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ANALYSIS OF A TWO DIMENSIONAL IDEAL JET FLOW

F. Guechi*

Department of Mathematics, Ferhat Abbas University, 19000, Sétif, Algeria

E-mail: f_guechi@yahoo.fr

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ABSTRACT

In this paper, a free surface flow of a liquid poured from a container is calculated analytically for various configurations of the lip. The flow is assumed to be steady, two dimensional, and irrotational. The liquid is treated as inviscid and incompressible. The effect of gravity is neglected. It is shown that there are jet like flows and other flows with one free surface which follow along the underside of the lip or spout. Some of the results are applicable also to flows over weirs and spillways. We use the method of Kirchhoff based on the hodograph method and Schwartz-Christoffel transformation technique to solve the problem for various values β of the inclination angle between the horizontal bottom and the inclined wall.

Keywords: *Free surface, flow, hodograph method, free streamlines theory.*

AMS Mathematics Subject Classification (2010): *76B07, 76D45, 76M40, 58C15, 30E15.*

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1. INTRODUCTION:

We consider a steady two dimensional potential channel flow against a wall of semi infinite length making an angle β with the horizontal. The flow domain is bounded below by an infinite rigid wall and above by a free surface. The fluid is assumed to be inviscid, incompressible and the flow is irrotationnal. The classical problem of a free streamline flow of an ideal fluid has been studied by many authors [4-12]. Toison et al. [10] used an iterative method and Vanden-Broeck et al. [11-12] used a series truncation. Jets impinging on walls were studied by many authors. Peng and Parker [9] considered a fluid jet impinging on an uneven wall. In their work, they considered different smooth geometry of the wall. Dias, Elcart and Trefethen [3] considered a jet emerging from a polygonal nozzle. Despite that the problem could theoretically be solved by the hodograph and *Schwartz-Christoffel* transform if the gravity and the surface tension are neglected, but when the nozzle has many corners, the *Schwartz-Christoffel* transform is obsolete. To remedy this mathematical limitation, the authors described an efficient mathematical procedure for computing two dimensional ideal jets issuing from an arbitrary polygonal container. Vanden Broeck and Tuck [11] calculated flow near the intersection of a vertical wall with a free surface taking into account only the gravity. In our case far upstream the flow is uniform with a constant velocity U and a constant depth H. The first step in this type of problem is characterized by the use of the method of *Kirchhoff*. The latter can treat the flows of border, which combines rectilinear wall and unknown free surface.

2. FORMULATION OF PROBLEM:

Let us consider the motion of a two-dimensional potential flow in a channel against a wall of semi infinite length. The inclined wall meets the horizontal bottom at the point O making an angle β. We assume that the fluid is inviscid, incompressible and the flow is irrotational and steady. Since the flow is considered to be potential the normal velocity vanishes on the rigid boundaries: the horizontal and the inclined walls. Far upstream, we assume that the flow is uniform so that the velocity approaches a constant U and the depth of the fluid tends to a constant H. The flow is limited by the free streamline ***ABC*****,** the horizontal wall ***AO*** and the inclined wall ***OC***. In the absence of gravity, the main flow extends to infinity in the direction of the bottom wall far upstream and in the direction of the inclined wall ***OC*** far downstream (Fig.1). Our formulation is made for a flow over various inclinations of the wall. We choose the Cartesian coordinates such that the x-axis is along the bottom streamline and passes through the point O and y-axis is vertically upward through the point O.

****Corresponding author: F. Guechi*,*E-mail: f_guechi@yahoo.fr***

Since the flow is irrotationnal and the fluid is incompressible, we define the complex variable $z = x + iy$ and the complex potential function $f = \Phi + i\psi$ where Φ and ψ are the potential and the stream functions respectively. Since Φ and ψ are conjugate solution of Laplace's equation, $f(z)$ is an analytic function of z within the flow region. The complex conjugate velocity is given by

$$\xi = \frac{df}{dz} = u - iv \quad and \quad q = \sqrt{u^2 + v^2} \tag{1}$$

Where u and v are the horizontal and vertical components of the fluid velocity, respectively, and may be expressed as

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2}$$

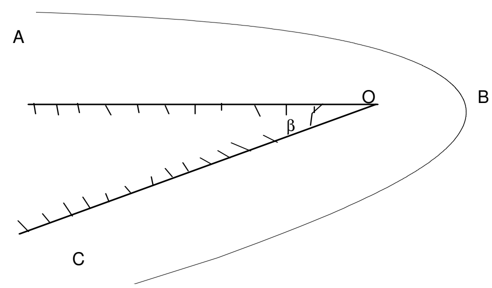

Figure 1. Sketch of the flow in the z-plan

Without loss of generality, we choose $\psi=0$ on the streamline AOC and $\Phi=0$ at the origin $O(x,y)=(0, 0)$. The complex potential f maps the flow domain conformally onto the infinite strip of width UH as shown in figure 2.

On the free streamline (free surface) ABC, the Bernoulli equation is to be satisfied, that is

$$\frac{1}{2}q^2 = Cte \quad \text{On} \quad \psi = UH, \quad -\infty \prec \phi \prec +\infty \tag{3}$$

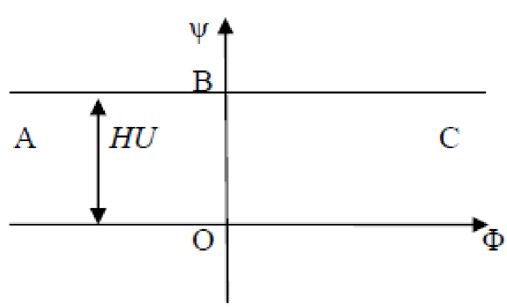

Figure 2. The flow configuration in the complex potential plan.

The physical flow problem as formulated above can be formulated as a boundary value problem in the potential function Φ which checks the following conditions:

(a) $\Delta\phi = 0$ In the flow domain

(b) $\frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right) + \frac{P}{\rho} = Cte$ On the free surface.

(c) $\frac{\partial \phi}{\partial y} = 0$ On AO.

(d) $\Phi(0, 0) = 0$.

3. METHODOLOGY:

To solve our problem, we initially use the method of the free surface streamline theory introduced by *Kirchhoff* [1-2], based on the hodograph transformation to find the form of the free surface. The complex transformation is defined by:

$$\Omega = \log\left(\frac{U}{df\,/\,dz}\right) = \log\left(\frac{U}{u - iv}\right) = \log\left(\frac{U}{q}\right) + i\theta \tag{4}$$

Where $z = x + iy$, q and θ are the module speed and the angle between the velocity vector and the x-axis, respectively. By this last transformation, the field occupied by the fluid in the z-plan is transformed into an infinite band in the Ω-plan (see Fig.3).

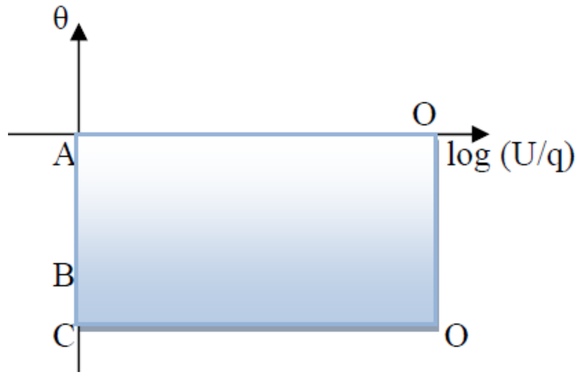

Figure 3. The flow domain in the Ω-plan

- The conform transformation of a semi-infinite band in the plan to the lower half-plan of another complex λ-plan, is given by the theorem of *Schwartz-Christoffel*, by respecting the direction and the orientation of the flow (see Fig.4).

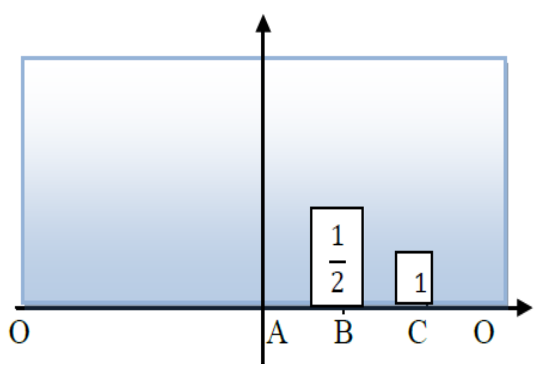

Figure 4. The flow domain in the λ-plan

This transformation is given by:

$$\frac{d\Omega}{d\lambda} = k\,\lambda^{-\frac{1}{2}}(\lambda - 1)^{-\frac{1}{2}} \tag{5}$$

After integration we obtain

$$\Omega = 2k_1 \arg ch\,(-2\lambda + 1) + k_2$$

We have λ = 0 on *A*, from or Ω = 0 and λ = 1 on *C*, with Ω = -*iβ*

Finally we obtain

$$\begin{aligned}\lambda &= \frac{1}{2}(1 - ch(-\frac{\pi}{\beta}\Omega) \\ &= -sh^2\left(\frac{\pi}{2\beta}\Omega\right)\end{aligned} \tag{6}$$

- The transformation which transforms the interior of the infinite band of the *f*-plan towards the lower half-plan of the λ-plan is:

$$\frac{df}{d\lambda} = k(\lambda - 1)^{-1}\lambda^{-1}$$

After calculations, we find a relation between λ and *f*:

$$\lambda = \frac{1}{1 - e^{\left(\frac{\pi f}{DU}\right)}} \tag{7}$$

Using the relation $U\frac{dz}{d\lambda} = U\frac{dz}{df}\frac{df}{d\lambda}$, we obtain

$$U\frac{dz}{d\lambda} = \left(\sqrt{1-\lambda} - i\sqrt{\lambda}\right)^{\frac{2\beta}{\pi}} \frac{-HU}{\pi\lambda(\lambda - 1)} \tag{8}$$

By integrating (8)

$$z - z_0 = -\frac{H}{\pi}\int \frac{\left(\sqrt{1-\lambda} - i\sqrt{\lambda}\right)^{\frac{2\beta}{\pi}}}{\lambda(\lambda - 1)}\, d\lambda \tag{9}$$

Where z_0 is a constant, we have $z_0 = (x_0, y_0) = (x(\lambda), y(\lambda)) = (x_0, 0)$ at *B*.

We give the shape of the free surface in the different following cases:

Case 1: $\beta = \pi/4$, the relation (9) becomes

$$z - z_0 = -\frac{H}{\pi}\int \frac{\left(\sqrt{1-\lambda} - i\sqrt{\lambda}\right)^{\frac{1}{2}}}{\lambda(\lambda - 1)}\, d\lambda \tag{10}$$

Case 2: $\beta = 3\pi/4$, we obtain

$$z - z_0 = -\frac{H}{\pi}\int \frac{\left(\sqrt{1-\lambda} - i\sqrt{\lambda}\right)^{\frac{3}{2}}}{\lambda(\lambda - 1)}\, d\lambda \tag{11}$$

Case 3: $\beta = -\pi/4$, then

$$z - z_0 = -\frac{H}{\pi}\int \frac{\left(\sqrt{1-\lambda} - i\sqrt{\lambda}\right)^{\frac{-1}{2}}}{\lambda(\lambda - 1)}\, d\lambda \tag{12}$$

Case 4: $\beta = 5\pi/4$, we have

$$z - z_0 = -\frac{H}{\pi}\int \frac{\left(\sqrt{1-\lambda} - i\sqrt{\lambda}\right)^{\frac{5}{2}}}{\lambda(\lambda - 1)}\, d\lambda \tag{13}$$

By integrating these equations, we obtain the form of the free surface for the different considered values of *β*. Figures 5-8 show the profiles of the free surface for the considered cases.

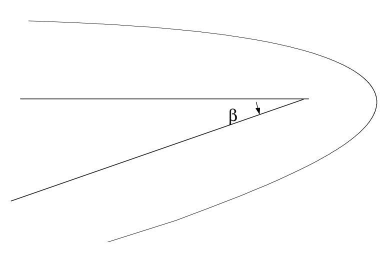

Figure 5. The form of the free surface with *β*= *π/4*

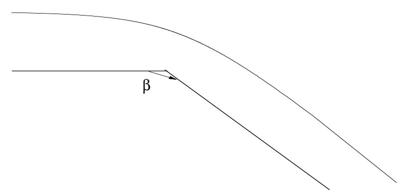

Figure 6. The form of the free surface with *β=3 π/4*

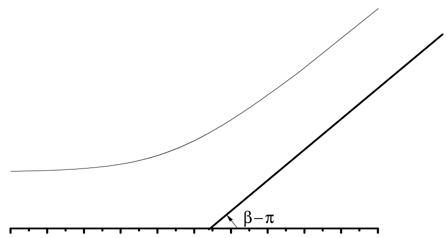

Figure 7. The form of the free surface with *β=π + π/4*

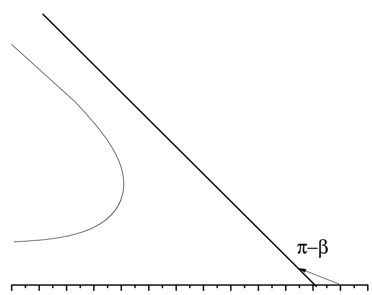

Figure 8. The form of the free surface with *β =- π/4*

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