b])$. *If* f *is* s_1*-convex function in the second sense and* g *is* s_2*-convex function in the second sense for some fixed* s_1, $s_2\in(0,1]$, *then we have the following inequality:*

$$\frac{2^{s_1-1}f\left(\frac{a+b}{2}\right)}{b-a}\int_a^b g(x)dx+\frac{2^{s_2-1}g\left(\frac{a+b}{2}\right)}{b-a}\int_a^b f(x)dx$$
$$\le\frac{1}{2(b-a)}\int_a^b f(x)g(x)dx+\frac{\Gamma(s_2+1)\Gamma(s_1+1)}{2\Gamma(s_1+s_2+2)}\cdot M(a,b)$$
$$+\frac{1}{2(s_1+s_2+1)}\cdot N(a,b)+2^{s_1+s_2-2}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right). \qquad (4)$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof: Since f is s_1-convex function in the second sense and g is s_2-convex function in the second sense for some fixed s_1, $s_2\in(0,1]$, we have that

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)$$
$$\le\frac{f(ta+(1-t)b)}{2^{s_1}}+\frac{f((1-t)a+tb)}{2^{s_1}}$$

and

$$g\left(\frac{a+b}{2}\right)=g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)$$
$$\le\frac{g(ta+(1-t)b)}{2^{s_2}}+\frac{g((1-t)a+tb)}{2^{s_2}}.$$

Arguing similarly as in Theorem 2, we get that

$$f\left(\frac{a+b}{2}\right)\left[\frac{g(ta+(1-t)b)}{2^{s_2}}+\frac{g((1-t)a+tb)}{2^{s_2}}\right]+g\left(\frac{a+b}{2}\right)\left[\frac{f(ta+(1-t)b)}{2^{s_1}}+\frac{f((1-t)a+tb)}{2^{s_1}}\right]$$
$$\le\left[\frac{f(ta+(1-t)b)}{2^{s_1}}+\frac{f((1-t)a+tb)}{2^{s_1}}\right]\left[\frac{g(ta+(1-t)b)}{2^{s_2}}+\frac{g((1-t)a+tb)}{2^{s_2}}\right]+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

which gives

$$\frac{1}{2^{s_2}}f\left(\frac{a+b}{2}\right)[g(ta+(1-t)b)+g((1-t)a+tb)]$$
$$+\frac{1}{2^{s_1}}g\left(\frac{a+b}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)]$$
$$\le\frac{1}{2^{s_1+s_2}}[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)]$$
$$+\frac{1}{2^{s_1+s_2}}\left[t^{s_2}(1-t)^{s_1}+t^{s_1}(1-t)^{s_2}\right][f(a)g(a)+f(b)g(b)]$$
$$+\frac{1}{2^{s_1+s_2}}\left[(1-t)^{s_1+s_2}+t^{s_1+s_2}\right][f(a)g(b)+f(b)g(a)].$$

Integrating both sides over $[0,1]$, we obtain

$$\frac{1}{2^{s_2}}f\left(\frac{a+b}{2}\right)\int_0^1[g(ta+(1-t)b)+g((1-t)a+tb)]dt$$

$$+\frac{1}{2^{s_1}}g\left(\frac{a+b}{2}\right)\int_0^1\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]dt$$

$$\le\frac{1}{2^{s_1+s_2}}\int_0^1\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right]dt$$

$$+\frac{1}{2^{s_1+s_2}}\int_0^1\left[t^{s_2}(1-t)^{s_1}+t^{s_1}(1-t)^{s_2}\right]\left[f(a)g(a)+f(b)g(b)\right]dt$$

$$+\frac{1}{2^{s_1+s_2}}\int_0^1\left[(1-t)^{s_1+s_2}+t^{s_1+s_2}\right]\left[f(a)g(b)+f(b)g(a)\right]dt$$

$$+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 dt \tag{5}$$

By making use of the substitution $x=at+(1-t)b$, $dx=(a-b)dt$ and $y=(1-t)a+tb$, $dy=(b-a)dt$, we observe that

$$\int_0^1 f\left(ta+(1-t)b\right)dt=\int_0^1 f\left((1-t)a+tb\right)dt=\frac{1}{b-a}\int_a^b f(x)dx,$$

$$\int_0^1 g\left(ta+(1-t)b\right)dt=\int_0^1 g\left((1-t)a+tb\right)dt=\frac{1}{b-a}\int_a^b g(x)dx$$

and

$$\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)dt=\int_0^1 f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)dt$$

$$=\frac{1}{b-a}\int_a^b f(x)g(x)dx.$$

We also notice that

$$\int_0^1 t^{s_2}(1-t)^{s_1}dt=\int_0^1 t^{s_1}(1-t)^{s_2}dt=\frac{\Gamma(s_2+1)\Gamma(s_1+1)}{\Gamma(s_1+s_2+2)}$$

and

$$\int_0^1(1-t)^{s_1+s_2}dt=\int_0^1 t^{s_1+s_2}dt=\frac{1}{s_1+s_2+1}.$$

Thus (5) reduces to

$$\frac{1}{2^{s_2-1}}\cdot\frac{f\left(\frac{a+b}{2}\right)}{b-a}1\int_a^b g(x)dx+\frac{1}{2^{s_1-1}}\cdot\frac{g\left(\frac{a+b}{2}\right)}{b-a}\int_a^b f(x)dx$$

$$\le\frac{1}{2^{s_1+s_2-1}}\cdot\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{2^{s_1+s_2-1}}\cdot\frac{\Gamma(s_2+1)\Gamma(s_1+1)}{\Gamma(s_1+s_2+2)}\cdot M(a,b)$$

$$+\frac{1}{2^{s_1+s_2-1}}\cdot\frac{N(a,b)}{s_1+s_2+1}+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right). \tag{6}$$

Multiplying both sides of (6) by $2^{s_1+s_2-2}$, we get the desired inequality.

Remark: 1 *If we choose $s_1=s_2=1$ in [3] and [4] we get those inequalities proved in [8].*

3. APPLICATIONS TO SOME SPECIAL MEANS:

In this section we consider the applications of our result to the following special means:

1. The power mean: $M_p=M_p(x_1,...,x_n):=\left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}, a,b\ge 0,$

2. The arithmetic mean: $A = A(a,b) := \frac{a+b}{2}, a,b \geq 0,$

3. The geometric mean $G = G(a,b) := \sqrt{ab}, a,b \geq 0,$

4. The Harmonic mean: $H = H(a,b) := \frac{2ab}{a+b}, a,b \geq 0,$

5. The quadratic mean: $K = K(a,b) := \sqrt{\frac{a^2+b^2}{2}}, a,b \geq 0,$

6. The logarithmic mean: $L = L(a,b) := \begin{cases} a, & \text{if} a = b \\ \frac{b-a}{\ln b - \ln a}, & \text{if} a \neq b, a,b > 0, \end{cases}$

7. The identric mean: $I = I(a,b) := \begin{cases} a, & \text{if} a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if} a \neq b, a,b \geq 0, \end{cases}$

8. The p -logarithmic mean: $L_p = L_p(a,b) := \begin{cases} a, & \text{if} a = b \\ \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} a \neq b \end{cases}, \ p \in \mathbb{R} \setminus \{-1,0\}, a,b > 0.$

The following inequality is well know in literature

$$H \leq G \leq L \leq I \leq A \leq K.$$

It is also know that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Now we quote a very important example from [4]:

Let $s \in (0,1)$, a, b, $c \in \mathbb{R}$. We define the function $f : [0,\infty) \to \mathbb{R}$ as

$$f(x) = \begin{cases} a, & \text{if} t = 0, \\ bt^s + c, & \text{if} t > 0. \end{cases}$$

If $b \geq 0$, $0 \leq c \leq a$ then $f \in K_s^2$. Therefore, for $a = c = 0$, $b = 1$, $s = \frac{1}{2}$, we have $f : [0,1] \to [0,1]$, $f(t) = t^{\frac{1}{2}}$, $f \in K_s^2$.

Proposition: 1 *Let* $0 < a < b$ *and* $0 < s < 1$. *Then we have*

$$\frac{A(a^s,b^s)A^s(a,b)}{2^s} \leq L_{2s}(a,b) + \frac{2}{2s+1}A(a^{2s},b^{2s}) + 2G^{2s}(a,b) \cdot \frac{(\Gamma(s+1))^2}{\Gamma(2(s+1))} \tag{7}$$

Proof: The inequality follows when we take the s -convex functions $f, g : [0,1] \to [0,1]$, $f(x) = x^s$, $g(x) = x^s$, $x \in [0,1]$, applied to (3) with $x = \frac{a+b}{2}$. The details are let to the interested readers.

Proposition: 2 *Let* $0 < a < b$ *and* $0 < s < 1$. *Then we have*

$$2^s A^s(a,b) L_s(a,b) \le \frac{1}{2} L_{2s}(a,b) + \frac{(\Gamma(s+1))^2}{\Gamma(2(s+1))} \cdot A(a^{2s}, b^{2s}) + \frac{1}{2s+1} G^{2s}(a,b)$$
$$+ 2^{2s-2} A^{2s}(a,b) \qquad (8)$$

Proof: The inequality follows when we take the s-convex functions $f, g : [0,1] \to [0,1]$, $f(x) = x^s$, $g(x) = x^s$, $x \in [0,1]$, applied to (4), however the details are left to the interested readers.

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