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Z^*-Open Sets And Z^*-Continuity In Topological Spaces

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ABSTRACT

The aim of this paper is to introduce and study the notion of Z-open sets and Z*-continuity. Some characterizations of these notions are presented. Also, some topological operations such as: Z*-boundary, Z*-border, Z*-exterior, Z*-limit point are introduced.*

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***Keyword:**. Z*-open sets , Z*-boundary, Z*-border, Z*-exterior, Z*-limit point, Z*-nbd and Z*-continuity.*

1. INTRODUCTION:

In 1963, N. Levine [11] introduced semiopen sets and semi-continuous mappings in topological spaces.The concept of δ-preopen sets and δ-almost continuity introduced by S.Raychaudhuri and M.N. Mukheriee in 1993 [16]. Also, in 1996 Andrijevi'c [3] (resp. J. Dontchev and M. Przemski [4], A. A. El-Atik [8]) introduced the notion b-open (resp. sp-open, γ-open) sets. In 2008, Ekici [5] introduced e-open sets and e-continuous map in topological spaces. The purpose of this paper is to introduce and study the notion of Z*-open sets and Z*-continuity. Some topological operations such as: Z*-limit point, Z*-boundary and Z*-exterior…atc are introduced. Also, some characterizations of these notions are presented.

2. PRELIMINARIES:

A subset A of a topological space (X,τ) is called regular open (resp. regular closed) [18] if A=int(cl(A)) (resp. A= cl(int(A))).The delta interior [18] of a subset A of X is the union of all reg ular open sets of X contained in A is denoted by δ-int(A). A subset A of a space X is called δ-open if it is the union of regular open sets. The complement of δ-open set is called δ -closed. Alternatively, a set A of (X,τ) is called δ-closed [18] if A= δ-cl(A), where δ-cl(A) = $\{x\in X : A \cap int(cl(U)) \neq \emptyset, U\in \tau \text{ and } x\in U\}$. Throughout this paper (X, τ) and (Y, σ) (simply X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned . For a subset A of a space (X, τ), cl(A), int(A) and X \ A denote the closure of A, the interior of A and the complement of A respectively. A subset A of a space X is called a-open[6] (resp. α-open [14], δ-semiopen [15], semiopen [11], δ-preopen [16], preopen [12], b-open [3] or γ-open [8] or sp-open [4], e-open [5], β-open [1] or semi-preopen [2], e*-open [7] or δ-β-open [10]) if $A \subseteq int(cl(\delta\text{-}int(A)))$, (resp. $A \subseteq int(cl(int(A)))$, $A \subseteq cl(\delta\text{-}int(A))$, $A \subseteq cl(int(A))$, $A \subseteq int(\delta\text{-}cl(A))$, $A \subseteq int(cl(A))$, $A \subseteq int(cl(A)) \cup cl(int(A))$, $A \subseteq cl(\delta\text{-}int(A)) \cup int(\delta\text{-}cl(A))$, $A \subseteq cl(int(cl(A)))$, $A \subseteq cl(int(\delta\text{-}cl(A)))$). The complement of a δ-semiopen (resp. semiopen, δ-preopen, preopen) set is called δ-semi-closed (resp. semi-closed, δ-pre-closed, pre-closed). The intersection of all δ-semi-closed (resp. semi-closed, δ -pre-closed, pre-closed)sets containing A is called the δ-semi-closure (resp. semi-closure, δ -pre-closure, pre-closure) of A and is denoted by δ-scl(A) (resp. scl(A), δ-pcl(A) , pcl(A)). The union of all δ-semiopen(resp. semiopen, δ-preopen, preopen) sets contained in A is called the δ-semi-interior (resp. semi-interior, δ-pre-interior, pre-interior) of A and is denoted by δ-sint(A) (resp. sint(A) , δ-pint(A), pint(A)). The family of all δ-open (resp. a-open, α-open, δ-semiopen, semiopen, δ-preopen, preopen, b-open, e-open, β-open, e*-open) is denoted by $\delta O(X)$ (resp. $aO(X)$, $\alpha O(X)$, $\delta SO(X)$, $SO(X)$, $\delta PO(X)$, $PO(X)$, $BO(X)$, $eO(X)$, $\beta O(X)$, $e^*O(X)$).

Lemma: 2.1 [18] Let A, B be two subsets of (X, τ). Then:
(1) A is δ-open if and only if A = δ-int(A),
(2) X \ (δ-int(A)) = δ-cl(X \ A) and δ-int(X \ A) = X \ (δ-cl(A)),
(3) cl(A) $\subseteq$ δ-cl (A) (resp. δ-int(A) $\subseteq$ int(A)), for any subset A of X,
(4) δ-cl $(A\cup B)$ = δ-cl(A) $\cup$ δ-cl (B) , δ-int$(A\cap B)$ = δ-int(A) $\cap$ δ-int(B).

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Proposition: 2.1 Let A be a subset of a space (X, τ). Then:
(1) $scl(A) = A \cup int(cl(A))$, $sint(A) = A \cap cl(int(A))$ [3],
(2) $pcl(A) = A \cup cl(int(A))$, $pint(A) = A \cap int(cl(A))$ [3],
(3) $\delta\text{-}scl(X \setminus A) = X \setminus \delta\text{-}sint(A)$, $\delta\text{-}scl(A \cup B) \subseteq \delta\text{-}scl(A) \cup \delta\text{-}scl(B)$ [15] ,
(4) $\delta\text{-}pcl(X \setminus A) = X \setminus \delta\text{-}pint(A)$, $\delta\text{-}pcl(A \cup B) \subseteq \delta\text{-}pcl(A) \cup \delta\text{-}pcl(B)$ [16].

Lemma: 2.2 The following hold for a subset H of a space (X, τ).
(1) $\delta\text{-}pcl(H) = H \cup cl(\delta\text{-}int(H))$ and $\delta\text{-}pint(H) = H \cap int(\delta\text{-}cl(H))$ [16] ,
(2) $\delta\text{-}scl(H) = H \cup int(\delta\text{-}cl(H))$ and $\delta\text{-}sint(H) = H \cap cl(\delta\text{-}int(H))$ [15],
(3) $cl(\delta\text{-}int(H)) = \delta\text{-}cl(\delta\text{-}int(H))$ and $int(\delta\text{-}cl(H)) = \delta\text{-}int(\delta\text{-}cl(H)$ [5].

Definition: 2.1 A function $f:(X, \tau) \to (Y, \sigma)$ is called α-continuous [13] (resp. pre-continuous [12], δ-almost continuous[16], γ-continuous [8], e-continuous [5], e*-continuous [7]) if $f^{-1}(V)$ is α-open(resp. preopen, δ-preopen, γ-open, e-open,e*-open) in X for each $V \in \sigma$.

3. Z*-open sets and Z*-closed:

Definition: 3.1 A subset A of a topological space (X, τ) is said to be:
(1) Z*-open if $A \subseteq cl(int(A)) \cup int(\delta\text{-}cl(A))$,
(2) Z*-closed if $int(cl(A)) \cap cl(\delta\text{-}int(A)) \subseteq A$.

The family of all Z*-open (resp. Z*-closed) subsets of a space (X, τ) will be as always denoted by Z*O(X) (resp. Z*C(X)).

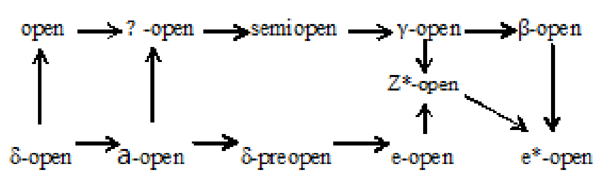
Remark: 3.1 The following holds for a space (X, τ).
(1) Every γ-open (resp. e-open) set is Z*-open,
(2) Every Z*-open set is e*-open.

But the converse of the a bovine are not necessarily true in general as shown by the following examples.

Example: 3.1 Let X = {a, b, c, d}, with topology τ = {∅,{a},{c},{a, b},{a, c},{a, b, c},{a, c, d}, X}. Then:
(1) the subset {a, d} is a Z*-open set but not e-open,
(2) the subset {b, c} is a Z*-open set but not γ-open.

Example: 3.2 Let X = {a, b, c, d, e} with topology τ = {∅, {a, b}, {c, d}, {a, b, c, d}, X}.Then, the subset {a, e} is e*-open set but not Z*-open.

Remark: 3.2 The following diagram holds for a subset of a space X:



Proposition: 3.1 Let (X, τ) be a topological space .Then the δ-closure of a Z*-open subset of (X, τ) is δ-semiopen.

Proof: Let $A \in Z^*O(X)$. Then $\delta\text{-}cl(A) \subseteq \delta\text{-}cl\,(cl(int(A)) \cup int(\delta\text{-}cl(A))) \subseteq \delta\text{-}cl(cl(int(A))) \cup \delta\text{-}cl(int(\delta\text{-}cl(A))) \subseteq \delta\text{-}cl(int(A)) \cup \delta\text{-}cl(int(\delta\text{-}cl(A))) = \delta\text{-}cl(int(\delta\text{-}cl(A))) = \delta\text{-}cl(\delta\text{-}int(\delta\text{-}cl(A))) = cl(\delta\text{-}int(\delta\text{-}cl(A)))$. Therefore $\delta\text{-}cl(A) \in \delta SO(X)$.

Lemma: 3.1 Let (X, τ) be a topological space .Then the following statements are hold.
(1) The union of arbitrary Z*-open sets is Z*-open,
(2) The intersection of arbitrary Z*-closed sets is Z*-closed.

Proof: (1) It is clear.

Remark: 3.2 By the following we show that the intersection of any two Z*-open sets is not Z*-open.

Example: 3.3 Let X = {a, b, c} with topology τ = {∅, {b}, {c}, {b, c}, X}.Then A = {a, c} and B = {a, b} are Z*-open sets, but $A \cap B$ = {a} is not Z*-open.

Definition: 3.2 Let (X, τ) be a topological space. Then:
(1) The union of all Z*-open sets of X contained in A is called the Z*-interior of A and is denoted by Z*-int(A),
(2) The intersection of all Z*-closed sets of X containing A is called the Z*-closure of A and is denoted by Z*-cl(A).

Theorem: 3.1 Let A, B be two subsets of a topological space (X, τ). Then the following are hold:
(1) $Z^*\text{-cl}(X \setminus A) = X \setminus Z^*\text{-int}(A)$,
(2) $Z^*\text{-int}(X \setminus A) = X \setminus Z^*\text{-cl}(A)$,
(3) If $A \subseteq B$, then $Z^*\text{-cl}(A) \subseteq Z^*\text{-cl}(B)$ and $Z^*\text{-int}(A) \subseteq Z^*\text{-int}(B)$,
(4) $x \in Z^*\text{-cl}(A)$ if and only if for each a Z*-open set U contains x, $U \cap A \neq \emptyset$,
(5) $x \in Z^*\text{-int}(A)$ if and only if there exist a Z*-open set W such that $x \in W \subseteq A$,
(6) A is Z*-open set if and only if $A = Z^*\text{-int}(A)$,
(7) A is Z*-closed set if and only if $A = Z^*\text{-cl}(A)$,
(8) $Z^*\text{-cl}(Z^*\text{-cl}(A)) = Z^*\text{-cl}(A)$ and $Z^*\text{-int}(Z^*\text{-int}(A)) = Z^*\text{-int}(A)$,
(9) $Z^*\text{-cl}(A) \cup Z^*\text{-cl}(B) \subseteq Z^*\text{-cl}(A \cup B)$ and $Z^*\text{-int}(A) \cup Z^*\text{-int}(B) \subseteq Z^*\text{-int}(A \cup B)$,
(10) $Z^*\text{-int}(A \cap B) \subseteq Z^*\text{-int}(A) \cap Z^*\text{-int}(B)$ and $Z^*\text{-cl}(A \cap B) \subseteq Z^*\text{-cl}(A) \cap Z^*\text{-cl}(B)$.

Remark: 3.3 By the following example we show that the inclusion relation in parts (9) and (10) of the above theorem cannot be replaced by equality.

Example: 3.4 Let $X = \{a, b, c, d\}$, with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.Then:
(1) If $A = \{a, d\}$, $B = \{b, d\}$, then $Z^*\text{-cl}(A) = A$, $Z^*\text{-cl}(B) = B$ and $Z^*\text{-cl}(A \cup B) = X$.

Thus $Z^*\text{-cl}(A \cup B) \nsubseteq Z^*\text{-cl}(A) \cup Z^*\text{-cl}(B)$,

(2) If $E = \{a, b\}$, $F = \{a, c\}$, then $Z^*\text{-cl}(E) = X$, $Z^*\text{-cl}(F) = F$ and $Z^*\text{-cl}(E \cap F) = \{a\}$.

Thus $Z^*\text{-cl}(E) \cap Z^*\text{-cl}(F) \nsubseteq Z^*\text{-cl}(E \cap F)$.

(3) If $M = \{c, d\}$, $N = \{b, d\}$, then $Z^*\text{-int}(M) = \emptyset$, $Z^*\text{-int}(N) = N$ and $Z^*\text{-int}(M \cup N) = \{b, c, d\}$.

Thus $Z^*\text{-int}(M \cup N) \nsubseteq Z^*\text{-int}(M) \cup Z^*\text{-int}(N)$.

Theorem: 3.2 Let A , B be two subsets of a topological space (X, τ). Then the following are hold:
(1) $Z^*\text{-cl}(\text{cl}(A) \cup B) = \text{cl}(A) \cup Z^*\text{-cl}(B)$,
(2) $Z^*\text{-int}(\text{int}(A) \cap B) = \text{int}(A) \cap Z^*\text{-int}(B)$.

Proof: (1) $Z^*\text{-cl}(\text{cl}(A) \cup B) \supseteq Z^*\text{-cl}(\text{cl}(A)) \cup Z^*\text{-cl}(B) \supseteq \text{cl}(A) \cup Z^*\text{-cl}(B)$.

The other inclusion, $\text{cl}(A) \cup B \subseteq \text{cl}(A) \cup Z\text{-cl}(B)$ which is Z*-closed. Hence,

$Z^*\text{-cl}(\text{cl}(A) \cup B) \subseteq \text{cl}(A) \cup Z^*\text{-cl}(B)$. Therefore, $Z^*\text{-cl}(\text{cl}(A) \cup B) = \text{cl}(A) \cup Z^*\text{-cl}(B)$, (2) It is follows from (1).

Theorem: 3.3 Let (X, τ) be a topological space and $A \subseteq X$. Then A is a Z*-open set if and only if $A = \text{sint}(A) \cup \delta\text{-pint}(A)$.

Proof: It is clear.

Proposition: 3.2 Let (X, τ) be a topological space and $A \subseteq X$. Then A is a Z*-closed set if and only if $A = \text{scl}(A) \cap \delta\text{-pcl}(A)$.

Proof: It follows from Theorem 3.3.

Proposition: 3.3 Let A be a subset of a space (X, τ).Then:
(1) $Z^*\text{-cl}(A) = \text{scl}(A) \cap \delta\text{-pcl}(A)$,
(2) $Z^*\text{-int}(A) = \text{sint}(A) \cup \delta\text{-pint}(A)$.

Lemma: 3.2 Let A be a subset of a topological space (X, τ).Then the following are hold:
(1) $\text{pcl}(\delta\text{-pint}(A)) = \delta\text{-pint}(A) \cup \text{cl}(\text{int}(A))$,
(2) $\text{pint}(\delta\text{-pcl}(A)) = \delta\text{-pcl}(A) \cap \text{int}(\text{cl}(A))$.

Proof: (1) By Lemma 2.2 and Proposition 2.1, $\text{pcl}(\delta\text{-pint}(A)) = \delta\text{-pint}(A) \cup \text{cl}(\text{int}(\delta\text{-pint}(A))) = \delta\text{-pint}(A) \cup \text{cl}(\text{int}(A \cap \delta\text{-cl}(\text{int}(A)))) = \delta\text{-pint}(A) \cup \text{cl}(\text{int}(A))$,
(2) It follows from (1).

Proposition: 3.4 Let A be a subset of a topological space (X, τ). Then:
(1) Z*-cl(A) = A ∪ pint(δ-pcl(A)),
(2) Z*-int(A) = A ∩ pcl(δ-pint(A)).

Proof: (1) By Lemma 3.2, A ∪ pint(δ-pcl(A)) = A∪ (δ-pcl(A) ∩ int(cl(A))) = (A ∪ δ-pcl(A)) ∩ (A ∪ int(cl(A))) = δ-pcl(A) ∩ scl(A) = Z*-cl(A).

(2) It follows from (1).

Theorem: 3.4 Let A be a subset of a topological space (X, τ).Then the following are equivalent:
(1) A is a Z*-open set,
(2) A ⊆ pcl(δ-pint(A)),
(3) there exists U ∈ δ-PO(X) such that U⊆ A ⊆ pcl(U),
(4) pcl(A) = pcl(δ-pint(A)).

Proof: (1) → (2). Let A be a Z*-open set. Then, A = Z*-int(A) and by Proposition 3.4, A = A ∩ pcl(δ-pint(A)) and hence , A ⊆ pcl(δ-pint(A)).

(3) → (1). Let A ⊆ pcl(δ-pint(A)).Then by Proposition 3.4, A ⊆ A∩pcl(δ-pint(A)) = Z*-int(A) and hence A = Z*-int(A). Thus A is Z*-open,

(2) → (3). It follows from putting U= δ-pint(A),

(3) →(2). Let there exists U ∈ δ-PO(X) such that U⊆ A ⊆ pcl(U). Since U ⊆ A, then pcl(U) ⊆ pcl(δ-pint(A)) therefore A⊆ pcl(U) ⊆ pcl(δ-pint(A)),

(4) ↔ (4). It is clear.

Theorem: 3.5 Let A be a subset of a topological space X. Then the following are equivalent:
(1) A is a Z*-closed set,
(2) δ-pint(pcl(A)) ⊆ A,
(3) there exists U ∈ δ-PC(X) such that pint(U) ⊆ A ⊆ U,
(4) pint(A) = pint(δ-pcl(A)).

Proof: It follows from Theorem 3.4.

Proposition: 3.5 If A is a Z*-open subset of a topological space (X, τ) such that A⊆ B ⊆ pcl(A), then B is Z*-open.

4. SOME TOPOLOGICAL OPERATIONS.

Definition: 4.1 Let (X, τ) be a topological space and A⊆ X. Then the Z*-boundary of A(briefly, Z*-b(A)) is defined by Z*-b(A) = Z*-cl(A) ∩ Z*-cl(X \ A).

Theorem: 4.1 If A is a subsets of a topological space (X, τ), then the following statement are hold:
(1) Z*-b(A) is Z*-closed,
(2) Z*-b(A) = Z*-b(X \ A),
(3) Z*-b(A) = Z*-cl(A) \ Z*-int(A),
(4) Z*-b(A) ∩ Z*-int(A) = ∅,
(5) Z*-b(A) ∩ Z*-int(A) = Z*-cl(A),
(6) Z*-b(Z*-b(A)) ⊆ Z*-b(A),
(7) Z*-b(Z*-int(A)) ⊆ Z*-b(A),
(8) Z*-b(Z*-cl(A)) ⊆ Z*-b(A),
(9) Z*-int(A) = A \ Z*-b(A),
(10) Z*-b(A ∩ B) ⊆ Z*-b(A) ∪ Z*-b(B).

Proof: (1) It is clear.

Theorem: 4.2 If A is a subset of a space X, then the following statement are hold:
(1) A is a Z*-open set if and only if A ∩ Z*-b(A) = ∅,
(2) A is a Z*-closed set if and only if Z*-b (A) ⊂ A,
(3) A is a Z*-clopen set if and only if Z*-b (A) = ∅ .

Proof: (1) It follows from Theorem 4.1.

Definition: 4.2 $Z^*\text{-Bd}(A) = A \setminus Z^*\text{-int}(A)$ is said to be Z*-border of A.

Theorem: 4.2 For a subset A of a space X, the following statements hold:
(1) $Z^*\text{-Bd}(A) \subseteq A$, for any $A \subseteq X$,
(2) $A = Z^*\text{-int}(A) \cup Z^*\text{-Bd}(A)$,
(3) $Z^*\text{-int}(A) \cap Z^*\text{-Bd}(A) = \emptyset$,
(4) A is Z*-open if and only if $Z^*\text{-Bd}(A) = \emptyset$,
(5) $Z^*\text{-Bd}(Z^*\text{-int}(A)) = \emptyset$,
(6) $Z^*\text{-int}(Z^*\text{-Bd}(A)) = \emptyset$,
(7) $Z^*\text{-Bd}(Z^*\text{-Bd}(A)) = Z^*\text{-Bd}(A)$,
(8) $Z^*\text{-Bd}(A) = A \cap Z^*\text{-cl}(X \setminus A)$.

Proof: (6) Let $x \in Z^*\text{-int}(Z^*\text{-Bd}(A))$. Then $x \in Z^*\text{-Bd}(A)$. Since, $Z^*\text{-Bd}(A) \subseteq A$, then $x \in Z^*\text{-int}(Z^*\text{-Bd}(A)) \subseteq Z^*\text{-int}(A)$. Hence, $x \in Z^*\text{-int}(A) \cap Z^*\text{-Bd}(A)$, which contradicts (3). Thus, $Z^*\text{-int}(Z^*\text{-Bd}(A)) = \emptyset$,
(8) $Z^*\text{-Bd}(A) = A \setminus Z^*\text{-int}(A) = A \setminus (X \setminus Z^*\text{-cl}(X \setminus A)) = A \cap Z^*\text{-cl}(X \setminus A)$.

Definition: 4.3 Let (X, τ) be a topological space and A□X. Then the set $X \setminus (Z^*\text{-cl}(A))$ is called the Z*-exterior of A and is denoted by Z*-ext(A). A point $p \in X$ is called a Z*- exterior point of A, if it is a Z*-interior point of $X \setminus A$.

Theorem: 4.3 If A and B are two subsets of a space (X, τ), then the following statement are hold:
(1) Z*-ext(A) is Z*-open,
(2) $Z^*\text{-ext}(A) = Z^*\text{-int}(X \setminus A)$,
(3) $Z^*\text{-ext}(Z^*\text{-ext}(A)) = Z^*\text{-int}(Z^*\text{-cl}(A))$,
(4) $Z^*\text{-ext}(X \setminus Z^*\text{-ext}(A)) = Z^*\text{-ext}(A)$,
(5) $Z^*\text{-int}(A) \subseteq Z^*\text{-ext}(Z^*\text{-ext}(A))$,
(6) $Z^*\text{-ext}(A) \cap Z^*\text{-b}(A) = \emptyset$,
(7) $Z^*\text{ext}(A) \cup Z^*\text{-b}(A) = Z^*\text{-cl}(X \setminus A)$,
(8) {Z*-int(A), Z*-b(A) and Z*-ext(A)} form a partition of X,
(9) If $A \subseteq B$, then $Z^*\text{-ext}(B) \subseteq Z^*\text{-ext}(A)$,
(10) $Z^*\text{-ext}(\emptyset) = X$ and $Z^*\text{-ext}(X)= \emptyset$,
(11) $Z^*\text{-ext}(A\cup B) \subseteq Z^*\text{-ext}(A) \cup Z^*\text{-ext}(B)$,
(12) $Z^*\text{-ext}(A \cap B) \supseteq Z^*\text{-ext}(A) \cap Z^*\text{-ext}(B)$,
(13) $Z^*\text{-ext}(A\cup B) \subseteq Z^*\text{-ext}(A \cap B)$.

Proof: It follows from Theorems 3.1 and 4.1.

Remark: 4.1 The inclusion relation in parts (11) and (12) of the above theorem cannot be replaced by equality as is shown by the following example.

Example: 4.1 Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a\}, \{b ,c\}, \{a, b, c\}, X\}$.

If $A=\{b, c\}$ and $B=\{a, c\}$, then $Z^*\text{-ext}(A) =\{a, d\}$, $Z^*\text{-ext}(B) = \{b, d\}$. But $Z^*\text{-ext}(A\cup B) =\emptyset$,

Therefore $Z^*\text{-ext}(A) \cup Z^*\text{-ext}(B) \nsubseteq Z^*\text{-ext}(A\cup B)$. Also, $Z^*\text{-ext}(A\cap B) = \{a, b, d\}$, hence,

$Z^*\text{-ext}(A\cap B) \nsubseteq Z^*\text{-ext}(A) \cap Z^*\text{-ext}(B)$.

Definition: 4.4 Let A be a subset of a topological space (X, τ). Then a point $P\in X$ is called a Z*-limit point of a set $A\subseteq X$ if every Z*-open set $G \subseteq X$ containing p contains a point of A other than p. The set of all Z*-limit points of A is called a Z*-derived set of A and is denoted by Z*-d(A).

Theorem: 4.4 If A and B are two subsets of a space X, then the following statement are hold:
(1) If $A\subseteq B$, then $Z^*\text{-d}(A) \subseteq Z^*\text{-d}(B)$,
(2) $Z^*\text{-d}(A) \cup Z^*\text{-d}(B) \subseteq Z^*\text{-d}(A\cup B$
(3) $Z^*\text{-d}(A \cap B) \subseteq Z^*\text{-d}(A) \cap Z^*\text{-d}(B)$,
(4) A is a Z*-closed set if and only if it contains each of its Z*-limit points,
(5) $Z^*\text{-cl}(A) = A \cup Z^*\text{-d}(A)$.

Proof: It is clear.

Definition: 4.5 A subset N of a topological space (X, τ) is called a Z*-neighbourhood (briefly, Z*-nbd) of a point $P\in X$ if there exists a Z*-open set W such that $P\in W\subseteq N$. The class of all Z*-nbds of $P\in X$ is called the Z*-neighbourhood system of P and denoted by $Z^*\text{-}N_p$.

Theorem: 4.5 A subset G of a space X is Z*-open if and only if it is Z*-nbd, for every point $P\in G$.

Proof: It is clear**.**

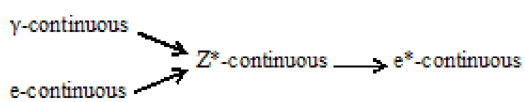
Theorem: 4.6 In a topological space (X, τ). Let $Z^*\text{-}N_p$ be the Z*-nbd System of a point $P\in X$. Then the following statement is hold:
(1) $Z^*\text{-}N_p$ is not empty and p belongs to each member of $Z^*\text{-}N_p$,
(2) Each superset of members of N_p belongs to $Z^*\text{-}N_p$,
(3) Each member $N \in Z^*\text{-}N_p$ is a superset of a member $W \in Z^*\text{-}N_p$, where W is Z*-nbd of each point $P \in W$.

Proof: Obvious.

5. Z***-continuous mappings.**

Definition: 5.1 A function $f:(X, \tau)\to(Y, \sigma)$ is called Z*-continuous if $f^{-1}(V)$ is Z*-open in X for each $V \in \sigma$.

Remark: 5.1. Let $f: (X, \tau) \to(Y, \sigma)$ be mapping from a space (X, τ) into a space (Y, σ), The following hold:



Now , the following examples show that these implication are not reversible.

Example: 5.1 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\},\{a, c\},\{a, b ,c\},\{a, c ,d\} , X\}$, $\sigma= \{\emptyset, \{b, c\},\{a, d\}, Y\}$. Then, the identity $f :(X, \tau) \to (Y, \sigma)$ is Z*-continuous but not γ-continuous and not e-continuous.

Example: 5.2 Let $X =Y= \{a, b, c, d, e\}$, $\tau = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma= \{\emptyset, \{a, e\}, Y\}$.Then the identity $f: (X, \tau)\to(Y, \sigma)$ is e*-continuous but not Z*-continuous.

Theorem: 5.1 Let $f:(X, \tau) \to(Y, \sigma)$ be a mapping. Then the following statements are equivalent:
(1) f is Z*-continuous.
(2) For each $x\in X$ and $V\in \sigma$ containing $f(X)$, there exists $U \in Z^*O(X)$ containing x such that $f(U) \subseteq V$,
(3) The inverse image of each closed set in Y is Z*-closed in X,
(4) $int(cl(f^{-1}(B))) \cap cl(\delta\text{-}int(f^{-1}(B))) \subseteq f^{-1}(cl(B))$, for each $B \subseteq Y$,
(5) $f^{-1}(int(B)) \subseteq cl(int(f^{-1}(B))) \cup int(\delta\text{-}cl(f^{-1}(B)))$, for each $B\subseteq Y$,
(6) If f is bijective, then $int(f(A)) \subseteq f(cl(int(A))) \cup f(int(\delta\text{-}cl(A)))$, for each $A\subseteq X$,
(7) If f is bijective, then $f(int(cl(A))) \cap f(cl(\delta\text{-}int(A))) \subseteq cl(f(A))$, for each $A\subseteq X$.

Proof: (1) ↔ (2) and (1) ↔(3) are obvious,

(3) → (4). Let $B\subseteq Y$, then by (3) $f^{-1}(cl(B))$ is Z*-closed. This means
$f^{-1}(cl(B)) \supseteq int(cl(f^{-1}(cl(B)))) \cap cl(\delta\text{-}int(f^{-1}(cl(B)))) \supseteq int(cl(f^{-1}(B))) \cap cl(\delta\text{-}int(f^{-1}(B)))$,

(4) → (5). By replacing $Y \setminus B$ instead of B in (4) ,we have
$int(cl(f^{-1}(Y \setminus B))) \cap cl(\delta\text{-}int(f^{-1}(Y \setminus B))) \subseteq f^{-1}(cl(Y \setminus B))$ and therefore
$f^{-1}(int(B)) \subseteq cl(int(f^{-1}(B))) \cup int(\delta\text{-}cl(f^{-1}(B)))$,

(5) → (6). Follows directly by replacing A instead of $f^{-1}(B)$ in (5) and applying the bijection of f,

(6) → (7). By complementation of (6) and applying the bijective of f , we have
$f(int(cl (X \setminus A))) \cap f(cl(\delta\text{-}int(X \setminus A))) \subseteq cl(f(X \setminus A))$ .We obtain the required by replacing A instead of $X \setminus A$,

(7) → (1). Let $V\in \sigma$ .But $W = Y \setminus V$, by (7), we have $f(int(cl(f^{-1}(W)))) \cap f(cl(\delta\text{-}int(f^{-1}(W)))) \subseteq cl(ff^{-1}(W)) \subseteq cl (W) = W$. So $int(cl (f^{-1}(W))) \cap cl(\delta\text{-}int(f^{-1} (W))) \subseteq f^{-1}(W)$ implies $f^{-1}(W)$ is Z*-closed and therefore $f^{-1}(V)\in Z^*O(X)$.

Theorem: 5.2. Let $f :(X, \tau) \to(Y, \sigma)$ be a mapping. Then the following statements are equivalent:
(1) f is Z*-continuous,

(2) $Z^*\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$, for each $B \subseteq Y$,
(3) $f(Z^*\text{-cl}(A)) \subseteq \text{cl}(f(A))$, for each $A \subseteq X$,
(4) $Z^*\text{-Bd}(f^{-1}(B)) \subseteq f^{-1}(\text{Bd}(B))$, for each $B \subseteq Y$,
(5) $f(Z^*\text{-d}(A)) \subseteq \text{cl}(f(A))$, for each $A \subseteq X$,
(6) $f^{-1}(\text{int}(B)) \subseteq Z^*\text{-int}(f^{-1}(B))$, for each $B \subseteq Y$,

Proof: (1) → (2). Let $B \subseteq Y$, $f^{-1}(\text{cl}(B))$ is Z^*-closed in X. Then $Z^*\text{-cl}(f^{-1}(B)) \subseteq Z^*\text{-cl}(f^{-1}(\text{cl}(B))) = f^{-1}(\text{cl}(B))$,

(2) → (3). Let $A \subseteq X$, then $f(A) \subseteq Y$, by (2), $f^{-1}(\text{cl}(f(A))) \supseteq Z^*\text{-cl}(f^{-1}(f(A))) \supseteq Z^*\text{-cl}(A)$.

Therefore, $\text{cl}(f(A)) \supseteq f f^{-1}(\text{cl}(f(A))) \supseteq f(Z^*\text{-cl}(A))$.

(3) → (1). Let $W \subseteq Y$ be open set. Then, $F = Y \setminus W$ is closed in Y and $f^{-1}(F) = X \setminus f^{-1}(W)$.

Hence, by (3), $f(Z^*\text{-cl}(f^{-1}(F))) \subseteq \text{cl}(f(f^{-1}(F))) \subseteq \text{cl}(F) = F$ thus, $Z^*\text{-cl}(f^{-1}(F)) \subseteq f^{-1}(F)$,

So, $f^{-1}(F) = X \setminus f^{-1}(W) \in Z^*C(X)$ and therefore $f^{-1}(W) \in Z^*O(X)$,

(4)→ (6). Let $B \subseteq Y$. Then by (4), $Z^*\text{-Bd}(f^{-1}(B)) = f^{-1}(B) \setminus Z^*\text{-int}(f^{-1}(B)) \subseteq f^{-1}(\text{Bd}(B)) =$
$f^{-1}(B \setminus \text{int}(B)) = f^{-1}(B) \setminus f^{-1}(\text{int}(B))$ this implies $f^{-1}(\text{int}(B)) \subseteq Z^*\text{-int}(f^{-1}(B))$,

(6) → (4). Let $B \subseteq Y$. Then by (6), $f^{-1}(\text{int}(B)) \subseteq Z^*\text{-int}(f^{-1}(B))$ we have
$f^{-1}(B) \setminus Z^*\text{-int}(f^{-1}(B)) \subseteq f^{-1}(B) \setminus f^{-1}(\text{int}(B)) \Rightarrow Z^*\text{-Bd}(f^{-1}(B)) \subseteq f^{-1}(\text{Bd}(B))$,

(1) → (5). It is obvious, since f is Z^*-continuous and by (3), $f(Z^*\text{-cl}(A)) \subseteq \text{cl}(f(A))$, for each $A \subseteq X$. So, $f(Z^*\text{-d}(A)) \subseteq f(Z^*\text{-cl}(A)) \subseteq \text{cl}(f(A))$.

(5) → (1). Let $U \subseteq Y$ be open set. Then, $F = Y \setminus U$ is closed in Y and $f^{-1}(F) = X \setminus f^{-1}(U)$.

Hence, by (5), $f(Z^*\text{-d}(f^{-1}(F))) \subseteq \text{cl}(f(f^{-1}(F))) \subseteq \text{cl}(F) = F$. Hence, $Z^*\text{-d}(f^{-1}(F)) \subseteq f^{-1}(F)$. By Theorem 4.4, $f^{-1}(F) = X \setminus f^{-1}(U)$ is Z^*-closed in X. Therefore, $f^{-1}(U)$ is Z^*-open in X,

(1) → (6). Let $B \subseteq Y$. Then $f^{-1}(\text{int}(B))$ is Z^*-open in X. Thus, $f^{-1}(\text{int}(B)) = Z^*\text{-int}(f^{-1}(\text{int}(B))) \subseteq Z^*\text{-int}(f^{-1}(B))$. Therefore, $f^{-1}(\text{int}(B)) \subseteq Z^*\text{-int}(f^{-1}(B))$,

(5) → (1). Let $U \subseteq Y$ be an open set. Then $f^{-1}(U) = f^{-1}(\text{int}(U)) \subseteq f^{-1}(Z^*\text{-int}(U))$.

Hence, $f^{-1}(U)$ is Z^*-open in X. Therefore, f is Z^*-continuous.

Remark: 5.3. The composition of two Z^*-continuous mappings need not be Z^*-continuous as show by the following example.

Example: 5.3. Let $X = Y = Z = \{a, b, c, d, e\}$ with topologies $\tau_x = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, an indiscrete topology $(Y, \Im)$ and $\tau_Z = \{\emptyset, \{a, e\}, Z\}$. Let the mappings $f:(X, \tau_X) \to (Y, □)$ and $g: (Y, □) \to (Z, \tau_Z)$ defined as identity mappings. It is clear that f and g are Z^*-continuous but g □ f is not Z^*-continuous.

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