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ON DISLOCATED METRIC SPACES

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ABSTRACT

The notion of dislocated metric is one of the various generalizations of metric that retains a variant of the illustrious Banach's Contraction principle and has useful applications in the semantic analysis of logic programming. The purpose of this note is to study topological aspects of a dislocated metric space and prove a dislocated metric version of Seghal's fixed point theorem which ultimately implies existence(and uniqueness in some cases) of a fixed point for self maps that satisfy conditions analogous to those of Banach, Kannan, Bianchini, Reitch and Rakotch [4].

Keywords: *dislocated metric, kuratowski's axioms, coincidence point, Contractive conditions.*

Subject classification: *47H10.*

INTRODUCTION:

Pascal Hitzler [1] presented variants of Banach's Contraction principle for various modified forms of a metric space including dislocated metric space and applied them to semantic analysis of logic programs. In this context Hitzler raised some related questions on the topological aspects of dislocated metrics.

In this paper we present results that establish existence of a topology induced by a dislocated metric and show that this topology is amortizable, by actually showing a metric that induces the topology.

Rhoades [4] collected a large number of variants of Banach's Contractive conditions on self maps on a metric space and proved various implications or otherwise among them. We pick up a good number of these conditions which ultimately imply Seghal's condition [4]. We prove that these implications hold good for self maps on a dislocated metric space and prove the dislocated metric version of Seghal's result there by deriving the dislocated analogue's of fixed point theorems of Banach, Kannan, Bianchini, Reich and others[4].

1. THE d -TOPOLOGY:

Definition 1.1: Let X be a set and $d : X \times X \to R$ be a mapping satisfying the following conditions for x, y, z in X.

(i) $d(x, y) \geq 0$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) = 0$ implies $x = y$ and
(iv) $d(x, y) \leq d(x, z) + d(z, y)$

Then d is called dislocated (simply d -) metric on X and the pair (X, d) is called a dislocated (d -) metric space. In what follows (X, d) stands for a d metric space.

If $x \in X$ and $\in > 0$ the set,

$B_{\in}(x) = \{ y / y \in X$ and $d(x, y) < \in \}$ is called the open ball with centre at x and radius $\in$.

Definition 1.2: We say that a net $(x_\alpha / \alpha \in \Delta)$ in X converges to x in (X, d) and write $\lim_{\alpha}(x_\alpha / \alpha \in \Delta) = x$ if $\lim_{\alpha} d(x_\alpha, x) = 0$.

Remark: The limit of a net in (X, d) is unique.

Notation: For $A \subset X$ we write $D(A) = \{ x \in X \,/\, x$ is a limit of a net in $(X, d)\}$

Proposition 1.3: Let $A \subseteq X$ and $B \subseteq X$ .Then
(i) $D(A) = \phi$ if $A = \phi$
(ii) $D(A) \subseteq D(B)$ if $A \subseteq B$
(iii) $D(A \cup B) = D(A) \cup D(B)$ and
(iv) $D(D(A)) \subseteq D(A)$

Proof: (i) and (ii) are clear. That $D(A) \cup D(B) \subseteq D(A \cup B)$ follows from (ii). To prove the reverse inclusion, Let $x \in D(A \cup B)$ and $x = \lim_{\alpha \in \Delta}(x_\alpha)$ where $(x_\alpha / \alpha \in \Delta)$ is a net in $A \cup B$. If $\exists\ \lambda \in \Delta$ such that $x_\alpha \in A$ for $\alpha \in \Delta$ and $\alpha \geq \lambda$ then $(x_\alpha / \alpha \geq \lambda, \alpha \in \Delta)$ is a cofinal subnet of $(x_\alpha / \alpha \in \Delta)$ and is in A and $\lim_{\alpha \geq \lambda} d(x, x_\alpha) = \lim_{\alpha \in \Delta} d(x, x_\alpha) = 0$ so that $x \in D(A)$. If no such λ exists in Δ then for every $\alpha \in \Delta$, choose $\beta(\alpha) \in \Delta$ such that $\beta(\alpha) \geq \alpha$ and $x_{\beta(\alpha)} \in B$ .Then $(x_{\beta(\alpha)} / \alpha \in \Delta)$ is a cofinal subnet in B of $(x_\alpha / \alpha \in \Delta)$ and $\lim_{\alpha \in \Delta} d(x_{\beta(\alpha)}, x) = \lim_{\alpha \in \Delta} d(x_\alpha, x) = 0$ so that $x \in D(B)$.It now follows that $D(A \cup B) \subseteq D(A) \cup D(B)$ and hence (iii) holds. To prove (iv) let $x \in D(D(A))$, $x = \lim_{\alpha \in \Delta} x_\alpha$, $x_\alpha \in D(A)$ for $\alpha \in \Delta$, and $\forall\ \alpha \in \Delta$, let $(x_{\alpha_\beta} / \beta \in \Delta(\alpha))$ be a net in $A \ni x_\alpha = \lim_{\beta \in \Delta(\alpha)} x_{\alpha_\beta}$ .For each positive integer $i \ \exists\ \alpha_i \in \Delta$ such that $d(x_{\alpha_i}, x) < \frac{1}{i}$,and $\beta_i \in \Delta(\alpha_i) \ni d(x_{\alpha_{i_{\beta_i}}}, x_{\alpha_i}) < \frac{1}{i}$. Write $\alpha_{i_{\beta_i}} = \gamma_i\ \forall\ i$, then $\{\gamma_1, \gamma_2, ...\}$ is directed set with $\gamma_i < \gamma_j$ if $i < j$, and $d(x_{\gamma_i}, x) \leq d(x_{\gamma_i}, x_{\alpha_i}) + d(x_{\alpha_i}, x) < \frac{2}{i}$. This implies that $x \in D(A)$.

As a corollary, we have the following

Corollary 1.4: If for $A \subset X$ and $\overline{A} = A \cup D(A)$, then the operation $A \to \overline{A}$ on P(X) satisfies Kuratowski's closure axioms [2]:
(i) $\overline{\phi} = \phi$
(ii) $A \subset \overline{A}$
(iii) $\overline{A} = \overline{\overline{A}}$ and
(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Consequently we have the following

Theorem 1.5: Let $\Im$ be the family of all subsets A of X for which $\overline{A} = A$ and τ be the complements of members of $\Im$. Then the $\Im$ is a topology for X and the $\Im$ -closure of a subset A of X is $\overline{A}$.

Definition 1.6: The topology $\Im$ obtained in Theorem 1.5 is called the topology induced by d and simply referred to as the d -topology of X and is denoted by $(X, d, \Im)$.

Proposition 1.7: Let $A \subseteq X$. Then $x \in D(A)$ iff for every $\delta > 0$, $B_\delta(x) \cap A \neq \phi$

Proof: If $x \in D(A)$,there exist a net $(x_\alpha / \alpha \in \Delta)$ in A such that lim $x_\alpha = x$, if $\delta > 0$ $\exists$ $\alpha_\delta \in \Delta$ such that $d(x_\alpha, x) < \delta$.If $\alpha \in \Delta$ and $\alpha \geq \alpha_\delta$.Hence $x_\alpha \in B_\delta(x) \cap A$ for $\alpha \geq \alpha_\delta$.
Conversely if for every $\delta > 0, B_\delta(x) \cap A \neq \phi$. We choose one x_n in $B_{\frac{1}{n}}(x) \cap A$ for every integer $n \geq 1$.Clearly $(x_n)$is a net in A and $d(x_n, x) < \frac{1}{n}$, so that $\lim_n x_n = x$. Hence $x \in \overline{A}$.

Corollary 1.8: $x \in \overline{A} \Leftrightarrow x \in A$ or $B_\delta(x) \cap A \neq \phi$ $\forall \delta > 0$.

Corollary 1.9: A is open in $(X, d, \Im)$ if and only if for every $x \in A$, $\exists$ $\delta > 0$ such that

$$\{x\} \cup B_\delta(x) \subseteq A.$$

Proposition 1.10: If $x \in X$ and $\delta > 0$ then $\{x\} \cup B_\delta(x)$ is an open set in$(X, d, \Im)$.

Proof: Let $A = \{x\} \cup B_\delta(x)$, $y \in B_\delta(x)$ and $0 < r < \delta - d(x, y)$
Then $B_r(y) \subseteq B_\delta(x) \subset A$ so that $B_r(y) \cap (X - A) = \phi$

Hence $(X - A)$ is closed in $(X, d, \Im)$.

Corollary 1.11: If $x \in X$ and $V_r(x) = B_r(x) \cup \{x\}$ for $r > 0$ then the collection $\{V_r(x) / x \in X\}$ is an open base at x in $(X, d, \Im)$.If d is a metric and $V = B_r(x)$, $\Im$ coincides with the metric topology.

Proposition 1.12: $(X, d, \Im)$ is a Hausdorff space.

Proof: If $x, y \in X$, and $d(x, y) > 0$ then $V_{\frac{\delta}{2}}(x) \cap V_{\frac{\delta}{2}}(y) = \phi$.

Corollory 1.13: If $x \in X$, the collection $\{V_r(x) / x \in X\}$ is an open base at x for $(X, d, \Im)$.Hence $(X, d, \Im)$ is first countable.

Remark: Corollory 1.13 enables us to deal with sequence instead of nets.

Proposition 1.14: Define ρ on X x X by $\rho(x, y) = \begin{cases} d(x,y) & if \;\; x \neq y \\ 0 & if \;\; x = y \end{cases}$ .Here ρ is a metric on X .
For $x \in X$ and $\in > 0$, $V_\in(x) = \{y \in X / \rho(x, y) < \in\}$.
Moreover a sequence (x_n) in X converges to x in (X, ρ) if and only if
(i) $x_n = x$ except for finitely many n or
(ii) lim $d(x_{n_k}, x) = 0$ for every subsequence(x_{n_k}) of (x_n) with $x_{n_k} \neq x$.

2. CONTINUITY:

Definition 2.1: Let (X, d) and (y, d') be d metric spaces. $f : X \to y$ is said to be d continuous at $x \in X$ if for every $\in > 0$ $\exists$ $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\in(f(x))$. f is d continuous if f is d continuous at every x in X .

Theorem 2.2: $f : (X, d) \to (y, d')$ is d continuous at x if and only if $f : (X, d, \Im) \to (y, d', \Im')$ is continuous at x where $\Im$ and $\Im'$ are the corresponding induced topologies.

Proof: Assume that f is d continuous at x. Let V be a neighbourhood of $f(x)$ in $(y, d', \Im')$.Then
$\exists \in > 0 \ni B_\in(f(x)) \subseteq V$. By hypothesis, $\exists$ $\delta > 0$ $\ni$ $f(B_\delta(x)) \subseteq B_\in(f(x))$
$\Rightarrow f(\{x\} \cup B_\delta(x)) \subseteq B_\in(\{f(x)\} \cup f(x)) \subseteq V$

Since $\{x\} \cup B_\delta(x)$ is open in $(X, d, \Im)$ into $(y, d' \Im')$. It follows that f from $(X, d, \Im)$ into $(y, d' \Im')$ is continuous at x. And $\in > 0$,

$$B_\in(f(x) \cup f(x)) \in \Im'. \text{ So } \exists\ \delta > 0 \ni f(\{x\} \cup B_\delta(x)\} \subseteq B_\in(f(x) \cup f(x))$$

Hence $f(B_\delta(x)) \subseteq B_\in(f(x))$
$\Rightarrow f : X \to y$ is d continuous.

Corollary 2.3: If (X, d), (y, d') are d metric spaces and ρ, ρ' are the induced metrics corresponding to d and d' respectively then, $f : (X, d) \to (y, d')$ is d continuous at x if and only if $f : (X, \rho) \to (\mathrm{Y}, \rho')$ is continuous.

Let (X, d) be a d metric space and $f : X \to X$ be a mapping. Write $V(x) = d(x, f(x))$ and Z(f)=\{ $x / V(x)$=0\}.Clearly every point of Z(f) is a fixed point of f but the converse is not necessarily true. We call points of Z(f) as coincidence points of f.The set Z(f) is a closed subset of X. Mathew's theorem [3] states that a contraction on a complete d metric space has a unique fixed point. The same theorem has been justified by an alternate proof by Pascal Hitzler [1].We present an extension of this theorem for coincidence points.

3. MAIN RESULTS:

Theorem 2.4: Let (X, d) be a complete d metric space and $f : X \to X$ be a contraction. Then there is a unique coincidence point for f.

Proof: For any $x \in X$ the sequence of iterates satisfies

$d(f^n(x), f^{n+1}(x)) \le \alpha^n d(x, f(x))$ where α is any contractive constant. Consequently if $n < m$

$$d(f^n(x), f^m(x)) \le (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1}) d(x, f(x))$$
$$= \alpha^n \frac{(1-\alpha^{m-n})}{1-\alpha} d(x, f(x))$$

Hence $\{ f^n(x) \}$ is Cauchy sequence in X.

If $\xi = \lim_n f^n(x)$ then $f(\xi) = \lim_n f^{n+1}(x)$
so $d(\xi, f(\xi)) = \lim_n d(f^n(x), f^{n+1}(x))$. Since $d(f^n(x), f^{n+1}(x)) < \alpha^n\ d(x, f(x))$

Since $0 < \alpha < 1$; $\lim \alpha^n\ d(x, f(x)) = 0$ Hence $d(\xi, f(\xi)) = 0$

Uniqueness : If $d(\xi, f(\xi)) = d(\eta, f(\eta)) = 0$, then $f(\xi) = \xi$ and $f(\eta) = \eta$ so that

$$d(\xi, \eta) \le d(\xi, f(\xi)) + d(f(\xi), f(\eta)) + d(f(\eta), \eta)$$
$$\le \alpha\ d(\xi, \eta)$$

so that $d(\xi, \eta)$=0. Hence $\xi = \eta$.

Theorem 2.5: Let (X, d) be a d metric space and $f : X \to X$ be continuous .Assume that $d(f(x), f(y)) < \max\{d(x, f(x)), d(y, f(y)), d(x, y)\}$ whenever $d(x, y) \neq 0$.Then f has a unique coincidence point whenever $clO(x)$ is nonempty for some $x \in X$.

Proof: Write $V(x) = d(x, f(x))$, $Z = \{x / V(x) = 0\}$; $O(x) = \{f^n(x) / n \ge 0\}$
Since f is continuous, V is continuous.

If $x \notin Z$ then $V(f(x)) = d(f(x), f^2(x)) < \max\{d(x, f(x)), d(f(x), f^2(x)), d(x, f(x))\}$
$= \max\{V(x), V(f(x))\}$

$\Rightarrow V(f(x)) < V(x)$ whenever $V(x) \neq 0$ i.e. $x \notin Z$ (1)

If $O(x) \cap Z = \phi$ then $V(f^{k+1}(x)) < V(f^k(x)) \quad \forall k$

Hence $V(f^n(x))$ is convergent. (2)

let ξ be a cluster point of $O(x)$, $\exists (n_i)\uparrow \ni \xi = \lim f^{n_i}(x)$

$\Rightarrow f^k(\xi) = \lim f^{n_i+k}(x)$
$\Rightarrow O(\xi) \leq clO(x)$.

Since V is continuous $V(f^k(\xi)) = \lim V(f^{n_i+k}(x))$

Since $O(x) \cap Z = \phi$ by (2) $\{V(f^n(x))\}$ is convergent.

Let $\gamma = \lim V(f^{n_i}(x)) = V(\lim f^{n_i}(x)) = V(\xi)$.

Also $\gamma = \lim V(f^{n_i+1}(x)) = V(f(\xi)) \quad ; \forall k$
$\Rightarrow V(f(\xi)) = V(\xi)$ (4)

From (1) and (3) it follows that $V(\xi) = 0$

If $V(\xi) = V(\eta) = 0$ then
$\xi = f(\xi)$, $\eta = f(\eta)$ if $d(\xi, \eta) \neq 0$
$d(\xi, \eta) = d(f(\xi), f(\eta)) < \max\{V(\xi), V(\eta), d(\xi, \eta)\} = d(\xi, \eta)$

which is a contradiction.

Hence $d(\xi, \eta) = 0$.

B.E Rhodes [4] presented a list of definitions of contractive type conditions for a self map on a metric space (X, d) and established implications and non implications among them ,there by facilitating to check the implication of any new contractive condition any one of the condition mentioned in [4] so as to derive a fixed point theorem. Among the conditions in [4], Seghal's condition is significant as a good number of Contractive conditions imply Seghal's condition. We now present the dislocated versions of these conditions.

Let (X, d) be a dislocated metric space and $f : X \to X$ be a mapping and x, y be any elements of X. Consider the following conditions

1. (Banach): there exists a number α , $0 \leq \alpha \leq 1$ such that for each $x, y \in X$
$$d(f(x), f(y)) \leq \alpha\, d(x, y).$$
2. (Rakotch): there exists a monotone decreasing function $\alpha : (0, \infty) \to [0,1)$ such that
$$d(f(x), f(y)) \leq \alpha\, d(x, y) \text{ whenever } d(x, y) \neq 0.$$
3. (Edelstein): $d(f(x), f(y)) < d(x, y)$ whenever $d(x, y) \neq 0$
4. (Kannan): there exists a number α , $0 < \alpha < \frac{1}{2}$ such that
$$d(f(x), f(y)) < \alpha\,[d(x, f(x)) + d(y, f(y))]$$
5. (Bianchini): there exists a number h ,$0 \leq h < 1$ such that

$$d(f(x), f(y)) \leq h \max\{d(x, f(x)), d(y, f(y))\}$$

6. $d(f(x), f(y)) < \max\{d(x, f(x)), d(y, f(y))\}$ whenever $d(x, y) \neq 0$
7. (Reich): there exist nonnegative numbers a, b, c satisfying a + b + c < 1 such that

$$d(f(x), f(y)) \leq a\,d(x, f(x)) + b\,d(y, f(y)) + c\,d(x, y)$$

8. (Reich): there exist monotonically decreasing functions a, b, c from $(0, \infty)$ to [0, 1) satisfying a(t) + b(t) + c(t) < 1 such that,

$$d(f(x), f(y)) \leq a\,d(x, y)\,d(x, f(x)) + b\,d(x, y)\,d(y, f(y)) + c\,d(x, y)\,d(x, y)$$

where $d(x, y) \neq 0$

9. There exist nonnegative functions a, b, c satisfying , $\sup_{x,y\in X}\{a(x, y)+b(x, y)+c(x, y)\} < 1$

such that $d(f(x), f(y)) \leq a(t)\,d(x, f(x)) + b(t)\,d(y, f(y)) + c(t)t$, where t=$(x, y)$.

10. (Sehgal): $d(f(x), f(y)) < \max\{d(x, f(x)), d(y, f(y)), d(x, y)\}$ if $d(x, y) \neq 0$

Theorem 2.6 If f is a self map on a dislocated metric space (X, d) and f satisfies any of the conditions (1) through (9) then f has a unique fixed point provided $cl\,O(x)$ is nonempty for some $x \in X$.

Proof: In [4] B.E Rhodes proved that when d is a metric

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (10)
(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (10)
(4) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (10)
(5) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$ (10)

With $d(x, y) \neq 0$ is replaced by $x \neq y$ .Consequently these implications hold good in a d metric space as well since $x \neq y \Rightarrow d(x, y) \neq 0$ in a d metric space .It now follows from 2.5 that f has a fixed point which is unique when $O(x)$ has a cluster point for some x .

Example 2.7: Define $d(x, y) = |x| + |y|$ For x, y in R. d is a dislocated metric on R.

If $\in > 0$, $B_0(\in) = (-\in, \in)$

If $x \neq 0$, $\in > 0$, $B_{\in}(x) = \begin{cases} |x| - \in, \in - |x| & \text{if } \in > |x| \\ \phi & \text{if } \in < |x| \end{cases}$. Also $0 \neq x \in B_{\in}(x)$ iff $|x| < \dfrac{\in}{2}$

Result 2.6 Define $f: R \to R$ by $f(x) = |x|$.Every non negative real number is a fixed point of f, but 0 is the only coincidence point. $d(f(x), x) = |x| + |x| = 0 \Leftrightarrow x = 0$ .Thus 0 is the only coincidence point while all non negative real numbers are fixed points.

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