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REFINEMENTS OF HERMITE-HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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ABSTRACT

In this paper, some inequalities of Hermite-Hdamard type for co-ordinated quasi-convex functions in two variables are given. The obtained results give refimenets of the Hermite-Hdamard type inequalities for co-ordinated quasi convex functions proved in [21].

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1 INTRODUCTION:

Let $f:I\to\mathbb{R}$, $\varnothing\neq I\subseteq\mathbb{R}$ be a convex on I, $a,b\in I$ with $a<b$. Then the inequalities:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}, \tag{1}$$

hold. The inequalities in (1) is known as the Hermite-Hadamard's inequalities for convex mappings. The inequalities in (1) hold in reversed order if f is a concave function.

In recent years, many authors have established several inequalities connected to Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see [12], [13], [17] and [24].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f:[a,b]\to\mathbb{R}$ is said to be quasi- convex on $[a,b]$ if

$$f(\lambda x+(1-\lambda)y)\leq\max\{f(x),f(y)\},$$

for any $x,y\in[a,b]$ and $\lambda\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [16]). For several results concerning inequalities for quasi-convex functions we refer the interested reader to [1]-[5], [16], [25, 26] and [28, 29].

Let us consider now a bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

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$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by S. S. Dragomir [11] as follows:

A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b], y\in[c,d]$.

Clearly, every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [11]).

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [11]:

Theorem: 1 *[11] Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx+\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (2)
\end{aligned}$$

The above inequalities are sharp.

In a recent paper [23], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

Definition: 1 *[23] A function $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to be quasi-convex on Δ if the inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le \max\{f(x,y),f(z,w)\},$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A function $f:\Delta\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are quasi-convex where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition of co-ordinated quasi-convex functions may be stated as:

Definition: 2 *A function $f:\Delta\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if*

$$f(tx+(1-t)z,sy+(1-s)w)\le \max\{f(x,y),f(x,w),f(z,y),f(z,w)\},$$

for all $(x,y),(z,w)\in\Delta$ and $s,t\in[0,1]$.

The class of co-ordinated quasi-convex functions on Δ is denoted by $QC(\Delta)$. It has been also proved in [23] that every quasi-convex functions on Δ is quasi-convex on the co-ordinates on Δ. We now give an example to show that there exists quasi-convex function on the co-ordinates which is not quasi-convex.

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Example: 1 *The function* $f:[-2,2]^2\to\mathbb{R}$, *defined by* $f(x,y)=\lfloor x\rfloor\lfloor y\rfloor$, *where* $\lfloor . \rfloor$ *is the floor function. This function is quasi-convex on the co-ordinates on* $[-2,2]^2$ *but is not quasi-convex on* $[0,1]^2$.

For example, take $(x,y)=(-2,1)$, $(z,w)=(1,-1)$ and $\lambda=\frac{1}{2}$, then

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)=f\left(-\frac{1}{2},0\right)=0,$$

on the other hand

$$\max\{f(x,y),f(z,w)\}=\max\{f(-2,1),f(1,-1)\}=-1,$$

which shows that

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)>\max\{f(x,y),f(z,w)\}.$$

For further results on several new classes of co-ordinated convex functions and related results we refer the interested reader to [6]-[9], [11], [15], [18]-[23] and [27]. Motivated by the results proved in [21, 27], the main purpose of the present paper is to establish some new inequalities for co-ordinated quasi-convex functions which are related to the rightmost terms of the Hermite-Hadamard type inequality (2) and to get refinements of the results for co-ordinated quasi-convex functions proved in [21].

2 MAIN RESULTS:

Throughout in this section, for convenience, we will use the notations:

$$L=\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|, M=\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|,\ N=\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|, O=\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|,$$

$$P=\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|,\ Q=\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|,\ R=\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|,$$

$$S=\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|$$

and

$$T=\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$

The following lemma is necessary and plays an important role in establishing our main results:

Lemma: 1 *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$, *then the following identity holds:*

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\\
=\frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\right.\\
+\int_0^1\int_0^1(-t)s\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\
+\int_0^1\int_0^1 t(-s)\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\
\left.+\int_0^1\int_0^1(-t)(-s)\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\right\},\end{aligned}\tag{3}$$

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where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx+\frac{1}{d-c}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy\right].$$

Proof: By integration by parts, we have

$$\begin{aligned}\int_0^1\int_0^1(-t)(-s)\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\ =\frac{4}{(b-a)(d-c)}f(a,c)-\frac{4}{(b-a)(d-c)}\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,c\right)dt\\ -\frac{4}{(b-a)(d-c)}f\left(a,\frac{1+s}{2}c+\frac{1-s}{2}d\right)ds\\ +\frac{4}{(b-a)(d-c)}\int_0^1\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt.\end{aligned}\tag{4}$$

Setting $x=\frac{1+t}{2}a+\frac{1-t}{2}b$ and $y=\frac{1+s}{2}c+\frac{1-s}{2}d$, we get from (4) the following inequality:

$$\begin{aligned}\int_0^1\int_0^1(-t)(-s)\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\ \le\frac{4}{(b-a)(d-c)}f(a,c)-\frac{8}{(b-a)^2(d-c)}\int_a^{\frac{a+b}{2}}f(x,c)dx\\ -\frac{8}{(b-a)(d-c)^2}\int_c^{\frac{c+d}{2}}f(a,y)dy+\frac{16}{(b-a)^2(d-c)^2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)dydx.\end{aligned}\tag{5}$$

In a similar way, we can have the following inequalities:

$$\begin{aligned}\int_0^1\int_0^1(-t)s\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\ \le\frac{4}{(b-a)(d-c)}f(a,d)-\frac{8}{(b-a)^2(d-c)}\int_a^{\frac{a+b}{2}}f(x,d)dx\\ -\frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^{d}f(a,y)dy+\frac{16}{(b-a)^2(d-c)^2}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}f(x,y)dydx,\end{aligned}\tag{6}$$

$$\begin{aligned}\int_0^1\int_0^1 t(-s)\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\ \le\frac{4}{(b-a)(d-c)}f(b,c)-\frac{8}{(b-a)^2(d-c)}\int_{\frac{a+b}{2}}^{b}f(x,c)dx\\ -\frac{8}{(b-a)(d-c)^2}\int_c^{\frac{c+d}{2}}f(b,y)dy+\frac{16}{(b-a)^2(d-c)^2}\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}f(x,y)dydx\end{aligned}\tag{7}$$

and

$$\begin{aligned}\int_0^1\int_0^1 ts\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\ \le\frac{4}{(b-a)(d-c)}f(b,d)-\frac{8}{(b-a)^2(d-c)}\int_{\frac{a+b}{2}}^{b}f(x,dd)dx\end{aligned}$$

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$$-\frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^{d} f(b,y)dy+\frac{16}{(b-a)^2(d-c)^2}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)dydx. \tag{8}$$

Substituting (5)-(8) in (4), simplifying and multiplying the resulting equality by $\frac{(b-a)(d-c)}{16}$, we get (3). Hence the proof of the Lemma is complete.

Now we begin with the following result:

Theorem: 2 *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is quasi-convex on the co-ordinates on* Δ, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\le\frac{(b-a)(d-c)}{64}\left[\sup\{O,Q,S,T\}+\sup\{N,Q,R,T\}+\sup\{L,P,R,T\}+\sup\{M,P,S,T\}\right], \tag{9}$$

where A is defined as in Lemma 1.

Proof: From Lemma 1, we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\le\frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|dsdt\right.$$
$$+\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|dsdt$$
$$+\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|dsdt$$
$$\left.+\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|dsdt\right] \tag{10}$$

By the quasi-convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ on $\Delta:=[a,b]\times[c,d]$, we observe that the following inequality holds:

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|dsdt$$
$$\le\int_0^1\int_0^1 st\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$

$$=\frac{1}{4}\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}. \tag{11}$$

Analogously, we also have that the following inequalities:

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|dsdt$$
$$\leq\frac{1}{4}\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}, \quad (12)$$

$$\int_0^1\int_0^1 st\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|dsdt$$
$$\leq\frac{1}{4}\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\} \quad (13)$$

and

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|dsdt$$
$$\leq\frac{1}{4}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}, \quad (14)$$

Substitution of (11)-(14) in (10) gives the desired inequality (9). This completes the proof.

Corollary: 1 *Suppose the conditions of the Theorem 2 are satisfied. Additionally, if*

1. $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is increasing on the co-ordinates on Δ, then

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|\leq\frac{(b-a)(d-c)}{64}[O+Q+S+T]. \quad (15)$$

2. $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is decreasing on the co-ordinates on Δ, then

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|\leq\frac{(b-a)(d-c)}{64}[L+R+P+T]. \quad (16)$$

Proof: It follows directly from Theorem 2.

Theorem: 3 *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is quasi-convex on the co-ordinates on* Δ *and* p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}\Big[\left(\sup\{L^q,P^q,R^q,T^q\}\right)^{\frac{1}{q}}+\left(\sup\{M^q,P^q,S^q,T^q\}\right)^{\frac{1}{q}}$$
$$+\left(\sup\{N^q,Q^q,R^q,T^q\}\right)^{\frac{1}{q}}+\left(\sup\{O^q,Q^q,S^q,T^q\}\right)^{\frac{1}{q}}\Big]. \quad (17)$$

where A is as given in Lemma 1.

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Proof: Suppose $p>1$. From Lemma 1 and well-known Hölder inequality for double integrals, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left(\int_0^1\int_0^1 t^p s^p dsdt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}\right]. \tag{18}$$

Now by the quasi-convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on Δ, we have that the following inequalities hold:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt$$
$$\leq\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{19}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt$$
$$\leq\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{20}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt$$
$$\leq\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\} \tag{21}$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt$$
$$\leq\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}. \tag{22}$$

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Also, we notice that

$$\int_0^1\int_0^1 t^p s^p dsdt = \frac{1}{(p+1)^2}. \tag{23}$$

Utilizing the inequalities (19)-(23) in (18), we get the required inequality (17), which completes the proof of the theorem.

Corollary: 2 *Suppose the conditions of the Theorem 3 are satisfied. Additionally, if*

1. $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is increasing on the co-ordinates on Δ, then

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|\le\frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}[O+Q+S+T]. \tag{24}$$

2. $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is decreasing on the co-ordinates on Δ, then

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|\le\frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}[L+R+P+T]. \tag{25}$$

Proof: It is a direct consequence of Theorem 3.

Our next result gives an improvement of the constant of the result given in Theorem 3.

Theorem: 4 *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is quasi-convex on the co-ordinates on* Δ *and* $q\ge 1$, *then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|\\ &\le\frac{(b-a)(d-c)}{64}\Bigg[\left(\sup\{L^q,P^q,R^q,T^q\}\right)^{\frac{1}{q}}+\left(\sup\{M^q,P^q,S^q,T^q\}\right)^{\frac{1}{q}}\\ &\quad+\left(\sup\{N^q,Q^q,R^q,T^q\}\right)^{\frac{1}{q}}+\left(\sup\{O^q,Q^q,S^q,T^q\}\right)^{\frac{1}{q}}\Bigg].\end{aligned} \tag{26}$$

where A is as given in Theorem 1.

Proof: Suppose $q\ge 1$. From using Lemma 1 and the power mean inequality, we have

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|\le\frac{(b-a)(d-c)}{16}\left(\int_0^1\int_0^1 tsdsdt\right)^{1-\frac{1}{q}}\\ &\times\Bigg[\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}\end{aligned}$$

$$+\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right)^{\frac{1}{q}}\right]. \tag{27}$$

Now by the quasi-convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on Δ, we have that the following inequalities hold:

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt$$
$$\le\frac{1}{4}\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{28}$$

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt$$
$$\le\frac{1}{4}\sup\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{29}$$

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt$$
$$\le\frac{1}{4}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\} \tag{30}$$

and

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt$$
$$\le\frac{1}{4}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}. \tag{31}$$

Also, we notice that

$$\int_0^1\int_0^1 tsdsdt=\frac{1}{4}.$$

Making use of the inequalities (28)-(31) in (27), we obtain the required inequality (26). This completes the proof.

Remark: 1 *Since* $2^p > p+1$ *if* $p>1$ *and accordingly*

$$\frac{1}{8}<\frac{1}{4(p+1)^{\frac{1}{p}}}$$

and hence we have that the following inequality:

$$\frac{1}{64}<\frac{1}{8}\cdot\frac{1}{8}<\frac{1}{4(p+1)^{\frac{1}{p}}}\cdot\frac{1}{4(p+1)^{\frac{1}{p}}}=\frac{1}{16(p+1)^{\frac{2}{p}}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Improvements of the inequalities of Corollary 2 are given in the following result:

Corollary: 3 *Suppose the conditions of the Theorem 4 are satisfied. Additionally, if*

1. $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is increasing on the co-ordinates on Δ, then (15) holds.
2. $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is decreasing on the co-ordinates on Δ, then (16) holds.

Proof: It follows from Theorem 4.

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