International Journal of Mathematical Archive-3(1), 2012, Page: 78-84
Available online through www.ijma.info ISSN 2229 – 5046

Merton's Type Portfolio Optimization Problem in Finite-Horizon Case with HARA Utility Function and Proportional Transaction costs, Explicit Solution

Dorj Nyamsuren*

School of Mathematics, Mongolian University of Science and Technology, Mongolia
E-mail: donyam@must.edu.mn

Tserendorj Batsukh

Department of Economics and Econometrics, Institute of Finance and Economics, Mongolia
E-mail: batsukh.ts@ife.edu.mn

(Received on: 02-01-12; Accepted on: 20-01-12)

ABSTRACT

***A** Merton's type portfolio optimization problem with HARA utility function and transaction costs in finite-horizon case is considered in this paper. One case for a particular class of utility and bequest function of the Merton's problem of an investor has been solved analytically.*

***Key words**: Transaction costs, Stochastic differential equation, Ito lemma, Bellman principles, Hamilton-Jacobi-Bellman equation, Brownian motion, Expected utility function, hyperbolic absolute risk aversion (HARA).*

1. INTRODUCTION:

Since von Neumann and Morgenstern (1944), many researchers have tried to model portfolio optimization problems within an expected utility maximization framework. Different utility function have been used in this approach, and the most notable recent works in this area are belonged to Long (1990) and Luenberger (1993), where log optimal portfolios are constructed and analyzed.

Merton has used stochastic control theory with continuous time dynamics to model multi-period portfolio optimization problems by reducing the problem into solving Hamilton-Jacobi-Bellman equations. His most important contributions include two papers: Merton (1969) and Merton (1971).

In general, all the studies on portfolio optimization with transaction costs different from each other through the modeling of transaction costs structure or with respect to the objective function of investors. Structures of transaction costs have been modeled in several ways.

Under a first approach, the investor has to pay a fixed fraction of his current wealth at the time of the transaction. This is called portfolio management fee approach. These types of models were investigated by Morton& Pliska (1995), Cadenillas & Pliska (1996), Atkinson & Willmot (1995).

In second approach, the transaction costs are assumed to be proportional to the trading volume of the risky assets, where the proportionality rate is constant and less than one. Under this cost formulation, the optimal consumption-investment policy has been studied by many authors in the continuous as well as the discrete time framework. Constantinides (1979) considers a discrete time of the proportional transaction cost model when there are one risky and one risk-free asset.

In the continuous time framework, when the price of the risky asset follows a geometric Brownian motion, Constantinides (1986), Davis &Norman (1990), and Dumas & Luciana (1991) investigate the problem for the investor maximizing his expected utility of the future consumption.

FORMULATION OF THE MODEL AND ITS SOLUTION NECESSARY CONDITION:

Merton's portfolio problem is a well known problem in continuous time finance. An investor with a finite lifetime must choose how must to consume and must allocate his wealth between stocks and a risk-free asset so as to maximize

Corresponding author: Dorj Nyamsuren*,*E-mail: donyam@must.edu.mn

expected lifetime utility. The problem was formulated and solved by Robert C. Merton in 1969, where research has continued to extend and generalize the model.

Objective functions and Budget Equation for Investor:

In our case the investor lives from time 0 to time T; his wealth at time t is denoted W_t. He starts with a known initial wealth W_o. At time t he must choose what fraction of his wealth to consume: c_t and what fraction to invest in a stock portfolio.

Let's assume that $U(c_t)$ is the period utility function for consumption at time *t*, and $B(W_T, T)$ is the utility function for bequest. Then the optimization problem for the investor is

$$\max_{\{c_t, s_i(t)\}} E\left\{\int_0^T e^{-\rho t} U(c_t)dt + B(T, W_T)\right\},$$

where E is the expectation operator, ρ is the subjective discount rate, T is the planning horizon.

Here the wealth evolves according to the stochastic differential equation.

Suppose there are n assets, and the price (P_i) of each asset follows correlated n-dimentional geometric Brownian motion

$$dP_i(t) = \mu_i P_i(t)dt + P_i(t)\sum_{j=1}^n \sigma_{ij} dB_j(t), \qquad i = 1,2,\cdots,n,$$

where μ_i-drift function, σ_{ij}-diffusion coefficient shock of j asset effect to i-asset price, $B_j(t)$-standard Brownian motion.

If we assume that, at time t the investor owns $N_i(t)$ units of asset$i$, then the total wealth is determined to be

$$W(t) = \sum_{i=1}^n N_i(t)P_i(t).$$

The portfolio $\{N_i(t)\}$ remains unchanged over the time interval$[t, t + dt[$. Then the change in wealth over that time interval is

$$dW(t) = \sum_{i=1}^n N_i(t)dP_i(t),$$

if there is no consumption. Assuming the consumption pattern is constant in interval $[t, t + dt)$ in the same way as for portfolio selection, the budget equation becomes

$$dW(t) = \sum_{i=1}^n N_i(t)dP_i(t) - c(t)dt. \tag{1}$$

Let

$$s_i(t) = \frac{N_i(t)P_i(t)}{W(t)},$$

the share of wealth in asset i, with

$$\sum_{i=1}^n s_i(t) = 1.$$

Then the budget equation (1) can be written as

$$dW(t) = \sum_{i=1}^n s_i(t)\mu_i W(t)dt - c(t)dt + \sum_{i=1}^n s_i(t)W(t)\sum_{j=1}^n \sigma_{ij} dB_j(t). \tag{2}$$

Suppose the n-th asset is rick-free ($\sigma_n = 0$) with constant rate of return $\mu_n = r > 0$. Then the budget is found as

$$dW(t) = \sum_{i=1}^{n-1}(\mu_i - r)s_i(t)W(t)dt + (rW(t) - c(t))dt + \sum_{i=1}^{n-1} s_i(t)W(t)\sum_{j=1}^{n}\sigma_{ij}dB_j(t). \tag{3}$$

Transaction costs are considered as comprising of two parts, an asset exchange or brokerage fee and a liquidity or marketability cost. Transaction costs are assumed to be proportional to the volume of the risky asset traded. In that case, the transaction cost is expressed following.

$$\varphi(t) = \gamma(1 - s_n)W(t) = \gamma\left(\sum_{i=1}^{n-1} s_i\right)W(t)$$

where $\varphi(t)$-transaction cost, γ-proportional coefficient.

Then the change in transaction costs is

$$d\varphi(t) = \gamma\left(\sum_{i=1}^{n-1} s_i\right)W(t)dt. \tag{4}$$

Substituting (4) into (3), we have

$$dW(t) = \sum_{i=1}^{n-1}(\mu_i - r - \gamma)s_i(t)W(t)dt + (rW(t) - c(t))dt + \sum_{i=1}^{n-1} s_i(t)W(t)\sum_{j=1}^{n-1}\sigma_{ij}dB_j(t).$$

Where we denote $s(t) = \left(s_1(t), \cdots, s_{n-1}(t)\right)^T$, $\nu = (\mu_1 - r - \gamma, \dots, \mu_{n-1} - r - \gamma)^T$, $\Sigma = (\sigma_{ij})_{i,j=1}^{n}$, $dB(t) = (dB_1(t), \dots, dB_{n-1}(t))^T$ respectively, change in wealth is determined following stochastic differential equation.

$$dW(t) = W(t)s^T\nu dt + \left(W(t)r - c(t)\right)dt + W(t)s^T\Sigma dB(t) \tag{5}$$

Finite-Horizon Problems, necessary condition of solution:

If we will suppose that the problem for investor is written below

$$\max_{c_t s_i(t)} E\int_0^T e^{-\rho t}U(c_t)dt + B(T, W_T), \quad \text{s.t (5)}, \qquad \text{with } \; W_0 = W \text{ given}, \tag{6}$$

then the solution to this problem is the well-known consumption-portfolio rule.

Let the present value of the indirect utility function at time *t* be

$$J(t, W_t) = \max_{\{c_\tau s_i(\tau)\}} E_{t,W_t}\int_t^T e^{-\tau}U(c_\tau)d\tau + B(T, W_T), \quad s.t\ (5).$$

Now we will write HJB equation for (6) problems. HJB equation is

$$0 = \max_{\{c_t, s_i\}}\left\{e^{-\rho t}U(c) + J_t + (Ws^T\nu + Wr - c)J_W + \frac{1}{2}(W^2 s^T Ds)J_{WW}\right\}, \tag{7}$$

which is a Bellman equation and it is necessary condition for stochastic optimal control problems. Where $D = \Sigma \cdot \Sigma^T$. Transversality condition is the boundary condition

$$J(T, W_T) = B(T, W_T).$$

In the absence of a bequest motive, the model assumes $J(T, W_T) = 0$.

The optimal consumption/portfolio rules are governed by

$$\frac{\partial(\cdot)}{\partial c} = e^{-\rho t} \cdot L_c + J_W = 0, \qquad \frac{\partial(\cdot)}{\partial s} = W\nu J_W + W^2 Ds J_{WW} = 0.$$

The first equation

$$e^{-\rho t}U'(c) = J_W, \tag{8}$$

says that consumption is so chosen that, in current values, the marginal utility of consumption equals the marginal utility of wealth. The second equation determines the optimal ratio for each risky asset:

$$s = \left(-\frac{J_W}{WJ_{WW}}\right)D^{-1}\nu. \tag{9}$$

The term of the bracket is the reciprocal of the Arrow-Pratt relative risk aversion. Substituting expressions (8), (9) into(7), we obtain

$$0 = e^{-\rho t}U(c) + J_t - \frac{1}{2}(\nu^T D^{-1}\nu)\frac{J_w^2}{J_{ww}} + (Wr - c)\,J_{W.} \tag{10}$$

2. MAIN RESULTS:

Proposition: In the case the utility function is of the form HARA, $U(c) = \frac{1}{\alpha-1}(\alpha c + \beta)^{1-\frac{1}{\alpha}}$, and the bequest function is of the form $B(T, W_T) = Ae^{-\rho T}\cdot\frac{1}{\alpha-1}(aW_T + b)^{1-\frac{1}{\alpha}}$, then solution of problem (6) and it's parameters are determined by followings:

- $c^* = \frac{1}{\alpha}[A^{-\alpha}(aW + b) - \beta], \quad s^* = D^{-1}\nu\cdot\frac{aW+b}{W}$
- $a = \alpha, \quad b = \frac{\beta}{r}, \quad A = \left[\alpha(\alpha-1)\left(\frac{\rho}{\alpha-1} - \frac{r}{\alpha} - \frac{1}{2}\nu^T D^{-1}\nu\right)\right]^{-\frac{1}{\alpha}}$

Proof: Using the first-order condition (8), we have

$$e^{-\rho t}U'(c) = J_W \implies U'(c) = J_W\,e^{\rho t} \Rightarrow c^* = (U')^{-1}(J_w\cdot e^{\rho t}).$$

Hence consumption is determined by the marginal utility function and the marginal value function as

$$c^* = (U')^{-1}(J_w\cdot e^{\rho t}).$$

For the utility function, we have chosen to be

$$U(c) = \frac{1}{\alpha-1}\cdot(e^{\rho t}\cdot J_W)^{1-\alpha}.$$

Substituting this into the Bellman equation (10), we have the following equation:

$$\left(rW + \frac{\beta}{\alpha} + \frac{1}{\alpha(\alpha-1)}(e^{\rho t}\cdot J_W)^{-\alpha}\right)J_W + J_t - \frac{1}{2}(\nu^T D^{-1}\nu)\frac{J_w^2}{J_{ww}} = 0. \tag{11}$$

The equation (11) is a nonlinear partial differential equation.

In the case that, the following system was hold, equation (11) can be solved analytically.

$$\begin{cases} rW + \dfrac{\beta}{\alpha} + \dfrac{1}{\alpha(\alpha-1)}(e^{\rho t}J_W)^{-\alpha} = pW + q \\ \dfrac{J_W^2}{J_{WW}} = (nW + m)J_W \\ \dfrac{J_W^2}{J_{WW}} = kJ_t \end{cases} \tag{12}$$

where p, q, m, n, k-are unknown coefficients.

From first equation of (12), we have found $(e^{\rho t}J_W)^{-\alpha} = \alpha(\alpha-1)\left((p-r)W + q - \frac{\beta}{\alpha}\right)$, and here we denote $a = \alpha(\alpha-1)\,(p-r)$ and $b = \alpha(\alpha-1)\left(q - \frac{\beta}{\alpha}\right)$ respectively. Hence the marginal value function becomes

$$J_W = e^{-\rho t}(aW + b)^{-\frac{1}{\alpha}}$$

By integrating both sides, we get the value function in general case to be

$$J(t,W) = Ae^{-\rho t}\frac{1}{\alpha-1}(aW+b)^{1-\frac{1}{\alpha}}. \tag{13}$$

Now we will show that above function (13) satisfies second and third equation of the system (12). Since

$$J_W = \frac{Aa}{\alpha}e^{-\rho t}(aW+b)^{-\frac{1}{\alpha}}, \qquad J_{WW} = -\frac{Aa^2}{\alpha^2}e^{-\rho t}(aW+b)^{-\frac{1}{\alpha}-1} \tag{14}$$
$$J_t = -A\rho\frac{1}{\alpha-1}e^{-\rho t}(aW+b)^{1-\frac{1}{\alpha}},$$

Substituting above derivatives into the second equation of the system (12)

$$\frac{J_W^2}{J_{WW}} = Ae^{-\rho t}(aW+b)^{1-\frac{1}{\alpha}} = -\frac{\alpha}{a}(aW+b)J_W.$$

Here if we will denote $m=\alpha,\ n=\frac{\alpha b}{a}$, then it becomes $\frac{J_W^2}{J_{WW}} = (nW+m)J_W$, and satisfies second equation of (12). Also if we will count

$$\frac{J_W^2}{J_{WW}} = Ae^{-\rho t}(aW+b)^{\frac{1}{\alpha}+1} = \left(-\frac{\alpha+1}{\rho}\right)J_t,$$

and denote $k = -\frac{\alpha+1}{\rho}$, then it becomes $\frac{J_W^2}{J_{WW}} = kJ_t$, and satisfies third equation of (12).

Therefore, we can choose the value function of the form(13).

Substituting J_W, J_{WW}, J_t and above result into(10), we have

$$e^{-\boldsymbol{\rho t}}\frac{1}{\alpha-1}(\alpha c+\beta)^{1-\frac{1}{\alpha}} - \frac{A\rho}{\alpha-1}e^{-\boldsymbol{\rho t}}(aW+b)^{1-\frac{1}{\alpha}} - \frac{1}{2}\left(-Ae^{-\boldsymbol{\rho t}}(aW+b)^{1-\frac{1}{\alpha}}\right)(\nu^T D^{-1}\nu) + (rW-c)e^{-\boldsymbol{\rho t}}U'(c) = 0.$$

Now we can find parameters A, a, b using condition (8)

$$A\frac{a}{\alpha}e^{-\rho t}(aW+b)^{-\frac{1}{\alpha}} = e^{-\rho t}(\alpha c+\beta)^{-\frac{1}{\alpha}}.$$

Hence

$$aW+b = \left(\frac{Aa}{\alpha}\right)^{\alpha}(\alpha c+\beta). \tag{15}$$

If we will account (15), then above equation is

$$\frac{1}{\alpha-1}(\alpha c+\beta)^{1-\frac{1}{\alpha}} - \frac{A^\alpha\rho}{\alpha-1}\left(\frac{a}{\alpha}\right)^{\alpha-1}(\alpha c+\beta)^{1-\frac{1}{\alpha}} - \frac{1}{2}(\nu^T D^{-1}\nu)A^\alpha\left(\frac{a}{\alpha}\right)^{\alpha-1}(\alpha c+\beta)^{1-\frac{1}{\alpha}} + (rW-c)(\alpha c+\beta)^{-\frac{1}{\alpha}}.$$
$$= 0.$$

All terms in the Bellman equation, except possibly the term$(rW-c)(\alpha c+\beta)^{-\frac{1}{\alpha}}$, have a common factor $(\alpha c+\beta)^{1-\frac{1}{\alpha}}$. Last term is of the order$(\alpha c+\beta)^{-\frac{1}{\alpha}}$, it suggests that we should choose a, b in such a way that $rW-c$ is proportional to $\alpha c+\beta$. Therefore, we have chosen $a=\alpha,\ b=\frac{\beta}{r}$ so

$$(rW-c) = \frac{r}{a}(aW+b) - \frac{1}{a}(rb+ac) = \frac{r}{a}\left((\alpha c+\beta)\left(\frac{Aa}{\alpha}\right)^{a} - \frac{1}{a}(rb+ac)\right).$$

Substituting them into the Bellman equation and dividing it by $(\alpha c+\beta)^{1-\frac{1}{\alpha}}$, the Bellman equation is obtained

$$\frac{1}{\alpha-1} - \frac{1}{\alpha} - A^\alpha\left(\frac{\rho}{\alpha-1} - \frac{r}{\alpha} - \frac{1}{2}(\nu^T D^{-1}\nu)\right) = 0$$

Where, this determines the constant A, i.e.,

$$A = \left[\alpha(\alpha-1)\left(\frac{\rho}{\alpha-1} - \frac{r}{\alpha} - \frac{1}{2}(\nu^T D^{-1}\nu)\right)\right]^{-\frac{1}{\alpha}}. \tag{16}$$

From (15), the optimal consumption is

$$c = \frac{1}{\alpha}(A^{-\alpha}(aW+b) - \beta),$$

and substituting (16), (14) into (9), the optimal portfolio rule is found to be

$$s = \frac{aW+b}{W} D^{-1}\nu .$$

It gives proof.

Now substituting above results into the budget equation(5), the wealth process is determined as

$$dW(t) = \left[r - \rho + \frac{1+\alpha}{2}\nu^T D^{-1}\nu\right]\left(\alpha W + \frac{\beta}{r}\right)dt + \left(\alpha W + \frac{\beta}{r}\right)\nu^T \Sigma^{-1} dB(t).$$

Using the Ito formula with $f(W) = \ln\left(\alpha W + \frac{\beta}{r}\right)$, we get

$$d\ln\left(\alpha W + \frac{\beta}{r}\right) = \frac{\alpha}{\left(\alpha W + \frac{\beta}{r}\right)} dW - \frac{1}{2}\cdot\frac{\alpha^2}{\left(\alpha W + \frac{\beta}{r}\right)^2}(dW)^2 == \alpha\left(r - \rho + \frac{1}{2}\nu^T D^{-1}\nu\right)dt + \alpha\nu^T\Sigma^{-1}dB(t) .$$

Integrating gives

$$\ln\left(\alpha W_t + \frac{\beta}{r}\right) - \ln\left(\alpha W_o + \frac{\beta}{r}\right) = \alpha\left(r - \rho + \frac{1}{2}\nu^T D^{-1}\nu\right)t + \alpha\nu^T\Sigma^{-1}B(t).$$

If we denote $X_t = \ln\left(\alpha W_t + \frac{\beta}{r}\right)$, then we have a strong solution

$$X_t = \ln\left(\alpha W_o + \frac{\beta}{r}\right) + \alpha\left(r - \rho + \frac{1}{2}\nu^T D^{-1}\nu\right)t + \alpha\nu^T\Sigma^{-1}B(t).$$

Using the Ito isometries, its mean and variance are

$$\mathbb{E}X_t = \ln\left(\alpha W_o + \frac{\beta}{r}\right) + \alpha\left(r - \rho + \frac{1}{2}\nu^T D^{-1}\nu\right)t,$$
$$\mathbb{E}(X_t - \mathbb{E}X_t)^2 = \alpha^2\nu^T D^{-1}\nu t.$$

Hence X_t is a normal distribution with following parameters

$$X_t \sim N\left[\ln\left(\alpha W_o + \frac{\beta}{r}\right) + \alpha\left(\frac{1}{2}\cdot\left(\frac{\mu-r}{\sigma}\right)^2 - \rho + r\right)t, \alpha^2\left(\frac{\mu-r}{\sigma}\right)^2 t\right]$$

or W_t is determined with e^{X_t}. I.e., W_t is a lognormal distribution.

CONCLUSION

- In this paper explicit solution of Merton's type portfolio optimization problem in finite horizon with HARA utility function and transactions costs was investigated.
- By assuming the utility function is of the form HARA, following results have been investigated;
  - investor's wealth are log-normally distributed with linear trend,
  - optimal consumption and the optimal portfolio rule are linear in wealth

REFERENCE

1. Atkinson, C., Wilmott, P.(1995), Portfolio Management with Transaction Costs: An Asymptotic Analysis of the Morton and Pliska Model, *Mathematical Finance*, 5, 357-367.
2. Cadenillas,A., Pliska, S.R. (1996), Optimal Trading od a Security when there are Taxes and Transaction Costs, *13th International Conference of the French Finance Association, Geneva*, June 24-26.
3. Constantinides, G.M. (1979), Multiperiod Consumption and Investment Behavior with Convex Transaction Costs, *Management Science*, 25, 1127-1137.

4. Constantinides, G.M. (1986), Capital Market Equilibrium wit Transaction Costs, *Journal of Political Economy*, 94, 842-862.
5. Davis, M.,Norman A. (1990), Portfolio Selection with Transaction Costs, *Math.Oper.Res*., 15, 676-713.
6. Dumas, B., Lusiano,E. (1991), An Exact Solution to a Dynamic Portfolio Choice Problem under Transaction Costs, *Journal of Finance*, 46, 577-595.
7. Fwu-Rang Chang (2004), *Stochastic Optimization in Continuous Time*, Cambridge university press, United Kingdom.
8. Long, J.B. The Numeraire Portfolio, *Journal of Financial Economics 26(1),* 1990, 29-69.
9. Luenberger, D.G., A Preference Foundation for Log Mean-variance Criteria in Portfolio Choice Problems, *Journal of Economic Dynamics and Control,* 17,1993, 887-906.
10. Merton, R.C. Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case, *Review of Economics and Statistics*, 51, 1969, 247-257.
11. Merton, R.C. Optimal Consumption and Portfolio Rules in a Continuous Time Model, *Journal of Economic Theory*, 3, 1971, 373-413.
12. Chang Hwan Sung (2007), Application of Modern Control Theory in Portfolio Optimization, *a dissertation for the degree of doctor of philosophy submitted to the department of management science and engineering and the committee on graduate studies of Stanford University.*
13. Fabio Trojani, Roberto G. Ferretti (2004), General Analytical solutions for Merton's Type Consumption-Investment Problems, *Discussion Paper no.2005-02*, University of St. Gallen.

****

