International Journal of Mathematical Archive-3(1), 2012, Page: 64-71
Available online through www.ijma.info ISSN 2229 – 5046

SEPARATION OF THE GENERAL TRICOMI DIFFERENTIAL OPERATOR IN HILBERT SPACE AND ITS APPLICATION

Saleh Omran*, Khaled Gepreel, and Sayed Khalil

[1]Math. Depart., Faculty of Science, Taif University, Kingdom of Saudi Arabia.
[2]Math. Depart., Faculty of Science, South valley University, Egypt.
[3]Math. Depart., Faculty of Science, Zagazig University, Egypt.
[4]Math. Depart., Faculty of Science, Minofia University, Egypt.

E-mail: salehomran@yahoo.com, kagepreel@yahoo.com

(Received on: 17-10-11; Accepted on: 01-11-11)

ABSTAACT

In this article, we separate the differential operator A of the form $Au(x,y) = Pu(x,y) + Vu(x,y)$ *for all* $x, y \in R$, *in the Hilbert space* $H = L_2(R^2, H_1)$ *with the operator potential* $V(x,y)$, *where* $L(H_1)$ *is the space of all bounded operators on an arbitrary Hilbert space* H_1 *and* $P = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{x^4}{4}\frac{\partial^2}{\partial y^2}\right)$ *is the general tricomi operator on* R^2. *Moreover, we study the existence and uniqueness of the solution of the differential equation* $Au = Pu + Vu = f$, *in the Hilbert space* H, *where* $f \in H$, *as an application of this separation.*

Keywords: *Separation; Tricomi differential operator; Hilbert space; Coercive estimate.*

---

1. INTRODUCTION:

The concept of separation for differential operators was first introduced by Everitt and Giertz $[6,7]$. They have obtained the separation results for the Sturm Liouville differential operator

$$Ay(x) = -y''(x) + V(x)y(x), \quad x \in R, \tag{1}$$

in the space $L_2(R)$. They have studied the following question: What are the conditions on $V(x)$ such that if $y(x) \in L_2(R)$ and $Ay(x) \in L_2(R)$ imply that both of $y''(x)$ and $V(x)y(x) \in L_2(R)$? More fundamental results of separation of differential operators were obtained by Everitt and Giertz [8,9]. A number of results concerning the property referred to the separation of differential operators was discussed by Biomatov [2], Otelbaev [16], Zettle [21] , Mohamed et al [10-15] and Zayed et al [17-19]. The separation for the differential operators with the matrix potential was first studied by Bergbaev [1]. Brown [3] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. Some separation criteria and inequalities associated with linear second order differential operators have been studied by many authers, see for examples [4, 5, 11, 13, 14, 15]. Recently, Zayed [20] has studied the separation for the following biharmonic differential operator in the Hilbert space H. In this article, we study the separation of the differential operator A of the form

$$Au(x,y) = -\frac{1}{2}\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{x^4}{4}\frac{\partial^2 u(x,y)}{\partial y^2}\right) + V(x,y)u(x,y). \tag{2}$$

Simply we denote the differential operator A by

$$Au(x,y) = Pu + V(x,y)u(x,y), \tag{3}$$

***Corresponding author:** Saleh Omran*, **E-mail:** salehomran@yahoo.com*

where $P=-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}-\frac{x^4}{4}\frac{\partial^2}{\partial y^2}\right)$ is the general Tricomi differential operator. We construct the coercive estimate for the differential operator (2). The existence and uniqueness theorem for the solution of the differential equation

$$Au(x,y)=Pu+V(x,y)u(x,y)=f(x,y),$$

in the Hilbert space $H=L_2\left(R^2,H_1\right)$ is also given, where $f(x,y)\in H$.

2. SOME NOTATIONS:

In this section we introduce the definitions that will be used in the subsequent section. We consider the Hilbert space H_1 with the norm $\|.\|_1$ and the inner product space $\langle u,v\rangle_1$. we introduce the Hilbert space $H=L_2\left(R^n,H_1\right)$ of all functions $u(x)$, $x\in R^n$ equipped with the norm

$$\|u\|^2=\int_{R^n}\|u(x)\|_1^2\,dx. \tag{5}$$

The symbol $\langle u,v\rangle$ where $u,v\in H$ denotes the scalar product in H which is defined by

$$\langle u,v\rangle=\int_{R^n}\langle u,v\rangle_1\,dx. \tag{6}$$

Let $W_2^2\left(R^n,H_1\right)$ be the space of all functions $u(x)$, $x\in R$ which have generalized derivatives $D^\alpha u(x)$, $\alpha\le 2$ such that $u(x)$ and $D^\alpha u(x)$ belong to H. We say that the function $u(x)$ for all $x\in R^n$ belongs to $W_{2,\,loc}^2\left(R^n,H_1\right)$ if for all functions $Q(x)\in C_0^\infty\left(R^n\right)$ the functions $Q(x)\,y(x)\in W_{2,\,loc}^2\left(R^n,H_1\right)$.

3. MAIN RESULTS:

Definition: The operator A of the form $Au(x,y)=-Pu(x,y)+V(x,y)u(x,y)$, $x,y\in R$ is said to be separated in the Hilbert space H if the following statement holds:

If $u(x,y)\in H\cap W_{2,\,loc}^2\left(R^2,H_1\right)$ and $Au(x)\in H$ imply that both $-Pu(x,y)$ and $V(x,y)u(x,y)\in H$.

Theorem: 1 If the following inequalities hold for all $x,y\in R,$

$$\left\|V_0^{\frac{1}{2}}\frac{\partial V}{\partial x}V^{-1}Vu\right\|\le\sigma_1\|Vu\|, \tag{7}$$

$$\left\|V_0^{-\frac{1}{2}}x^2\frac{\partial V}{\partial y}V^{-1}Vu\right\|\le\sigma_2\|Vu\|, \tag{8}$$

where σ_1 and σ_2 are positive constants satisfying $\sigma_1+\sigma_2<4$, then the coercive estimate

$$\|Vu\|+\|Pu\|+\left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\|+\left\|V_0^{\frac{1}{2}}x^2\frac{\partial u}{\partial y}\right\|\le N\|Au\|, \tag{9}$$

is valid, where $V_0=Re(V)$ and

$$\begin{aligned}N=1&+2\left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-1}+\left[\frac{1}{2}-\frac{\beta}{4}\sigma_1\right]^{-\frac{1}{2}}\left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-1}\\&+\left[\frac{1}{8}-\frac{\beta}{16}\sigma_2\right]^{-\frac{1}{2}}\left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-1},\end{aligned} \tag{10}$$

is a constant independent on $u(x,y)$ while $\in$ is given by

$$\frac{4\sigma_1+\sigma_2}{16}<\beta<\frac{4}{\sigma_1+\sigma_2}.$$

Then the operator A given by (4) is separated in the Hilbert space H.

Proof: From the definition of the inner product in the Hilbert space H, we can obtain

$$\left\langle \frac{\partial u}{\partial x_i},v\right\rangle=-\left\langle u,\frac{\partial v}{\partial x_i}\right\rangle, i=1,2,3,\dots,n \qquad \text{for all} \quad u,v\in C_0^\infty\left(R^n\right).$$

Hence

$$\begin{aligned}\langle Au,Vu\rangle&=-\frac{1}{2}\left\langle\frac{\partial^2 u}{\partial x^2},\ Vu\right\rangle-\frac{1}{8}\left\langle x^4\,\frac{\partial^2 u}{\partial y^2},\ Vu\right\rangle+\langle Vu,Vu\rangle\\ &=\frac{1}{2}\left\langle\frac{\partial u}{\partial x},\frac{\partial(Vu)}{\partial x}\right\rangle-\frac{1}{8}\left\langle\frac{\partial^2 u}{\partial y^2},\ x^4Vu\right\rangle+\langle Vu,Vu\rangle \qquad (11)\\ &=\frac{1}{2}\left\langle\frac{\partial u}{\partial x},V\frac{\partial u}{\partial x}\right\rangle+\frac{1}{2}\left\langle\frac{\partial u}{\partial x},u\frac{\partial V}{\partial x}\right\rangle+\frac{1}{8}\left\langle\frac{\partial u}{\partial y},x^4V\frac{\partial u}{\partial y}\right\rangle+\frac{1}{8}\left\langle\frac{\partial u}{\partial y},x^4u\frac{\partial V}{\partial y}\right\rangle+\langle Vu,Vu\rangle.\end{aligned}$$

Equating the real parts of the two sides of (11), we get

$$\begin{aligned}Re\langle Au,Vu\rangle&=\left\langle V_0^{\frac{1}{2}}\frac{\partial u}{\partial x},V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\rangle+\frac{1}{2}Re\left\langle V_0^{\frac{1}{2}}\frac{\partial u}{\partial x},V_0^{-\frac{1}{2}}\frac{\partial V}{\partial x}V^{-1}Vu\right\rangle+\frac{1}{8}\left\langle x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y},x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\rangle\\ &+\frac{1}{8}Re\left\langle x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y},V_0^{-\frac{1}{2}}x^2\frac{\partial V}{\partial y}V^{-1}Vu\right\rangle+\langle Vu,Vu\rangle,\end{aligned} \qquad (12)$$

Since for any complex number Z, we have

$$-|Z|\le Re Z\le|Z|, \qquad (13)$$

then by the Cauchy- Schwarz inequality, we obtain

$$Re\langle Au,Vu\rangle\le|\langle Au,Vu\rangle|\le\|Au\|\|Vu\|. \qquad (14)$$

With the help of (14) and conditions (7), (8), we have

$$Re\left\langle V_0^{\frac{1}{2}}\frac{\partial u}{\partial x},V_0^{\frac{1}{2}}\frac{\partial V}{\partial x}V^{-1}Vu\right\rangle\ge-\left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\|\left\|V_0^{\frac{1}{2}}\frac{\partial V}{\partial x}V^{-1}Vu\right\|\ge-\sigma_1\left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\|\|Vu\|, \qquad (15)$$

$$Re\left\langle x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y},x^2V_0^{-\frac{1}{2}}\frac{\partial u}{\partial y}\right\rangle\ge-\left\|x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\|\left\|x^2V_0^{-\frac{1}{2}}\frac{\partial V}{\partial y}V^{-1}Vu\right\|\ge-\sigma_2\left\|x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\|\|Vu\|, \qquad (16)$$

It is easy to see that for any $\beta>0$ and $y_{1,}y_2\in R^2$, then we have the inequality

$$|y_1||y_2|\le\frac{\beta|y_1|^2}{2}+\frac{|y_2|^2}{2\beta}. \qquad (17)$$

Applying (17) to (15) and (16), we have

$$Re\left\langle V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}, V_0^{\frac{1}{2}}\frac{\partial V}{\partial x}V^{-1}Vu\right\rangle \geq -\frac{\beta\sigma_1}{2}\left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\|^2 - \frac{\sigma_1}{2\beta}\|Vu\|^2, \tag{18}$$

$$Re\left\langle x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y}, x^2V_0^{-\frac{1}{2}}\frac{\partial V}{\partial y}V^{-1}Vu\right\rangle \geq -\frac{\beta\sigma_2}{2}\left\|x^2V_0^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\|^2 - \frac{\sigma_2}{2\beta}\|Vu\|^2, \tag{19}$$

From (13),(14) and (18) - (19), we conclude that

$$\left[1-\frac{\beta}{2}\sigma_1)\right]\left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\|^2 + \left[1-\frac{\beta}{2}\sigma_2\right]\left\|V_0^{\frac{1}{2}}y^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\|^2 + \left[1-\frac{1}{2\beta}(\sigma_1+\sigma_2)\right]\|Vu\|^2 \leq \|Vu\|\|Au\|. \tag{20}$$

Choosing $\frac{4\sigma_1+\sigma_2}{16} < \beta < \frac{4}{\sigma_1+\sigma_2}$, we obtain from (20) that

$$\|Vu\| \leq \left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-1}\|Au\|, \tag{21}$$

$$\left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\| \leq \left[\frac{1}{2}-\frac{\beta}{4}\sigma_1\right]^{-\frac{1}{2}}\left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-\frac{1}{2}}\|Au\|, \tag{22}$$

$$\left\|V_0^{\frac{1}{2}}x^2\frac{\partial u}{\partial y}\right\| \leq \left[\frac{1}{8}-\frac{\beta}{16}\sigma_2\right]^{-\frac{1}{2}}\left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-\frac{1}{2}}\|Au\|. \tag{23}$$

From (3) and (21) we get

$$\|Pu\| \leq \|Vu\| + \|Au\| \leq \left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-1}\|Au\|. \tag{24}$$

Consequently, we deduce from (21)-(24) that

$$\|Vu\| + \|Pu\| + \left\|V_0^{\frac{1}{2}}\frac{\partial u}{\partial x}\right\| + \left\|V_0^{\frac{1}{2}}x^2\frac{\partial u}{\partial x}\right\| \leq N\|Au\|, \tag{25}$$

where

$$N = 1+2\left[1-\frac{1}{16\beta}(4\sigma_1+\sigma_2)\right]^{-1} + \left[\frac{1}{2}-\frac{\beta}{4}\sigma_1\right]^{-\frac{1}{2}}\left[1-\frac{1}{16\beta}(\sigma_1+\sigma_2)\right]^{-1}$$
$$+\left[\frac{1}{8}-\frac{\beta}{16}\sigma_2\right]^{-\frac{1}{2}}\left[1-\frac{1}{16\beta}(\sigma_1+\sigma_2)\right]^{-1}.$$

This completes the proof..

Theorem: 2 If the operator A given by (3) is separated in the Hilbert space H and if there are positive functions $\psi(x,y)$ and $t(x,y)\in C^1(R^2)$ satisfying

$$\left\|t^{-1}\frac{\partial t}{\partial x}V_0^{-\frac{1}{2}}\right\| \leq 2\sqrt{\sigma_1}, \qquad \left\|x^2t^{-1}\frac{\partial t}{\partial y}V_0^{-\frac{1}{2}}\right\| \leq 2\sqrt{\sigma_2}, \tag{26}$$

where $0<\sigma_1+\sigma_2<\frac{\beta}{2}$, while β is defined in theorem1. Then the differential equation $Au=-Pu+Vu=f$, for all $f\in H$ has a unique solution in the Hilbert space H.

Proof: First, we prove the differential equation

$$Au=-Pu+Vu=0, \tag{27}$$

has the only zero solution $u(x,y)=0$ for all $x,y\in R$. To this end, we assume that $t(x,y)$ and $\psi(x,y)$ are two positive functions belonging to $C^1(R^2)$.

$$\begin{aligned}\langle Vu,t\psi u\rangle &= \langle Pu,t\psi u\rangle = \left\langle \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{x^4}{4}\frac{\partial^2 u}{\partial y^2}\right), t\psi u\right\rangle = \frac{1}{2}\left\langle \frac{\partial^2 u}{\partial x^2}, t\psi u\right\rangle + \frac{1}{8}\left\langle x^4\frac{\partial^2 u}{\partial y^2}, t\psi u\right\rangle \\ &= -\frac{1}{2}\left\langle \frac{\partial u}{\partial x}, \frac{\partial (t\psi u)}{\partial x}\right\rangle + \frac{1}{8}\left\langle \frac{\partial^2 u}{\partial y^2}, x^4 t\psi u\right\rangle = -\frac{1}{2}\left\langle \frac{\partial u}{\partial x}, t\psi\frac{\partial u}{\partial x}\right\rangle - \frac{1}{2}\left\langle \frac{\partial u}{\partial x}, tu\frac{\partial \psi}{\partial x}\right\rangle - \frac{1}{2}\left\langle \frac{\partial u}{\partial x}, \psi u\frac{\partial t}{\partial x}\right\rangle \\ &\quad -\frac{1}{8}\left\langle \frac{\partial u}{\partial y}, x^4 t\psi\frac{\partial u}{\partial y}\right\rangle - \frac{1}{8}\left\langle \frac{\partial u}{\partial y}, x^4 tu\frac{\partial \psi}{\partial y}\right\rangle - \frac{1}{8}\left\langle \frac{\partial u}{\partial y}, x^4\psi u\frac{\partial t}{\partial y}\right\rangle.\end{aligned} \tag{28}$$

Equating the real parts of both sides of (28), we have

$$\begin{aligned}\langle V_0u,t\psi u\rangle &= \left\langle t^{\frac12}\psi^{\frac12}V_0^{\frac12}u, t^{\frac12}\psi^{\frac12}V_0^{\frac12}u\right\rangle = -\frac{1}{2}\left\langle t^{\frac12}\psi^{\frac12}\frac{\partial u}{\partial x}, t^{\frac12}\psi^{\frac12}\frac{\partial u}{\partial x}\right\rangle - \frac{1}{2}Re\left\langle \frac{\partial u}{\partial x}, tu\frac{\partial \psi}{\partial x}\right\rangle \\ &\quad -\frac{1}{2}Re\left\langle \frac{\partial u}{\partial x}, \psi u\frac{\partial t}{\partial x}\right\rangle - \frac{1}{8}\left\langle t^{\frac12}\psi^{\frac12}x^2\frac{\partial u}{\partial y}, t^{\frac12}\psi^{\frac12}x^2\frac{\partial u}{\partial y}\right\rangle - \frac{1}{8}Re\left\langle \frac{\partial u}{\partial y}, x^4 tu\frac{\partial \psi}{\partial y}\right\rangle \\ &\quad -\frac{1}{8}Re\left\langle \frac{\partial u}{\partial y}, x^4\psi u\frac{\partial t}{\partial y}\right\rangle.\end{aligned} \tag{29}$$

By using Cauchy- Schwarz inequality, we obtain

$$\begin{aligned}2\,Re\left\langle \frac{\partial u}{\partial x}, tu\frac{\partial \psi}{\partial x}\right\rangle &= \left\langle \frac{\partial u}{\partial x}, tu\frac{\partial \psi}{\partial x}\right\rangle + \overline{\left\langle \frac{\partial u}{\partial x}, tu\frac{\partial \psi}{\partial x}\right\rangle} = \left\langle \frac{\partial u}{\partial x}, tu\frac{\partial \psi}{\partial x}\right\rangle + \left\langle u, t\frac{\partial \psi}{\partial x}\frac{\partial u}{\partial x}\right\rangle \\ &= -\left\langle u, \frac{\partial}{\partial x}(tu\frac{\partial \psi}{\partial x})\right\rangle + \left\langle u, t\frac{\partial \psi}{\partial x}\frac{\partial u}{\partial x}\right\rangle \\ &= -\left\langle u, tu\frac{\partial^2 \psi}{\partial x^2}\right\rangle - \left\langle u, t\frac{\partial \psi}{\partial x}\frac{\partial u}{\partial x}\right\rangle - \left\langle u, u\frac{\partial \psi}{\partial x}\frac{\partial t}{\partial x}\right\rangle + \left\langle u, t\frac{\partial \psi}{\partial x}\frac{\partial u}{\partial x}\right\rangle \\ &= -\left\langle u, tu\frac{\partial^2 \psi}{\partial x^2}\right\rangle - \left\langle u, u\frac{\partial \psi}{\partial x}\frac{\partial t}{\partial x}\right\rangle = -\left\langle u, (tu\frac{\partial^2 \psi}{\partial x^2} + u\frac{\partial \psi}{\partial x}\frac{\partial t}{\partial x})\right\rangle \\ &= -\left\langle u, (t\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial t}{\partial x})u\right\rangle = -\left\| (t\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial t}{\partial x})^{\frac12}u\right\|^2,\end{aligned} \tag{30}$$

On the other hand, we have

$$\begin{aligned}Re\left\langle \frac{\partial u}{\partial x}, u\psi\frac{\partial t}{\partial x}\right\rangle &= Re\left\langle t^{\frac12}\psi^{\frac12}\frac{\partial u}{\partial x}, t^{\frac12}\psi^{\frac12}\left[t^{-1}\frac{\partial t}{\partial x}V_0^{-\frac12}\right]V_0^{\frac12}u\right\rangle \\ &\ge -\left\| t^{\frac12}\psi^{\frac12}\frac{\partial u}{\partial x}\right\| \left\| t^{\frac12}\psi^{\frac12}\left[t^{-1}\frac{\partial t}{\partial x}V_0^{-\frac12}\right]V_0^{\frac12}u\right\| \\ &\ge -\frac{\beta}{2}\left\| t^{\frac12}\psi^{\frac12}\frac{\partial u}{\partial x}\right\|^2 - \frac{1}{2\beta}\left\| t^{\frac12}\psi^{\frac12}\left[t^{-1}\frac{\partial t}{\partial x}V_0^{-\frac12}\right]V_0^{\frac12}u\right\|^2 \\ &\ge -\frac{\beta}{2}\left\| t^{\frac12}\psi^{\frac12}\frac{\partial u}{\partial x}\right\|^2 - \frac{2\sigma_1}{\beta}\left\| t^{\frac12}\psi^{\frac12}V_0^{\frac12}u\right\|^2,\end{aligned} \tag{31}$$

$$\begin{aligned}2\,Re\left\langle \frac{\partial u}{\partial y}, tx^4 u\frac{\partial\psi}{\partial y}\right\rangle &= \left\langle \frac{\partial u}{\partial y}, tx^4 u\frac{\partial\psi}{\partial y}\right\rangle + \overline{\left\langle \frac{\partial u}{\partial y}, tx^4 u\frac{\partial\psi}{\partial y}\right\rangle} = \left\langle \frac{\partial u}{\partial y}, tx^4 u\frac{\partial\psi}{\partial y}\right\rangle + \left\langle u, tx^4 \frac{\partial\psi}{\partial y}\frac{\partial u}{\partial y}\right\rangle \\ &= -\left\langle u, \frac{\partial}{\partial y}(tx^4 u\frac{\partial\psi}{\partial y})\right\rangle + \left\langle u, tx^4 \frac{\partial\psi}{\partial y}\frac{\partial u}{\partial y}\right\rangle = -\left\langle u, tx^4 u\frac{\partial^2\psi}{\partial y^2}\right\rangle - \left\langle u, tx^4 \frac{\partial\psi}{\partial y}\frac{\partial u}{\partial y}\right\rangle \\ &\quad -\left\langle u, x^4 u\frac{\partial\psi}{\partial y}\frac{\partial t}{\partial y}\right\rangle + \left\langle u, tx^4 \frac{\partial\psi}{\partial y}\frac{\partial u}{\partial y}\right\rangle = -\left\langle u, tx^4 u\frac{\partial^2\psi}{\partial y^2} + x^4 u\frac{\partial\psi}{\partial y}\frac{\partial t}{\partial y}\right\rangle \\ &= -\left\langle u, x^4 (t\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\psi}{\partial y}\frac{\partial t}{\partial y})u\right\rangle = -\left\| x^2 (t\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\psi}{\partial y}\frac{\partial t}{\partial y})^{\frac{1}{2}} u\right\|^2,\end{aligned} \tag{32}$$

$$\begin{aligned}Re\left\langle \frac{\partial u}{\partial y}, x^4\psi u\frac{\partial t}{\partial y}\right\rangle &= Re\left\langle x^2\psi^{\frac{1}{2}}t^{\frac{1}{2}}\frac{\partial u}{\partial y}, x^2\psi^{\frac{1}{2}}t^{\frac{1}{2}}\left[t^{-1}\frac{\partial t}{\partial y}V_0^{\frac{-1}{2}}\right]V_0^{\frac{1}{2}}u\right\rangle \\ &= Re\left\langle x^2\psi^{\frac{1}{2}}t^{\frac{1}{2}}\frac{\partial u}{\partial y}, \psi^{\frac{1}{2}}t^{\frac{1}{2}}\left[x^2 t^{-1}\frac{\partial t}{\partial y}V_0^{\frac{-1}{2}}\right]V_0^{\frac{1}{2}}u\right\rangle \\ &\ge -\left\| x^2\psi^{\frac{1}{2}}t^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\| \left\| \psi^{\frac{1}{2}}t^{\frac{1}{2}}\left[x^2 t^{-1}\frac{\partial t}{\partial y}V_0^{\frac{-1}{2}}\right]V_0^{\frac{1}{2}}u\right\| \\ &\ge -\frac{\beta}{2}\left\| x^2\psi^{\frac{1}{2}}t^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\|^2 - \frac{1}{2\beta}\left\| \psi^{\frac{1}{2}}t^{\frac{1}{2}}\left[x^2 t^{-1}\frac{\partial t}{\partial y}V_0^{\frac{-1}{2}}\right]V_0^{\frac{1}{2}}u\right\|^2 \\ &\ge -\frac{\beta}{2}\left\| x^2\psi^{\frac{1}{2}}t^{\frac{1}{2}}\frac{\partial u}{\partial y}\right\|^2 - \frac{2\sigma_2}{\beta}\left\| \psi^{\frac{1}{2}}t^{\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2.\end{aligned} \tag{33}$$

Putting $\beta = 2,$ in the above inequalities and using Eqs (30)-(33), we have:

$$[1-\sigma_1-\sigma_2]\left\| t^{\frac{1}{2}}\psi^{\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2 \le \frac{1}{2}\left\| (t\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial x}\frac{\partial t}{\partial x})^{\frac{1}{2}} u\right\|^2 + \frac{1}{2}\left\| x^2(t\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\psi}{\partial y}\frac{\partial t}{\partial y})^{\frac{1}{2}} u\right\|^2. \tag{34}$$

Choosing $\psi(x, y) = 1, t(x, y) = 1$ for all $x, y \in R$, then if $0 < \sigma_1 + \sigma_2 < 1$, we deduce from (34) that

$$[1-\sigma_1-\sigma_2]\left\| t^{\frac{1}{2}}\psi^{\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2 \le 0. \tag{35}$$

From (5) and (35), we deduce that

$$0 < [1-\sigma_1-\sigma_2]\int_{R^2}\left\| t^{\frac{1}{2}}\psi^{\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|_1^2 dxdy \le 0. \tag{36}$$

The inequality (36) holds only for $u(x, y) = 0$. This proves that $u(x, y) = 0$ is the only solution of Eq.(27). Second, we know that the linear manifold $M = \{ f \ :\ Au = f$ for all $u \in C_0^\infty (R^2)\}$ is dense everywhere in H. So there exist a sequence of functions $\{u_r\} \in C_0^\infty(R^2)$, such that for all $f \in H,$ we have $\| Au_r - f \| \to 0,$ as $r \to \infty$. By applying the coercive estimate (25), we find that

$$\| V(u_p - u_r)\| + \| p(u_p - u_r)\| + \left\| V_0^{\frac{1}{2}}\frac{\partial(u_p - u_r)}{\partial x}\right\| + \left\| V_0^{\frac{1}{2}}x^2\frac{\partial(u_p - u_r)}{\partial y}\right\| \le N\| A(u_p - u_r)\|, \tag{37}$$

where $u = u_p - u_r$ for all $p, r = 1,2,,....$ As $p \to \infty, r \to \infty$. It follows that the sequences $\{Vu_r\}, \{Pu_r\},$ $\left\{ V_0^{\frac{1}{2}}\frac{\partial u_r}{\partial x}\right\}$

and $\left\{V_0^{\frac{1}{2}} x^2 \frac{\partial u_r}{\partial y}\right\}$ converge in H. Then there exist a vector functions $w_0, w_1,$ w_2 and w_3 in H such that $\|V(u_r - w_0)\|, \|P(u_r - w_1)\|$, $\left\|V_0^{\frac{1}{2}} \frac{\partial(u_r - w_2)}{\partial x}\right\|$ and $\left\|V_0^{\frac{1}{2}} x^2 \frac{\partial(u_r - w_2)}{\partial y}\right\|$ are convergent to zero, as $r \to \infty$.

This implies that $u_r \to V^{-1} w_0 = u$, $Pu_r \to Pu$, $V_0^{\frac{1}{2}} \frac{\partial u_r}{\partial x} \to V_0^{\frac{1}{2}} \frac{\partial u}{\partial y}$ and $V_0^{\frac{1}{2}} x^2 \frac{\partial u_r}{\partial y} \to V_0^{\frac{1}{2}} x^2 \frac{\partial u}{\partial y}$ as $r \to \infty$. Hence for any $f \in H$ there exist $u \in H \cap W_{2,loc}^2(R^2, H_1)$, such that $Au = f$.

Suppose that $\bar{u}$ is another solution of the equation $Au = f$, then $A(u - \bar{u}) = 0$. But $Au = 0$ has only the zero solution, then $u = \bar{u}$ and the uniqueness is proved. Hence, the proof of theorem is completed.

REFERENCES:

[1] A. Bergbaev, Smooth solution of non-linear differential equation with matrix potential, the VII Scientific Conference of Mathematics and Mechanics AlmaAta. (1989). (Russian).

[2] K. Kh. Biomatov., Coercive estimates and separation for second order elliptic differential equations, Soviet Math. Dok1. 38 (1989). English trans1. in American Math. Soc., (1989) 157-160.

[3] R.C. Brown, Separation and disconjugacy, J. Inequal. Pure and Appl. Math., 4 (2003) Art. 56.

[4] R.C. Brown and D. B. Hinton, Two separation criteria for second order ordinary or partial differential operators, Math. Bohemica, 124 (1999) 273 - 292.

[5] R.C. Brown and D. B. Hinton and M. F. Shaw, Some separation criteria and inequalities associated with linear second order differential operators, in function spaces and applications, Narosa publishing House, New Delhi, (2000) 7-35.

[6] W. N. Everitt and M. Giertz, Some properties of the domains of certain differential operators, Proc. London Math. Soc., 23 (1971) 301-324.

[7] W. N. Everitt and M. Giertz, Some inequalities associated with certain differential operators, Math. Z., 126 (1972) 308-326.

[8] W. N. Everitt and M. Giertz, On some properties of the powers of a familly self-adjoint differential expressions, Proc. London Math. Soc., 24 (1972) 149-170.

[9] W. N. Everitt and M. Giertz, Inequalities and separation for Schrodinger-type operators in $L_2\left(R^n\right)$, Proc. Roy. Soc. Edin., 79A (1977) 257-265.

[10] A. S. Mohamed, Separation for Schrodinger operator with matrix potential, Dok1. Acad. Nauk Tajkctan , 35 (1992) 156-159. (Russian).

[11] A. S. Mohamed and B. A. El-Gendi, Separation for ordniary differential equation with matrix coefficient, Collect. Math., 48 (1997) 243-252.

[12] A. S. Mohamed, Existence and uniqueness of the solution, separation for certain second order elliptic differential equation, Applicable Analysis, 76 (2000) 179-185.

[13] A. S. Mohamed and H. A. Atia, Separation of the Sturm-Liouville differential operator with an operator potential, Applied Mathematics and Computation, 156 (2004) 387-394.

[14] A. S. Mohamed and H. A. Atia, Separation of the Schrodinger operator with an operator potential in the Hilbert spaces, Applicable Analysis, 84 (2005) 103-110.

[15] A. S. Mohamed and H. A. Atia, Separation of the general second order elliptic differential operator with an operator potential in the weighted Hilbert spaces, Applied Mathematics and Computation, 162 (2005) 155-163.

[16] M. Otelbaev, On the separation of elliptic operator, Dok1. Acad. Nauk SSSR, 234 (1977) 540-543. (Russian).

[17] E. M. E. Zayed, A. S. Mohamed and H. A. Atia, Separation for Schrodinger type operators with operator potential in Banach spaces, Applicable Analysis, 84 (2005) 211-220.

[18] E. M. E. Zayed, A. S. Mohamed and H. A. Atia, On the separation of elliptic differential operators with operator potential in weighted Hilbert spaces, Panamerican Mathematical Journal, 15 (2005) 39-47.

[19] E. M. E. Zayed, A. S. Mohamed and H. A. Atia, Inequalities and separation for the Laplace Beltrami differential operator in Hilbert spaces, J. Math. Anal. Appl., 336 (2007) 81-92.

[20] E. M. E. Zayed, Separation for the biharmonic differential operator in the Hilbert space associated with the existence and uniqueness Theorem, J. Math. Anal. Appl., 337 (2008) 659-666.

[21] A. Zettle, Separation for differential operators and the L_p spaces, Proc. Amer. Math. Soc., 55(1976) 44-46.

****