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PROPERTIES P AND Q IN NON-ARCHIMEDEAN G-FUZZY METRIC SPACES

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ABSTRACT

In this paper we introduce the concept of non-Archimedean G-fuzzy metric space and obtain some results for two semi-compatible mappings in this newly defined space. Our results improve and generalize the results of Mustafa et. al. [13] and Abbas & Rhoades [1] in non-Archimedean G-fuzzy metric space. Moreover, we prove that these mappings satisfy Properties P and Q.

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1. INTRODUCTION AND PRELIMINARIES:

In 1965, Zadeh [18] introduced the concept of Fuzzy set. Since that time a substantial literature has been developed on this subject. Several authors [2, 4, 7, 10] proved fixed point theorems for fuzzy metric space in different ways. In 1975, Kramosil and Michalek [11] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situtation. After that George and Veeramani[4-6] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [11]. They also showed that every metric induces a fuzzy metric. Grabiec [7] proved fuzzy Banach contraction theorem on fuzzy metric space. Singh and Chauhan [17] proved some common fixed point theorems in fuzzy metric spaces in the sense of George and Veeramani. Recently, Dorel Mihet [12] introduced the concept of non-Archimedean fuzzy metric space and proved Banach Contraction theorem in this space. In 2006, Mustafa and Sims [15] introduced the concept of G-metric space by generalizing the concept of metric space. Then, based on the notion of generalized metric spaces, several authors have obtained some fixed point results for a self-mapping under various contractive conditions, (see[1,3,13]).

Motivated by the concepts of G-metric space, Non-Archimedean metric space and Fuzzy metric space, we introduce the concept of non-Archimedean G-fuzzy metric space and obtain two common fixed point theorems for two semi-compatible mappings. Our results improve and generalize the results of Mustafa et. al.[13] and Abbas & Rhoades [1] in non-Archimedean G-fuzzy metric space. We also establish properties P and Q for these mappings. An interesting fact about maps satisfying properties P and Q is that they have no nontrivial periodic points. Some papers dealing with properties P and Q are ([8, 9, 16]).

We first give some definitions and results that will be needed in the sequel.

Definition: 1.1([15]) Let X be a nonempty set and $G : X \times X \times X \to R^+$ a function satisfying the following axioms:
(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

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Definition: 1.2 ([15]) Let (X, G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\varepsilon > 0$, there exists a $k \in \mathbf{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all n, m $\geq$ k (throughout this paper we mean by **N** the set of all natural numbers). We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

Proposition: 1.3 ([15]) Let (X, G) be a G-metric space. Then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$,
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Example: 1.4 ([15]). Let (X, d) be a usual metric space, then (X,G_S) and (X,G_m) are G-metric spaces, where

$$G_S(x, y, z) = d(x, y) + d(y, z) + d(x, z), \text{ for all } x, y, z \in X,$$

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \text{ for all } x, y, z \in X.$$

Definition: 1.5 A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:
(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0, 1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Now, we introduce the concept of Non-Archimedean G-fuzzy metric space (briefly as N. A. G-fuzzy metric space) as follows:

Definition: 1.6 A 3-tuple $(X, M_G, *)$ is called a non-Archimedean G-fuzzy metric space if X is an arbitrary(non-empty) set, $*$ is a continuous t-norm and M_G is a G-fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each x, y, z, $a \in X$ and t, s > 0

(M_G1) $M_G(x, x, y, t) > 0$ with $x \neq y$;
(M_G2) $M_G(x, x, y, t) \geq M_G(x, y, z, t) > 0$ with $z \neq y$;
(M_G3) $M_G(x, y, z, t) = 1$ iff $x= y= z$;
(M_G4) $M_G(x, y, z, t) = M_G(p\{x, y, z\}, t)$ (symmetry) where p is a permutation function;
(M_G5) $M_G(x, a, a, t) * M_G(a, y, z, s) \leq M_G(x, y, z, \max\{t, s\})$;
(M_G6) $M_G(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous.

Example: 1.7 Let X =**R** with G-metric on X defined by

$$G(x, y, z) = |x - y|+|y - z|+|z - x|.$$

Denote $a * b = ab$ for all $a, b \in [0, 1]$. For all $x, y, z \in X$ and $t > 0$, define M_G on $X^3 \times (0, \infty)$ as follows:

$$M_G(x, y, z, t) = \left(\frac{t}{t+1}\right)^{G(x,y,z)}.$$

Then, $(X, M_G, *)$ is a non -Archimedean G-fuzzy metric space.

Definition: 1.8 Let $(X, M_G, *)$ be a non-Archimedean G-fuzzy metric space. Then,

(1) A sequence $\{x_n\}$ in X is said to be convergent to x iff $M_G(x_m, x_n, x, t) \to 1$ as $n \to \infty$, *for each* $t > 0$.
(2) A sequence $\{x_n\}$ in X is said to be a cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M_G(x_m, x_n, x_l, t) > 1- \varepsilon$ for each $l, m, n \geq n_0$.
(3) The G-fuzzy metric space is called complete if every cauchy sequence is convergent.

Following similar argument in G-metric space, the sequence $\{x_n\}$ in X also converges to x iff $M_G(x_n, x_n, x, t) \to 1$ as $n \to \infty$, *for each* $t > 0$ and it is Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$M_G(x_m, x_n, x_n, t) > 1- \varepsilon$ for each m, $n \geq n_0$.

Definition: 1.9 Denote by Φ the class of continuous functions $\emptyset$: [0, 1] → [0, 1] such that $\emptyset(t) > t$ for all $0 \leq t < 1$ and $\emptyset(1) = 1$.

Lemma: 1.10 Let (X, M_G, ∗) be a non-Archimedean G-fuzzy metric space. Then $M_G(x, y, z, t)$ is non-decreasing with respect to t for all x, y, z in X.

Throughout this paper, we assume that $\lim_{t\to\infty} M_G(x,y,z,t) = 1$and that **N** is the set of all natural numbers.

Lemma: 1.11 Let (X, M_G, ∗) be a non-Archimedean G-fuzzy metric space. Let $\{y_n\}$ be a sequence in X, where ∗ is a continuous t-norm satisfying $t * t \geq t$ for all t ∈ [0, 1]. If there exists t > 0 and $\emptyset \in \Phi$ such that

$M_G(y_{n+1}, y_{n+2}, y_{n+2}, t) \geq \emptyset(M_G(y_n, y_{n+1}, y_{n+1}, t))$, n ∈ N, then $\{y_n\}$ is a Cauchy sequence in X.

Proof: If we define $r_n = M_G(y_{n+1}, y_{n+2}, y_{n+2}, t)$, then

(1.11.1) $\qquad r_n \geq \emptyset(r_{n-1}) > r_{n-1}$.

So that the sequence $\{r_n\}$ is an increasing sequence of positive real numbers in [0, 1] and tends to a limit $r \leq 1$. We claim that r =1. If r <1, on taking $n \to \infty$ in (1.11.1), we get $r \geq \emptyset(r) > r$, which is a contradiction. Hence r =1.

Now, for any positive integer p, we have

$M_G(y_n, y_{n+p}, y_{n+p}, t) \geq M_G(y_n, y_{n+1}, y_{n+1}, t) * \ldots * M_G(y_{n+p-1}, y_{n+p}, y_{n+p}, t)$.

Taking the limit as $n \to \infty$, we get

$\lim_{n\to\infty} M_G(y_n, y_{n+p}, y_{n+p}, t) = 1$. Hence, $\{y_n\}$ is a Cauchy sequence.

Now, we introduce the concept of weakly compatible maps and semi-compatible maps in non-Archimedean G-fuzzy metric space as follows:

Definition: 1.12 Let f and g be self maps on a non-Archimedean G-fuzzy metric space (X, M_G, ∗). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, fx = gx implies that fgx = gfx.

Definition1.13. A pair (f, g) of self mappings of a non-Archimedean G-fuzzy metric space is said to be semi-compatible if $\lim_{n\to\infty} fgx_n$ = gx, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = x, for some x ∈X.

It follows that (f, g) is semi-compatible and fy = gy, then fgy = gfy.

Note that every pair of semi-compatible maps is weakly compatible but converse need not be true.

Example: 1.14 Let X =[0, 1] with G-metric on X defined by

$$G(x, y, z) = |x-y| + |y-z| + |z-x|.$$

Denote a ∗ b = ab for all a, b ∈ [0, 1]. For all x, y, z ∈ X and t > 0, define M_G on $X^3 \times (0, \infty)$ as follows:

$$M_G(x, y, z, t) = \left(\frac{t}{t+1}\right)^{G(x,y,z)}.$$

Then, (X, M_G, ∗) is a non -Archimedean G-fuzzy metric space. Define a self map on X as follows:

$$Sx = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$$ and let I be the identity map on X.

If $x_n = \frac{1}{2} - \frac{1}{n}$ .Then $\{Ix_n\} = x_n \rightarrow \frac{1}{2}$ and $\{Sx_n\} \rightarrow \frac{1}{2}$. Again $\{ISx_n\} \rightarrow \frac{1}{2} \neq S\left(\frac{1}{2}\right)$.

Thus (I, S) is not semi-compatible. But (I, S) is weakly compatible.

Definition: 1.15 ([1]) Let f and g be self maps on a set X and if w = fx = gx for some x in X, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Proposition: 1.16 Let f and g be semi-compatible self-maps of a set X. If f and g have a unique point of coincidence fx = gx = w, then w is the unique common fixed point of f and g.

Proof: Since fx = gx = w and f and g are semi-compatible, we have

fw = fgx = gfx =gw, implies that, fw = gw. Thus, w is a point of coincidence of f and g. But w is the only point of coincidence of f and g, so w = fw = gw. Moreover, if z = fz = gz, then z is a point of coincidence of f and g. Therefore, z = w, by uniqueness. Thus, w is the unique common fixed point of f and g.

Definition: 1.17 Let $(X, M_G, *)$ be a non-Archimedean G-fuzzy metric space and $T : X \rightarrow X$ be a mapping with fixed point set $F(T) \neq \emptyset$. Then T has property P if $F(T^n) = F(T)$, for each n N.

Definition: 1.18 Let $(X, M_G, *)$ be a non-Archimedean G-fuzzy metric space and $T, S : X \rightarrow X$ be two mappings with $F(S) \cap F(T) \neq \emptyset$. Then, S and T have property Q if $F(S^n) \cap F(T^n) = F(S) \cap F(T)$, for each n N.

2. FIXED POINT RESULTS:

Now, we generalize the results of Abbas & Rhoades [1] to non-Archimedean G-fuzzy metric space for semi-compatible maps as follows:

Theorem: 2.1 Let $(X, M_G, *)$ be a non-Archimedean G-fuzzy metric space with $t * t \geq t$. Suppose f and g be a self-map of X satisfying for all $x, y, z \in X$

(2.1.1) $M_G(fx, fy, fz, t) \geq \emptyset(M_G(gx, gy, gz, t))$

where $\emptyset \in \Phi$, $t > 0$. If $f(X) \subset g(X)$ and g(X) is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are semi-compatible, then f and g have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, so we choose a point x_1 in X such that $f(x_0) = g(x_1)$.Continuing this process, having chosen x_n in X, we can find x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Inductively, construct sequence $\{y_n\}$ in X such that

(2.1.2) $y_n = fx_n = gx_{n+1}$, n = 0, 1, 2 . . .

Now, we prove that $\{y_n\}$ is a Cauchy sequence.Then, by (2.1.1), we have

$M_G(y_n, y_{n+1}, y_{n+1}, t) = M_G(fx_n, fx_{n+1}, fx_{n+1}, t)$

$$\geq \emptyset(M_G(gx_n, gx_{n+1}, gx_{n+1}, t)) = \emptyset(M_G(y_{n-1}, y_n, y_n, t)).$$

Then, by lemma 1.11, $\{y_n\}$ is a cauchy sequence.This implies that $\{gx_n\}$ is a cauchy sequence. Since g(X) is complete, so there exists $u \in g(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u.$$

Since $u \in g(X)$, so there exists $p \in X$ such that gp = u. Let $fp \neq u$. From (2.1.1)

$M_G(fx_n, fp, fp, t) \geq \emptyset(M_G(gx_n, gp, gp, t))$. As $n \rightarrow \infty$, we get

$M_G(u, fp, fp, t) \geq \emptyset(M_G(gp, gp, gp, t)) = \emptyset(1) = 1$.

This implies that $M_G(u, fp, fp, t) = 1$,which is a contradiction, since $fp \neq u$.

Thus, fp = gp = u. Hence, p is a coincidence point of f and g.

Now, we will show that p is unique. Assume that there exists another point q in X such that fq = gq.If fp ≠fq, then

$M_G(fq, fp, fp, t) \geq \emptyset(M_G(gq, gp, gp, t)) = \emptyset(M_G(fq, fp, fp, t)) > M_G(fq, fp, fp, t)$.

By lemma 1.10, we obtain a contradiction. Hence fp = fq.

Moreover, if f and g are semi-compatible, then from proposition 1.16, f and g have a unique common fixed point.

If we take g = I in Theorem 2.1, we obtain the following result:

Corollary: 2.2 Let $(X, M_G, *)$ be a complete non-Archimedean G-fuzzy metric space with $t * t \geq t$. Suppose f be a self-map of X satisfying for all x, y, z ∈ X

$$M_G(fx, fy, fz, t) \geq \emptyset(M_G(x, y, z, t))$$

where t > 0 and $\emptyset \in \Phi$. Then f has a unique fixed point.

Theorem: 2.3 Let $(X, M_G, *)$ be a non-Archimedean G-fuzzy metric space with $t * t \geq t$. If the mappings $f, g : X \to X$ satisfy either

(2.3.1) $M_G(fx, fy, fz, t) \geq \emptyset(\min\{M_G(gx, fx, fx, t), M_G(gy, fy, fy, t), M_G(gz, fz, fz, t)\})$

or

(2.3.2) $M_G(fx, fy, fz, t) \geq \emptyset(\min\{M_G(gx, gx, fx, t), M_G(gy, gy, fy, t), M_G(gz, gz, fz, t)\})$,

for all x, y, z ∈ X where $\emptyset \in \Phi$, t > 0. If $f(X) \subset g(X)$ and g(X) is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are semi-compatible, then f and g have a unique common fixed point.

Proof: Suppose that f and g satisfy (2.3.1). Let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, so we choose a point x_1 in X such that $f(x_0) = g(x_1)$.Continuing this process, having chosen x_n in X, we can find x_{n+1} in X such that $f(x_n) = g(x_{n+1})$.Inductively, construct sequence $\{y_n\}$ in X such that

(2.3.3) $y_n = fx_n = gx_{n+1}$, n = 0, 1, 2, . . .

Now, we prove that $\{y_n\}$ is a Cauchy sequence. Then, by (2.3.1), we have

$$\begin{aligned} M_G(y_n, y_{n+1}, y_{n+1}, t) &= M_G(fx_n, fx_{n+1}, fx_{n+1}, t) \\ &\geq \emptyset(\min\{M_G(gx_n, fx_n, fx_n, t), M_G(gx_{n+1}, fx_{n+1}, fx_{n+1}, t), M_G(gx_{n+1}, fx_{n+1}, fx_{n+1}, t)\}) \\ &= \emptyset(\min\{M_G(y_{n-1}, y_n, y_n, t), M_G(y_n, y_{n+1}, y_{n+1}, t), M_G(y_n, y_{n+1}, y_{n+1}, t)\}). \end{aligned}$$

Thus, we obtain

$M_G(y_n, y_{n+1}, y_{n+1}, t) \geq \emptyset(\min\{M_G(y_{n-1}, y_n, y_n, t), M_G(y_n, y_{n+1}, y_{n+1}, t)\})$.

Without loss of generality assume $y_n \neq y_{n+1}$ for each n. (Since, if there exists an n such that $y_n = y_{n+1}$, then $y_n = fx_n = gx_{n+1} = fx_{n+1} = gx_{n+2}$, implies that, $gx_{n+1} = fx_{n+1}$.

Then, f and g have a coincidence point.) Therefore, if in the above inequality

$M_G(y_n, y_{n+1}, y_{n+1}, t) \geq \emptyset(M_G(y_n, y_{n+1}, y_{n+1}, t)) > M_G(y_n, y_{n+1}, y_{n+1}, t)$.

By lemma 1.10, which is a contradiction. Hence,

$$M_G(y_n, y_{n+1}, y_{n+1}, t) \geq \emptyset(M_G(y_{n-1}, y_n, y_n, t)).$$

Thus, by lemma 1.11, $\{y_n\}$ is a cauchy sequence, which implies that $\{gx_n\}$ is a cauchy sequence. Since g(X) is complete, so there exists u ∈g(X) such that

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u.$

Since $u \in g(X)$, so there exists $p \in X$ such that $gp = u$. Let $fp \neq u$. From (2.3.1)

$M_G(fx_n, fp, fp, t) \geq \emptyset(\min\{M_G(gx_n, fx_n, fx_n, t), M_G(gp, fp, fp, t), M_G(gp, fp, fp, t)\})$.

As $n \to \infty$, we get $M_G(u, fp, fp, t) \geq \emptyset(\min\{M_G(u, u, u, t), M_G(u, fp, fp, t)\})$
$\geq \emptyset(\min\{1, M_G(u, fp, fp, t)\})$

Now, if $M_G(u, fp, fp, t) \geq \emptyset(1) = 1$, this implies that $M_G(u, fp, fp, t) = 1$

which is a contradiction, since $fp \neq u$.

Hence $M_G(u, fp, fp, t) \geq \emptyset(M_G(u, fp, fp, t)) > M_G(u, fp, fp, t)$

By lemma 1.10, which is absurd. Hence, $fp = u$. Thus, $fp = gp = u$.

Hence, p is a coincidence point of f and g.

Now, we show that p is unique. Assume that there exists another point q in X such that $fq = gq$. If $fp \neq fq$, then

$M_G(fq, fp, fp, t) \geq \emptyset(\min\{M_G(gq, fq, fq, t), M_G(gp, fp, fp, t), M_G(gp, fp, fp, t)\})$
$\geq \emptyset(\min\{M_G(fq, fq, fq, t), M_G(fp, fp, fp, t)\}) \geq \emptyset(1) = 1.$

This implies that $M_G(fq, fp, fp, t) = 1$. By lemma 1.10, which is a contradiction as $fp \neq fq$. Hence $fp = fq$.

Moreover, if f and g are semi-compatible, then from proposition 1.16, f and g have a unique common fixed point. The proof using (2.3.2) is similar.

If we take g = I in Theorem 2.3, we obtain the following result as a generalization of Theorem 2.3 of Mustafa et. al.[13] to non-Archimedean G-fuuzzy metric spaces:

Corollary: 2.4 Let $(X, M_G, *)$ be a complete non Archimedean G-fuzzy metric space with $t * t \geq t$. If the mappings $f : X \to X$ satisfy for all $x, y, z \in X$ either

$M_G(fx, fy, fz, t) \geq \emptyset(\min\{M_G(x, fx, fx, t), M_G(y, fy, fy, t), M_G(z, fz, fz, t)\})$
or

$M_G(fx, fy, fz, t) \geq \emptyset(\min\{M_G(x, x, fx, t), M_G(y, y, fy, t), M_G(z, z, fz, t)\})$

where $t > 0$ and $\emptyset \in \Phi$. Then f has a unique fixed point.

Example: 2.5 Let $(X, M_G, *)$ be a non-Archimedean G-fuzzy metric space defined in example (1.7). Define $f, g: X \to X$ as follows:

$fx = \frac{x}{6}$ and $gx = \frac{x}{3}$. and define $\emptyset : [0,1] \to [0,1]$ as $\emptyset(t) = \sqrt{t}$.

Then all of the hypothesis of Theorems (2.1) holds. Also f and g satisfy condition (2.1.1) for all $x, y, z \in \mathbf{R}$ and 0 is the unique common fixed point of f and g.

3. PROPERTIES P AND Q:

In this section, we shall show that maps satisfying the conditions of Theorem 2.1 , 2.3 and corollary 2.2 , 2.4 possess Properties Q and P respectively.

Theorem: 3.1 Under the conditions of Theorem 2.1, f and g have Property Q.

Proof: From Theorem 2.1, $F(f) \cap F(g) \neq \emptyset$.Therefore, $F(f^n) \cap F(g^n) \neq \emptyset$ for each positive integer n. Let n be a fixed positive integer greater than 1 and suppose that

$u \in F(f^n) \cap F(g^n)$. We claim that $u \in F(f) \cap F(g)$.

Let $u \in F(f^n) \cap F(g^n)$. Then, for any positive integers i, j,k, r, l, s satisfying $0 \le i, r, j, k, l, s \le n$, we have

$$M_G(f^i g^j u, f^r g^l u, f^s g^k u, t) \ge \emptyset(M_G(g(f^{i-1} g^j u), g(f^{r-1} g^l u), g(f^{s-1} g^k u), t))$$
$$\ge \emptyset(M_G(f^{i-1} g^{j+1} u, f^{r-1} g^{l+1} u, f^{s-1} g^{k+1} u, t)).$$

Define $\delta = \min_{0\le i,r,j,l,s,k\le n} M_G(f^i g^j u, f^r g^l u, f^s g^k u, t)$ where $t > 0$.

Assume that $0 \le \delta < 1$, then it follows from (2.1.1) $\delta \ge \emptyset(\delta) > \delta$,

which is a contradiction and hence $\delta = 1$.

In particular, $M_G(fu, u, u, t) = 1$ and $M_G(gu, u, u, t) = 1$ for each $t > 0$ and hence

$fu = gu = u$, implies that, $u \in F(f) \cap F(g)$. Hence f and g have Property Q.

Corollary: 3.2 Under the conditions of Corollary 2.2, f has Property P.

Theorem: 3.3 Under the conditions of Theorem 2.3, f and g have Property Q.

Proof: From Theorem 2.3, $F(f) \cap F(g) \neq \emptyset$.Therefore, $F(f^n) \cap F(g^n) \neq \emptyset$ for each positive integer n. Let n be a fixed positive integer greater than 1 and suppose that

$u \in F(f^n) \cap F(g^n)$. We claim that $u \in F(f) \cap F(g)$.

Let $u \in F(f^n) \cap F(g^n)$. Then, for any positive integers i, j, r, l, s, k satisfying $0 \le i, r, j, l, s, k \le n$, we have

$$M_G(f^i g^j u, f^r g^l u, f^s g^k u, t) \ge \emptyset(\min\{M_G(g(f^{i-1} g^j u), f(f^{i-1} g^j u), f(f^{i-1} g^j u), t), M_G(g(f^{r-1} g^l u), f(f^{r-1} g^l u), f(f^{r-1} g^l u), t), M_G(g(f^{s-1} g^k u), f(f^{s-1} g^k u), f(f^{s-1} g^k u), t)\}$$

$$\ge \emptyset(\min\{M_G(f^{i-1} g^{j+1} u, f^i g^j u, f^i g^j u, t), M_G(f^{r-1} g^{l+1} u, f^r g^l u, f^r g^l u, t), M_G(f^{s-1} g^{k+1} u, f^s g^k u, f^s g^k u, t)\}.$$

Define $\delta = \min_{0\le i,r,j,l,s,k\le n} M_G(f^i g^j u, f^r g^l u, f^s g^k u, t)$ where $t > 0$.

Assume that $0 \le \delta < 1$, then it follows from (2.3.1) $\delta \ge \emptyset(\min\{\delta, \delta, \delta\}) = \emptyset(\delta) > \delta$, which is a contradiction and hence $\delta = 1$.

In particular, $M_G(fu, u, u, t) = 1$ and $M_G(gu, u, u, t) = 1$ for each $t > 0$ and hence

$fu = gu = u$, implies that, $u \in F(f) \cap F(g)$. Hence f and g have Property Q.

Corollary: 3.4 Under the conditions of Corollary 2.4, f has Property P.

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