International Journal of Mathematical Archive-3(1), 2012, Page: 233-239
Available online through www.ijma.info ISSN 2229 – 5046

A NOTE ON UNIFORM MATRIX SUMMABILITY

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(Received on: 07-12-11; Accepted on: 23-12-11)

ABSTRACT

The purpose of the present paper is to establish a new result concerning uniform matrix summability of conjugate series of a Fourier series. Relevant connections of the results presented herewith various known results are briefly indicated.

Key Words: *Uniform triangular matrix summability, Conjugate series of Fourier series, Fourier coefficients, Nörlund summability.*

AMS 2010 Mathematics Subject Classification: *42A24, 40C05.*

1. INTRODUCTION AND PRELIMINARIES:

Let f be 2π periodic, Lebesgue integrable function with Fourier series given by

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.1}$$

The conjugate series of series (1.1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt) = -\sum_{n=1}^{\infty} B_n(t). \tag{1.2}$$

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying Silverman-Töeplitz [9] conditions of regularity i.e.

(i) $\sum_{k=0}^{n} a_{n,k} \to 1 \quad as \ n \to \infty$

(ii) $a_{n,k} = 0 \quad for \quad k > n$ and

(iii) $\sum_{k=0}^{n} |a_{n,k}| \leq M$ where M is finite constant.

Let $\sum_{n=0}^{\infty} u_n(x)$ be an infinite series defined in $[a,b] \subset [-\pi,\pi]$. The n^{th} partial sum of the series $\sum_{n=0}^{\infty} u_n(x)$ is given by $S_n(x) = \sum_{\nu=0}^{n} u_\nu(x) \ \forall \ x \in [a,b]$.

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If there exists a bounded function $S(x)$ such that

$$t_n(x) = \sum_{k=0}^{n} a_{n,k} \left\{ S_k(x) - S(x) \right\},$$
$$= o(1), as \ \ n \to \infty,$$

uniformly $\forall\, x \in [a,b]$ then we say that the series $\sum_{n=0}^{\infty} u_n(x)$ is summable (*T*) uniformly in $a \le x \le b$ to the sum $S(x)$.

Particular Cases:

Several authors such as ([1]-[4]), (see also [5]) studied the matrix summability method and obtained many interesting results.

The important particular cases of the triangular matrix means are:

(*i*) Cesàro mean of order 1 or (*C*, 1) mean if $a_{n,k} = \dfrac{1}{n+1} \quad \forall\, k$.

(*ii*) Harmonic means when $a_{n,k} = \dfrac{1}{(n-k+1)\ \log n}$.

(*iii*) (*C*, δ) means when $a_{n,k} = \dfrac{\binom{n-k+\delta-1}{\delta-1}}{\binom{n+\delta}{\delta}}$.

(*iv*) (*H, p*) means when $a_{n,k} = \dfrac{1}{(\log)^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^q (k+1)$.

(*v*) Nörlund means when $a_{n,k} = \dfrac{p_{n-k}}{P_n}$, where $P_n = \sum_{k=0}^{\infty} p_k$, $P_n \ne 0$.

(*vi*) Riesz means $(\overline{N}, p_n)$ when $a_{n,k} = \dfrac{p_k}{P_n}$, $P_n \ne 0$.

(*vii*) Generalised Nörlund Means (*N, p, q*) when $a_{n,k} = \dfrac{p_{n-k} q_k}{R_n}$. where $R_n = \sum_{k=0}^{\infty} p_k q_{n-k}$, $R_n \ne 0$.

We denote $\overline{S_n(x)}$, the n^{th} partial sum of the series (1.2).

Let $$\overline{f_n} = \overline{f_n(x)} = -\frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \psi(t) \cot \frac{t}{2} dt, \tag{1.3}$$

$$\overline{f(x)} = \lim_{n \to \infty} \overline{f_n(x)}, \tag{1.4}$$

$$\psi(t) = f(x+t) - f(x-t), \tag{1.5}$$

$$\Psi(t) = \int_0^t |\psi(u)| du, \tag{1.6}$$

$$A_{n,\tau} = \sum_{k=o}^{\tau} a_{n,n-k} = \sum_{k=n-\tau}^{n} a_{n,k}, \tag{1.7}$$

where $$\tau = \left[\frac{1}{t} \right] = \text{integral part of } \frac{1}{t}, \tag{1.8}$$

and $$\overline{K_n(t)} = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}, \tag{1.9}$$

Saxena [6] discussed the uniform harmonic summability of conjugate series of a Fourier series in the following form:

Theorem: 1.1 If $\Psi(t) = \int_0^t |\psi(u)|du = o\left(\frac{t}{\log\frac{1}{t}}\right)$, uniformly in a set *E*, as $t \to +0$, then the series (1.2) is summable by harmonic means uniformly in *E* to the sum $\overline{f(x)}$, provided the limit (1.4) exists uniformly in *E*.

Tripathi and Singh [8] extended the above result to the case of uniform Nörlund summability in the following form:

Theorem: 1.2 If the sequence $\{q_n\}$ is real, non-negative and monotonic non-increasing sequence of coefficients such that $Q_n \to \infty$ as $n \to \infty$ and the function $\lambda(t)$, $\beta(t)$ and $\frac{t\lambda(t)}{\beta(t)}$ increase monotonically with t and $\lambda(n)Q_n = O[\beta(Q_n)]$ as $n \to \infty$, then if

$$\Psi(t) = \int_0^t |\psi(u)|du = o\left(\frac{\lambda(\frac{1}{t})q_\tau}{\beta(Q_\tau)}\right),$$

uniformly in a set *E*, as $t \to +0$, then the series (1.2) is summable (N, q_n) uniformly in *E* to the sum $\overline{f(x)}$, at the point $t = x$, provided the limit (1.4) exists uniformly in *E*.

2 MAIN THEOREM:

The purpose of this paper is to generalize the result of Saxena [6] and Tripathi and Singh [8] for uniform matrix summability method. In fact, we prove the following interesting result.

Theorem: 2.1 Let $T = (a_{n,k})$ be an infinite triangular matrix such that the elements $(a_{n,k})$ are non-negative and non-decreasing with $k \le n$ and if

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left(\frac{t\beta(1/t)}{\log(1/t)}\right), \tag{2.1}$$

as $t \to +0$, uniformly in a set $E = [a,b]$, where $\beta(t)$ is a positive function of *t* such that $\frac{\beta(n)}{\log n} \to 0$ as $n \to \infty$, then the conjugate series of a Fourier series (1.2) is summable (*T*) uniformly in *E* to $\overline{f(x)} = -\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{t}{2}dt$ provided the limit (1.4) exist uniformly in $E = [a,b]$.

To prove our main theorem we require the following lemmas.

3. LEMMAS:

Lemma: 3.1 If $(a_{n,k})$ is a non-negative and non-decreasing with $k \le n$, then

$$\left|\sum_{k=0}^{n} a_{n,k}\cos(k+\tfrac{1}{2})t\right| = O(A_{n,\tau}) \text{ for } 0 < \frac{1}{n} \le t < \delta < \pi.$$

Proof: $$\left|\sum_{k=0}^{n} a_{n,k}\cos(k+\tfrac{1}{2})t\right| \le \left|\sum_{k=0}^{n-\tau} a_{n,k}\cos(k+\tfrac{1}{2})t\right| + \left|\sum_{k=n-\tau}^{n} a_{n,k}\cos(k+\tfrac{1}{2})t\right|$$

$$\le 2a_{n,n-\tau} \max_{0\le k\le r\le n-\tau}\left|\sum_{k=0}^{r}\cos(k+\tfrac{1}{2})t\right| + \sum_{k=n-\tau}^{n} a_{n,k}\left|\cos(k+\tfrac{1}{2})t\right|,$$

(by Abel's Lemma)

$$\le 2a_{n,n-\tau}\left|\frac{\sin(r+1)\frac{t}{2}}{\sin\frac{t}{2}}\right| + A_{n,\tau}$$

$$\left|\sum_{k=0}^{n} a_{n,k}\cos(k+\tfrac{1}{2})t\right| \le \frac{2a_{n,n-\tau}}{t} + A_{n,\tau}. \tag{3.1}$$

Now
$$A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k} = \sum_{k=n-\tau}^{n} a_{n,k}$$
$$= a_{n,n-\tau} + a_{n,n-\tau+1} + \ldots.. + a_{n,n}$$
$$\ge (\tau+1)a_{n,n-\tau}$$
$$\ge \frac{a_{n,n-\tau}}{t} \quad (\text{ since } \tau = \lfloor \tfrac{1}{t} \rfloor).$$

Therefore
$$\frac{a_{n,n-\tau}}{t} = O(A_{n,\tau}). \tag{3.2}$$

By (3.1) and (3.2), we have $\left|\sum_{k=0}^{n} a_{n,k}\cos(k+\tfrac{1}{2})t\right| = O(A_{n,\tau})$.

Lemma: 3.2 If $(a_{n,k})$ is non-negative and non-decreasing with $k \le n$ and $\overline{K_n(t)}$ is given by (1.9) then $\overline{K_n(t)} = O\left(\frac{A_{n,\tau}}{t}\right)$ for $0 < \frac{1}{n} \le t < \delta < \pi$.

Proof: Since for $0 < \frac{1}{n} \le t < \delta < \pi$, $\sin t \ge \frac{t}{\pi}$,

We have
$$\left|\overline{K_n(t)}\right| = \frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,k}\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right|$$
$$\le \frac{1}{2\pi\sin\frac{t}{2}}\left[\left|\sum_{k=0}^{n} a_{n,k}\cos(k+\tfrac{1}{2})t\right|\right]$$
$$\le \frac{1}{2\pi}\cdot\frac{2\pi}{t}\left[O(A_{n,\tau})\right] \quad \text{from Lemma 3.1}$$
$$\left|\overline{K_n(t)}\right| = O\left(\frac{A_{n,\tau}}{t}\right).$$

Hence the lemma is proved.

4. PROOF OF THE MAIN THEOREM:

It is well known that the integral formula for the k^{th} partial sum of conjugate series of a Fourier series (1.2) is given by, (see [7]):

$$\overline{S_k(x)} = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(k+\frac{1}{2})t - \cos\frac{t}{2}}{\sin\frac{t}{2}}dt$$

$$\overline{S_k(x)} - \overline{f(x)} = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\,\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

Now $\sum_{k=0}^{n} a_{n,k}\left\{\overline{S_k(x)} - \overline{f(x)}\right\} = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\sum_{k=0}^{n} a_{n,k}\,\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}dt$

$$= \int_0^{\pi} \psi(t).\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\,\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

$$= \int_0^{\pi} \psi(t)\overline{K_n(t)}dt \tag{4.1}$$

So in order to prove our main theorem, we have to show that

$$\int_0^{\pi} \psi(t)\overline{K_n(t)}dt = o(1)\text{, uniformly in E.} \tag{4.2}$$

We set

$$\int_0^{\pi} \psi(t)\overline{K_n(t)}dt = \int_0^{\frac{1}{n}} \psi(t)\overline{K_n(t)}dt + \int_{\frac{1}{n}}^{\delta} \psi(t)\overline{K_n(t)}dt + \int_{\delta}^{\pi} \psi(t)\overline{K_n(t)}dt$$

$$= I_1 + I_2 + I_3\text{, say} \tag{4.3}$$

Since limit (1.4) exists uniformly in *E* so

$$\frac{1}{2\pi}\int_0^{\frac{1}{n}} \psi(t)\overline{K_n(t)}dt = o(1)\text{, uniformly in } E. \tag{4.4}$$

Also, for $0 < t < \dfrac{1}{n}$,

$$\left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\,\frac{\cos(k+\frac{1}{2})t - \cos\frac{t}{2}}{\sin\frac{t}{2}}\right| = \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\,\frac{2\sin(k+1)\frac{t}{2}\sin k\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{\pi}\sum_{k=0}^{n} a_{n,k}\,\frac{(k+1)\left|\sin\frac{t}{2}\right|.k\left|\sin\frac{t}{2}\right|}{\left|\sin\frac{t}{2}\right|} \quad \text{(Since } |\sin kt| \le k|\sin t|$$

for $0 < t < \frac{1}{n}$)

$$= O\left(\sum_{k=0}^{n} a_{n,k}.k(k+1)t\right)$$

$$= O\left(n^2 t\sum_{k=0}^{n} a_{n,k}\right)$$

$$= O(n^2 t)\text{,} \tag{4.5}$$

Hence

$$I_1 = \int_0^{\frac{1}{n}} \psi(t)\overline{K_n(t)}dt$$

$$= \int_0^{\frac{1}{n}} \psi(t)\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\left[\frac{\cos(k+\frac{1}{2})t - \cos\frac{t}{2}}{\sin\frac{t}{2}}\right]dt + \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\int_0^{\frac{1}{n}} \psi(t)\cot\tfrac{t}{2}\,dt$$

$$|I_1| \le \int_0^{\frac{1}{n}} |\psi(t)|.O(n^2 t)dt + o(1)$$, uniformly in *E*, (using (4.4) and (4.5))

$$\le O(n).\int_0^{\frac{1}{n}} |\psi(t)|dt + o(1)$$, uniformly in *E*,

$$\le O(n).o\left(\frac{\beta(n)}{n\log n}\right) + o(1)$$, uniformly in *E*, (by condition (2.1))

$$\le o\left(\frac{\beta(n)}{\log n}\right) + o(1)$$, uniformly in *E*,

$= o(1)$, as $n \to \infty$, uniformly in *E* (by hypothesis of theorem) (4.6)

Now $I_2 = \int_{\frac{1}{n}}^{\delta} \psi(t)\overline{K_n(t)}dt$

$$|I_2| = O(1).\int_{\frac{1}{n}}^{\delta}\left(\frac{A_{n,\tau}}{t}\right).|\psi(t)|dt$$ (by using Lemma 3.2)

$$= O(1).\left[\left(\frac{A_{n,\tau}}{t}.\Psi(t)\right)_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta}\frac{d}{dt}\left(\frac{A_{n,\tau}}{t}\right).\Psi(t)dt\right]$$

$$= o(1).\left[\left(\frac{A_{n,\tau}}{t}.\frac{t\beta(1/t)}{\log(1/t)}\right)_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta}\left(\frac{A_{n,\tau}}{t^2}\right).\frac{t\beta(1/t)}{\log(1/t)}dt + \int_{\frac{1}{n}}^{\delta}\frac{1}{t}.\frac{t\beta(1/t)}{\log(1/t)}d(A_{n,\tau})\right]$$

$$= o\left(\frac{A_{n,[1/\delta]}\beta\left[\frac{1}{\delta}\right]}{\log\left[\frac{1}{\delta}\right]}\right) + o\left(\frac{A_{n,n}\beta(n)}{\log n}\right) + o\left(\int_{\frac{1}{\delta}}^{n}\frac{A_{n,u}\beta(u)}{u\log u}du\right) + o\left(\int_{\frac{1}{\delta}}^{n}\frac{\beta(u)}{\log u}d(A_{n,u})\right)$$

$$= o(1) + o(1) + o\left(\frac{A_{n,n}\beta(n)}{\log n}\right) + o\left(\frac{\beta\left[\frac{1}{\delta}\right]}{\log\left[\frac{1}{\delta}\right]}.\sum_{k=\left[\frac{1}{\delta}\right]+1}^{n} a_{n,k}\right) + o\left(\frac{\beta(n)}{\log n}.\sum_{k=\left[\frac{1}{\delta}\right]+1}^{n} a_{n,k}\right)$$

(Applying the Mean Value Theorem for integrals)

$$= o(1) + o(1) + O(1).o\left(\frac{\beta(n)}{\log n}\right) + o(1) + o(1)$$

$= o(1)$, as $n \to \infty$, uniformly in *E*. (4.7)

Also by the virtue of Riemann Lebesgue theorem and regularity of method of summation, we have $I_3 = o(1)$, as $n \to \infty$, (4.8)

Thus from (4.1), (4.3), (4.6), (4.7) and (4.8), we have

$$\sum_{k=0}^{n} a_{n,k} \left\{ \overline{S_k(x)} - \overline{f(x)} \right\} = o(1), \text{ as } n \to \infty, \text{ uniformly in } E.$$

This completes the proof of the theorem.

Remark: 1 If we put $a_{n,k} = \dfrac{1}{(n-k+1)\log n}$, $\beta(t) = 1\ \forall\, t$, $[a,b] = E$ then we obtain the corresponding result of Saxena [6].

Remark: 2 The result of Tripathi and Singh [8] is a particular case of our theorem if $a_{n,k} = \dfrac{q_{n-k}}{Q_n}$, $Q_n = \sum_{k=0}^{n} q_k$, $\beta(t) = \dfrac{t\lambda(t) q_t \log t}{\alpha(Q_t)}$ and $[a,b] = E$.

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