International Journal of Mathematical Archive-3(1), 2012, Page: 318-323
IJMA Available online through www.ijma.info ISSN 2229 – 5046

$\mathscr{LF}$- N-NORMED LINEAR SPACE

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(Received on: 02-01-12; Accepted on: 20-01-12)

ABSTRACT

This paper introduces the notion of Cauchy sequence, convergent sequence and completeness in $\mathscr{LF}$n-normed linear space.

Key Words: *$\mathscr{L}$- fuzzy normed space, $\mathscr{LF}$n-normed linear space, Continuous t- norm.*

AMS Subject Classification No: *03E72, 46S40, 54A40.*

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1. INTRODUCTION:

Gahler [4] introduced the theory of n-norm on a linear space. For a systematic development of n-normed linear space one may refer to [5, 6, 8, 9]. In [5], Hendra Gunawan and Mashadi have also discussed the Cauchy sequence and convergent sequence in n-normed linear space. A detailed theory of fuzzy normed linear space can be found in [1, 2, 3, 7, 11]. In [10, 13], Vijayabalaji extended n-normed linear space to fuzzy n-normed linear space and complete fuzzy n-normed linear space.

Our object in this paper is to introduce the notion of Cauchy sequence and convergent sequence in $\mathscr{L}$-fuzzy n-normed linear space and to study the completeness of the $\mathscr{L}$-fuzzy n-normed linear space.

2. PRELIMINARIES:

Definition: 2.1 Let $n \in N$ (natural numbers) and X be a real linear space of dimension $d \geq n$. (Here we allow d to be infinite). A real valued function $\|\bullet \ldots \bullet\|$ on $X \times X \times \ldots \times X$ (n times) $= X^n$ satisfying the following four properties:

(1) $\| x_1, x_2,\ldots,x_n \| = 0$ if any only if $x_1, x_2,\ldots,x_n$ are linearly dependent .

(2) $\| x_1, x_2,\ldots,x_n \|$ is invariant under any permutation of $x_1, x_2,\ldots,x_n$.

(3) $\| x_1, x_2,\ldots,cx_n \| = |c| \, \| x_1, x_2,\ldots,x_n \|$, for any real c .

(4) $\| x_1, x_2,\ldots,x_{n-1}, y + z \| \leq \| x_1, x_2,\ldots,x_{n-1}, y \| + \| x_1, x_2,\ldots,x_{n-1}, z \|$.

is called an n-norm on X and the pair (X, $\|\bullet,\ldots,\bullet\|$) is called an n-normed linear space.

Definition: 2.2 A sequence $\{x_n\}$ in an n-normed linear space (X, $\|\bullet,\ldots, \bullet\|$) is said to converge to an $x \in X$ (in the n-norm) whenever $\lim\limits_{n\to\infty} \| x_1, x_2, \ldots x_{n-1}, x_n - x \| = 0.$

Definition: 2.3 A sequence $\{x_n\}$ in an n-normed linear space (X, $\|\bullet,\ldots,\bullet\|$) is called a Cauchy sequence if sequence if $\lim\limits_{n,k\to\infty} \| x_1, x_2, \ldots x_{n-1}, x_n - x_k \| = 0.$

Definition: 2.4 An n-normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition: 2.5 Let $\mathscr{L} = (L, \leq_L)$ be a complete lattice and U a non – empty set called universe. An $\mathcal{L}$-fuzzy set in U is defined as a U→L mapping. For each u in U, A(u) represents the degree (in L) to which u satisfies A.

Lemma: 2.1 Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition: 2.6 An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{ x, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u) : u \in U \}$, where for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [1,0]$ and $\eta_{\mathcal{A}}(u) \in [1,0]$ are called the membership degree and the non-membership degree, respectively, u in set $\mathcal{A}_{\zeta,\eta}$, and furthermore satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Classically, a triangular norm T on ([0,1], $\leq$) is defined as an increasing, commutative, associative mapping $T:[0,1]^2 \to [0,1]$ satisfying T(1,x)=x, for all x∈ [0,1].These definitions can be straight forwardly extended to any lattice $\mathscr{L} = (L, \leq_L)$

Definition: 2.7 A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathscr{T}: L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(\mathscr{T}(x, 1_{\mathscr{L}}) = x)$ (boundary condition);

(ii) $(\forall (x, y) \in L^2)(\mathscr{T}(x, y) = \mathscr{T}(y, x))$ (commutative);

(iii) $(\forall (x, y, z) \in L^3)(\mathscr{T}(x, \mathscr{T}(y, z)) = \mathscr{T}(\mathscr{T}(x, y), z))$ (associativity);

(iv) $(\forall (x, x', y, y') \in L^4(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathscr{T}(x, y) \leq_L \mathscr{T}(x', y'))$ (monotonicity)

There are recursively defined by $\mathscr{T} \doteq \mathscr{T}$ and $\mathscr{T}(x_{(1)},\ldots,x_{(n+1)}) = \mathscr{T}(\mathscr{T}^{n-1}(x_{(1)},\ldots,x_{(n+1)})$ for n≥2 and $x_{(i)} \in L$.

Definition: 2.8 A continuous t- norm $\mathscr{T}$ on $\mathscr{L}^*$ is called continuous t- representable if and only if there exist a continuous t-norm * and a continuous t-conorm ◊ on [0, 1] such that,

for all $x = ((x_1, x_2, \ldots, x_n), (x_1', x_2', \ldots x_n'))$, $y = ((y_1, y_2, \ldots, y_n), (y_1', y_2', \ldots y_n')) \in \mathscr{L}^*$

$\mathscr{T}(x, y) = ((x_1 * y_1, x_2 * y_2, \ldots, x_n * y_n), (x_1' \lozenge y_1', x_2' \lozenge y_2', \ldots x_n' \lozenge y_n'))$.

Definition: 2.9 A negator on $\mathcal{L}$ is any decreasing mapping $\mathscr{N}: L \to L$ satisfying $\mathscr{N}(0_{\mathscr{L}}) = 1_{\mathscr{L}}$ and $\mathscr{N}(1_{\mathscr{L}}) = 0_{\mathcal{L}}$. If $\mathscr{N}(\mathscr{N}((x_1, x_2, \ldots, x_n))) = (x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathscr{L}$, then $\mathscr{N}$ is called an involutive negator. The negator $\mathcal{N}_s$ on ([0, 1], $\leq$) defined as, for all $x_1, x_2, \ldots, x_n \in [0, 1]$, $\mathscr{N}_s(x) = 1 - (x_1, x_2, \ldots, x_n)$ is called the standard negator on ([0, 1], $\leq$).

3. Complete $\mathscr{L}$– fuzzy n – normed linear space:

Definition: 3.1 Let X be a linear space over a field F, $\mathscr{T}$ is a continuous t-norm on $\mathscr{L}$ and $\mathscr{P}$ is an $\mathscr{L}$- fuzzy set on $X^n \times [0, \infty)$ is called a $\mathscr{L}$- fuzzy n-norm on X if and only if :

(N_1) $\mathscr{P}(x_1, x_2, \ldots, x_n, t) >_L 0_{\mathscr{L}}$.

(N_2) $\mathscr{P}(x_1, x_2, \ldots, x_n, t) = 1_{\mathscr{L}}$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.

(N_3) $\mathscr{P}(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

(N_4) $\mathscr{P}(x_1, x_2, \ldots, c\, x_n, t) = \mathscr{P}(x_1, x_2, \ldots, x_n, t/|c|)$ if $c \neq 0$, c∈ F(field).

(N_5) $\mathscr{P}((x_1,x_2,..., x_n) + (y_1,y_2,..., y_n), s + t) \geq_{\mathscr{L}} \mathcal{F}(\mathscr{P}(x_1,x_2,..., x_n, s), \mathscr{P}(y_1,y_2,..., y_n, t))$.

(N_6) $\mathscr{P}(x_1, x_2,...,x_n, t)$ is left continuous and non-decreasing function such that $\lim_{t\to\infty} \mathscr{P}(x_1,x_2,...,x_n, t) = 1_{\mathscr{L}}$.

Then $(X, \mathscr{P}, \mathscr{T})$ is called a $\mathscr{L}$- fuzzy n-normed linear space or in short $\mathscr{L}\mathscr{T}$n-NLS.

To strengthen the above definition, we present the following example.

Example: 3.1 Let $(X, \|\bullet,...,\bullet\|)$ be an n-normed linear space.

Define $\mathscr{T}(a, b) = \min\{a, b\}$ and $\mathscr{P}(x_1, x_2,...,x_n, t) = t / (t + \| x_1, x_2,...,x_n \|)$. Then is a $(X, \mathscr{P}, \mathscr{T})$ is a $\mathscr{L}\mathscr{T}$-n-NLS.

Proof:

(N_1) Clearly $\mathscr{P}(x_1, x_2,...,x_n, t) > 0$.

(N_2) $\mathscr{P}(x_1, x_2,...,x_n, t) = 1_{\mathscr{L}}$

$\Leftrightarrow$ $t / (t + \| x_1, x_2,...,x_n \|) = 1_{\mathscr{L}}$

$\Leftrightarrow$ $\| x_1, x_2,...,x_n \| = 0_{\mathscr{L}}$

$\Leftrightarrow$ $x_1, x_2,...,x_n$ are linearly dependent.

(N_3) $\mathscr{P}(x_1, x_2,...,x_n, t)$

$= t / (t + \| x_1, x_2,...,x_n \|)$

$= t / (t + \| x_1, x_2,...,x_n, x_{n-1} \|)$

$= \mathscr{P}(x_1,x_2,...,x_n, x_{n-1}, t)$.

It follows similarly for the rest.

(N_5) Without loss of generality assume that $\mathscr{P}(y_1,y_2,..., y_n, t) \leq \mathscr{P}(x_1,x_2,..., x_n, s)$. Then

$t/ (t + \| y_1,y_2,..., y_n \|) \leq s / (s + \| x_1,x_2,..., x_n \|)$

$\Rightarrow$ $t (s + \| x_1,x_2,..., x_n \|) \leq s (t + \| y_1,y_2,..., y_n \|)$

$\Rightarrow$ $t \| x_1,x_2,..., x_n \| \leq s \| y_1,y_2,..., y_n \|$

$\Rightarrow$ $\| x_1,x_2,..., x_n \| \leq (s / t) \| y_1,y_2,..., y_n \|$.

Therefore,

$$\begin{aligned} \| x_1,x_2,..., x_n \| + \| y_1,y_2,..., y_n \| &\leq (s / t) \| y_1,y_2,..., y_n \| + \| y_1,y_2,..., y_n \| \\ &\leq ((s / t) + 1) \| y_1,y_2,..., y_n \| \\ &= ((s + t) / t) \| y_1,y_2,..., y_n \|. \end{aligned}$$

But, $$\begin{aligned} \| (x_1,x_2,..., x_n) + (y_1,y_2,..., y_n) \| &\leq \| x_1,x_2,..., x_n \| + \| y_1,y_2,..., y_n \|. \\ &\leq ((s + t) / t) \| y_1,y_2,..., y_n \|. \end{aligned}$$

$\Rightarrow$ $\| (x_1,x_2,..., x_n) + (y_1,y_2,..., y_n) \| / (s+t) \leq \| y_1,y_2,..., y_n \| / t$

$\Rightarrow$ $1 + (\| (x_1,x_2,..., x_n) + (y_1,y_2,..., y_n) \| / (s+t)) \leq 1 + (\| y_1,y_2,..., y_n \|) / t$

$\Rightarrow$ $((s + t) + \| (x_1,x_2,..., x_n) + (y_1,y_2,..., y_n) \|) / (s+t) \leq (t + \| y_1,y_2,..., y_n \|)/ t$

$\Rightarrow \quad (s+t)/(s+t) + \| (x_1,x_2,..., x_n) + (y_1,y_2,..., y_n) \| \geq t / (t + \| y_1,y_2,..., y_n \|)$

$\Rightarrow \quad \mathscr{P}((x_1,x_2,..., x_n) + (y_1,y_2,..., y_n), s + t) \geq \mathscr{T}\{ \mathscr{P}(x_1,x_2,..., x_n, s), \mathscr{P}(y_1,y_2,..., y_n, t)\}.$

(N_6) Clearly $\mathscr{P}(x_1, x_2,...,x_n, t)$ is a left continuous function.

Suppose that $t_2 > t_1 > 0$ with $t_1, t_2 \in [0, \infty)$ then,

$t_2 / (t_2 + \| x_1,x_2,..., x_n\|) - t_1/ (t_1 + \| x_1,x_2,..., x_n \|) = \| x_1,x_2,..., x_n\| (t_2 - t_1) / ((t_2 + \| x_1,x_2,..., x_n\|) ((t_1 + \| x_1,x_2,..., x_n \|)) \geq 0$

for all $(x_1,x_2,..., x_n) \in X^n$.

$\Rightarrow \quad t_2 / (t_2 + \| x_1,x_2,..., x_n \|) \geq t_1 / (t_1 + \| x_1,x_2,..., x_n \|)$

$\Rightarrow \quad \mathscr{P}(x_1, x_2,...,x_n, t_2) \geq \mathscr{P}(x_1, x_2,...,x_n, t_1).$

Thus $\mathscr{P}(x_1, x_2,...,x_n, t)$ is a non-decreasing function of $t \in [0, \infty)$.

Also,

$$\lim_{t\to\infty} \mathscr{P}(x_1,x_2,...,x_n, t) = \lim_{t\to\infty} t / (t + \| x_1,x_2,..., x_n \|)$$
$$= \lim_{t\to\infty} t / t (1 + (1/t) \| x_1,x_2,..., x_n \|)$$
$$= 1_{\mathscr{L}}$$

Thus $(X, \mathscr{P}, \mathscr{T})$ is a $\mathscr{LF}$-n-NLS.

Definition: 3.2 A sequence $\{x_n\}$ in a $\mathscr{LF}$-n-NLS $(X, \mathscr{P}, \mathscr{T})$ is said to converge to x if given $\varepsilon, t > 0$, $\varepsilon \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$, there exists an integer $n_0 \in N$ such that $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) >_L \mathscr{N}(\varepsilon)$ for all $n \geq n_0$.

Theorem: 3.1 In a $\mathscr{LF}$-n-NLS $(X, \mathscr{P}, F)$ is a sequence $\{x_n\}$ converges to x if and only if $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) \to 1_{\mathscr{L}}$ as $n \to \infty$.

Proof: Fix $t > 0$. Suppose $\{x_n\}$ converges to x. Then for a given ε, $\varepsilon \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$, there exists an integer $n_0 \in N$ such that $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) >_L \mathscr{N}(\varepsilon)$.

Thus $1 - \mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) < \varepsilon$ and hence $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) \to 1_{\mathscr{L}}$ as $n \to \infty$.

Conversely, if for each $t > 0$, $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) \to 1_{\mathscr{L}}$ as $n \to \infty$, then for every ε, $\varepsilon \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$, there exists an integer n_0 such that $1 - \mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) <_L \varepsilon$ for all $n \geq n_0$. Thus $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t) >_L 1 - \varepsilon$ for all $n \geq n_0$.

Hence $\{x_n\}$ converges to x in $(X, \mathscr{P}, \mathscr{T})$

Definition: 3.3 A sequence $\{x_n\}_{n\in N}$ in a $\mathscr{LF}$-n-NLS $(X, \mathscr{P}, \mathscr{T})$ is said to be Cauchy sequence if given $\varepsilon \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$, $t > 0$, there exists an integer $n_0 \in N$ such that $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x_m, t) >_L \mathscr{N}(\varepsilon)$ for all $n, m \geq n_0$.

Theorem: 3.2 In a $\mathscr{LF}$-n-NLS $(X, \mathscr{P}, \mathscr{P})$ every convergent sequence is a Cauchy sequence.

Proof: Let $\{x_n\}$ be a convergent sequence in $(X, \mathscr{P}, \mathscr{T})$. Suppose $\{x_n\}$ converges to x.

Let $t > 0$ and $\varepsilon \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$. Choose $r \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that $\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) > \mathscr{N}(\varepsilon)$.

Since $\{x_n\}$ converges to x, we have an integer n_0 such that $\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x, t/2) >_L \mathscr{N}(r)$

Now,

$$\mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x_m, t) = \mathscr{P}(x_1,x_2,..., x_{n-1}, x_n - x + x - x_m, t)$$
$$= \mathscr{T}(\mathscr{P}(x_1, x_2... x_{n-1}, x_n - x, t/2), \mathscr{P}(x_1,x_2,..., x_{n-1}, x - x_k, t/2))$$
$$\geq \mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \text{ for all } n, m \geq n_0$$

$> \mathcal{N}(\varepsilon)$ for all n, m $\geq n_0$

Therefore $\{x_n\}$ is a Cauchy sequence in (X , $\mathcal{P}$,$\mathcal{F}$).

Definition: 3.4 A $\mathcal{L}\mathcal{F}$-n-NLS is said to be complete if every Cauchy sequence in it is convergent.

The following example shows that there may exist Cauchy sequence in a $\mathcal{L}\mathcal{F}$-n-NLS which is not convergent.

Example: 3.2 Let (X, $\|\bullet,\ldots,\bullet\|$) be an n-normed linear space and define $\mathcal{T}$(a , b) = min {a, b} for all a, b ∈ [0,1] and $\mathcal{P}(x_1,x_2,\ldots,x_n, t) = t \,/\, (t + \| x_1,x_2,\ldots, x_n \|)$. Then (X, $\mathcal{P}$, $\mathcal{F}$) is shown to be a $\mathcal{L}\mathcal{F}$-n-NLS.

Let $\{x_n\}$ be a sequence in $\mathcal{L}\mathcal{F}$-n-NLS, then

(a) $\{x_n\}$is a Cauchy sequence in (X, $\|\bullet,\ldots,\bullet\|$) if and only if $\{x_n\}$ is a Cauchy sequence in (X , $\mathcal{P}$,$\mathcal{F}$).
(b) $\{x_n\}$ is a convergent sequence in (X, $\|\bullet,\ldots,\bullet\|$) if and only if $\{x_n\}$ is a convergent sequence in (X , $\mathcal{P}$,$\mathcal{F}$).

Proof:

(a) $\{x_n\}$ is a Cauchy sequence in (X, $\|\bullet,\ldots,\bullet\|$)

$$\Leftrightarrow \lim_{n,m\to\infty} \| x_1,x_2,\ldots,x_{n-1} ,x_n - x_m \| = 0.$$

$$\Leftrightarrow \lim_{n,m\to\infty} \mathcal{P}(x_1,x_2,\ldots, x_{n-1}, x_n - x_m , t)$$

$$\Leftrightarrow \lim_{n,m\to\infty} t \,/\, (t + \| x_1,x_2,\ldots, x_n - x_m \|) = 1_{\mathcal{L}}$$

$$\Leftrightarrow \mathcal{P}(x_1,x_2,\ldots, x_{n-1}, x_n - x_m , t) \to 1_{\mathcal{L}} \quad \text{as} \quad n \to \infty.$$

$$\Leftrightarrow \mathcal{P}(x_1,x_2,\ldots, x_{n-1}, x_n - x_m , t) > \mathcal{N}(\varepsilon), \text{ for all } n ,m \geq n_0.$$

$\Leftrightarrow$ $\{x_n\}$ is a Cauchy sequence in (X , $\mathcal{P}$,$\mathcal{F}$).

(b) $\{x_n\}$ is a convergent sequence in (X, $\|\bullet,\ldots,\bullet\|$)

$$\Leftrightarrow \lim_{n\to\infty} \| x_1, x_2,\ldots,x_{n-1} ,x_n - x \| = 0.$$

$$\Leftrightarrow \lim_{n\to\infty} \mathcal{P}(x_1, x_2\ldots x_{n-1}, x_n - x_m , t)$$

$$\Leftrightarrow \lim_{n\to\infty} t \,/\, (t + \| x_1,x_2,\ldots x_{n-1}, x_n - x_m \|) = 1_{\mathcal{L}}$$

$$\Leftrightarrow \mathcal{P}(x_1,x_2,\ldots, x_{n-1}, x_n - x , t) \to 1_{\mathcal{L}} \quad \text{as} \quad n \to \infty.$$

$$\Leftrightarrow \mathcal{P}(x_1,x_2,\ldots, x_{n-1}, x_n - x , t) > \mathcal{N}(\varepsilon), \text{ for all } n \geq n_0.$$

$\Leftrightarrow$ $\{x_n\}$ is a convergent sequence in (X , $\mathcal{P}$,$\mathcal{F}$).

Thus if there exists an n-normed linear space (X, $\|\bullet,\ldots,\bullet\|$) which is not complete, then the $\mathcal{L}$ - fuzzy n-norm induced by such a crisp n-norm $\|\bullet,\ldots,\bullet\|$ on an incomplete n-normed linear space X is an incomplete $\mathcal{L}$ - fuzzy n-normed linear space.

Theorem: 3.3 A $\mathcal{L}\mathcal{F}$-n-NLS (X , $\mathcal{P}$,$\mathcal{F}$) in which every Cauchy sequence has a convergent subsequence is complete.

Proof: Let $\{x_n\}$ be a Cauchy sequence in $(X, \mathscr{P}, \mathscr{T})$ and $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $\{x_n\}$ converges to x. Let $t > 0$ and $\varepsilon \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$. Choose $r \in L/\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that $\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) > \mathscr{N}(\varepsilon)$.

Since $\{x_n\}$ is a Cauchy sequence, there exists an integer $n_0 \in N$ such that $\mathscr{P}(x_1, x_2, \ldots, x_{n-1}, x_n - x_m, t/2) >_L \mathscr{N}(r)$ for all $n, m \geq n_0$.

Since $\{x_{n_m}\}$ converges to x, there is a positive integer $i_m > n_0$ such that

$$\mathscr{P}(x_1, x_2, \ldots, x_{n-1}, x_{i_m} - x, t/2) >_L \mathscr{N}(r).$$

Now,

$$\begin{aligned} \mathscr{P}(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) &= \mathscr{P}(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_m} + x_{i_m} - x + x - x_m, t/2 + t/2) \\ &\geq_L \mathscr{T}(\mathscr{P}(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_m}, t/2), \mathscr{P}(x_1, x_2, \ldots, x_{n-1}, x_{i_m} - x, t/2)) \\ &>_L \mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \\ &>_L \mathscr{N}(\varepsilon) \end{aligned}$$

Therefore $\{x_n\}$ converges to x in $(X, \mathscr{P}, \mathscr{T})$ and hence it is complete.

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