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THE DEFICIENT DISCRETE QUARTIC SPLINE INTERPOLATION

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ABSTRACT

In this paper, we have obtained a precise error estimate of the deficient discrete Quartic spline interpolate matching the given function values at the mesh point and its difference at mid points and also boundary points.

Key word's: *Error Bounds, Interpolation, Deficient Discrete Quartic splines.*

1. INTRODUCTION:

Deficient splines are more useful than usual splines as they require less continuity requirement at mesh points. Discrete splines have been introduced by Mangasarian and Schumaker [7] in connection with certain studies of minimization problems involving difference. Rana and Dubey [8] have obtains asymptotically precise estimate of the difference between discrete cubic spline interplant and the function interpolated, which is sometimes used to smooth a histogram. In the Direction of some constructive aspect's of discrete splines, we refer to Astor and Duris [1], Jia [4], Dikshit and Powar [3] and Rana [9]. The object of the present paper is to study the existence, uniqueness and convergence properties of deficient discrete quartic spline matching the given functional values at mesh points and its defferecne at mid points and also boundary point..

Let us consider a mesh P on $[0,1]$, which is defined by

$$0 = x_0 < x_1 < ... < x_n = 1.$$

such that $x_i - x_{i-1} = P_i$, for $i = 1,2,..., n.$ and $P = \max_{1 \le i \le n} P_i$, Throughout, h will be represent a given positive real number. Consider a real continuous function $S(x,h)$ defined over $[0,1]$ which is such that its restriction S_i on $[x_{i-1}, x_i]$ is a polynomial of degree 4 or less for $i = 1,2,..., n,$ then $S(x,h)$ defines a discrete quartic spline if

$$D_h^{\{j\}} S_i(x \ ,h) = D_h^{\{j\}} S_{i+1}(x \ ,h) \quad j = 0,1,2.3. \tag{1.1}$$

where the difference operator D_h are defined as

$$D_h^{\{0\}} f(x) = f(x), \ \ D_h^{\{1\}} f(x) = (f(x+h) - f(x-h))/2h$$

$$D_h^{\{2\}} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

2. EXISTENCE AND UNIQUENESS

The class of deficient discrete quartic splines is denoted by $S(4,2,\Delta,h)$ where $S^*(4,2,\Delta,h)$ denotes the class of all discrete deficient quartic splines which satisfies the boundary condition

$$\begin{aligned} D_h^{\{1\}} S(x_0, h) &= D_h^{\{1\}} f(x_0), \\ D_h^{\{1\}} S(x_n, h) &= D_h^{\{1\}} f(x_n), \end{aligned} \tag{2.1}$$

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We introduce the following interpolating conditions

$$\left.\begin{aligned} S(x_i) &= f(x_i) & i &= 0,\dots,n-1,n. \\ D_h^{\{1\}}S(Z_i) &= D_h^{\{1\}}f(Z_i) & i &= 0,\dots,n-1. \end{aligned}\right\} \tag{2.2}$$

where $Z_i = \dfrac{x_i + x_{i-1}}{2}$

Infact, we shall prove following:

Theorem: 2.1 For any $h > 0$, there exists a unique deficient discrete quartic spline $S(x,h) \in S^*(4,2,P,h)$ which satisfies the conditions (2.1) and (2.2).

Let $E(Z)$ be a quartic spline polynomial defined on [0,1]. It can be easily verified that

$$E(Z) = E(0)P_1(Z) + D_h^{\{1\}}E'(1/2)P_2(Z) + E(1)P_3(Z) + D_h^{\{1\}}E'(0)P_4(Z) + D_h^{\{1\}}E'(1)P_5(Z) \tag{2.4}$$

where $P_1(Z) = \dfrac{(1-z)^2}{2}(2+4z-z^2)$

$$P_2(Z) = -z^2(1-Z)^2$$

$$P_3(Z) = \frac{z^2}{2}(7+6z-z^2)$$

$$P_4(Z) = z\ (1-z)^2(1+z/4)$$

$$P_5(Z) = \frac{z^2(z-1)\ (3+z)}{4}$$

Let $S(x,h)$ be a discrete quartic polynomial $[x_i, x_{i+1}]$ and $\dfrac{x-x_i}{P_i} = Z$ then we can write

$$S_i(x,h)=f(x_i)P_1(z)+p_iD_n^{\{1\}}f(z_{i+1})P_2(z)+f(x_{i+1})P_3(z)+p_iD_n^{\{1\}}f(x_i)P_4(z)+p_iD_n^{\{1\}}f(x_{i+1})P_5(z) \tag{2.5}$$

which is clearly satisfy the condition (2.1) and (2.2).

Let $\Im_i(a,b) = ap_i^2 + bh^2$, where a,b are real numbers and $m_i = D_h^{\{1\}}S_i(x,h)$, we apply the continuity of the second difference of $S_i(x,h)$ at x_i in (2.5) to see that

$$\boldsymbol{p_i^3 m_{i-1}\Im_{i-1}(5,1) + \{p_i^3\Im_{i-1}(9,1) + p_{i-1}^3\Im_i(7,-1)\}m_i + p_{i-1}^3 m_{i+1}\Im_{i-1}(3,-1) = \frac{p_{i-1}^3}{p_i}F_i + \frac{p_i^3}{p_{i-1}}F_i^*} \tag{2.6}$$

where $F_i = (f_i - f_{i-1})\Im_i(10,-2) + p_{i-1}f^{\{1\}}(Z_i)\Im_{i-1}(4,4)$

We can easily see that excess of the absolute value of the coefficient of m_i over the sum of absolute value of the coefficients of m_{i-1} and m_{i+1} in (2.6) under the condition of theorem 2.1 is given by

$$\boldsymbol{G_i(h) = [p_i^3\Im_{i-1}(4,0) + p_{i-1}^3\Im_i(4,0)]}$$

Thus the coefficient matrix of the system of equations (2.6) is diagonally dominant and hence invertible, therefore the system of equation (2.6) has unique solution, which completes the proof of the theorem 2.1.

REMARK: In the case, when $h \to 0$ theorem 2.1 gives the corresponding results for continuous C^2 quartic spline interpolation under condition (2.1) and (2.2).

3. ERROR BOUNDS:

Now system of equation (2.6) may be written as

$$A(h)\cdot M(h) = F.$$

where $A(h)$ is coefficient matrix and $M(h) = m_i(h)$, However, as already shown in the proof of theorem 2.1 $A(h)$ is invertible. Denoting the inverse of $A(h)$ by $A^{-1}(h)$ we note that row max norm $A^{-1}(h)$ satisfies the following inequality

$$\| A^{-1}(h) \| \le y(h). \tag{3.1}$$

where $y(h) = \max\{l_i(h)\}^{-1}$ for convenience, we assume in this section that $1 = Nh$ when N is positive integer. It is also assumed that the mesh points $[x_i]$ are such that

$$x_i \in [0,1]_h, \quad for \quad i = 0,1,...,n.$$

where discrete interval $[0,1]_h$ is the set of points $\{0, N,..., Nh\}$ for a function f and two distinct points x_1, x_2 in it's domain, the first divided difference is defined by

$$[x_1, x_2]f = [f(x_1) - f(x_2)]/(x_1 - x_2)$$

and

$$[x_1, x_2, x_3]f = \frac{[x_2, x_3]_f - [x_1, x_2]_f}{(x_3 - x_1)}. \tag{3.2}$$

For convenience, we write $f^{\{1\}}$ for $D_h^{\{1\}} f$, $f_i^{\{1\}}$ for $D_h^{\{1\}} f(x_i)$ and $w(f, p)$ for modules of continuity of f, the discrete norm of a function f over the interval $[0,1]_h$ is defined by

$$\| f \| = \max_{x \in [0,1]_h} | f(x) | \tag{3.3}$$

We shall obtain in the following the bound of error functions $e(x) = s(x,h) - f(x)$ over the discrete interval $[0,1]h$.

Theorem: 3.1 Suppose $S(x,h)$ is the discrete quartic splines interpolant of theorem 2.1 then

$$\| e(x) \| \le \ k(p,h) w(f^{\{1\}}, p) \tag{3.4}$$

and

$$\| e^{\{1\}}(x) \| \le \ y(h) k^*(p,h) w(f^{\{1\}}, p) \tag{3.5}$$

$$\| e^{\{2\}}(x) \| \le \ k^{**}(p,h) w(f^{\{1\}}, p) \tag{3.6}$$

where $k(p,h)$, $k^*(p,h)$ and $k^{**}(p,h)$ are some positive functions of p and h.

Proof: Writing $f(x_i) = f_i$. Equation (3.1) may be written as

$$A(h) \cdot e^{\{1\}}(x_i) = (f_i^{\{1\}}) A(h) = (L_i) \ \ (say) \tag{3.7}$$

when we replace $m_i(h)$ by

$$e_i^{\{1\}}(x_i) = s^{\{1\}}(x_i, h) - f_i^{\{1\}}$$

we need following lemma due to Lyche [5,6], to obtain inequlity (3.4).

Lemma: 3.1 Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ be given sequence of non-negative real numbers such that $\sum a_i = \sum b_j$ then for any real valued function f, defined on discrete interval $[0,1]_h$, we have

$$\left| \sum_{i=1}^m a_i [x_{i_0}, x_{i_1}, ..., x_{i_k}] f - \sum_{j=1}^n b_j [y_{j_0}, y_{j_1}, ..., y_{j_k}]_f \right| \le W(f^{(k)}, |1 - kh|) \frac{\sum a_i}{k!} \tag{3.8}$$

where x_{ik}, $y_{jk} \in [0,1]_h$ for relevant values of i, j and k.

We can write the equation (2.6) is of the form of error function as follows.

$$[p_i^3 e_{i-1}^{\{1\}} \Im_{i-1}(5,1) + \{p_i^3 \Im_{i-1}(9,1) + p_{i-1}^3 \Im_i(7,-1)\} e_i^{\{1\}} + p_{i-1}^3 e_{i+1}^{\{1\}} \Im_i(3,-1)] = \frac{p_{i-1}^3}{p_i} F_i + \frac{p_i^3}{p_{i-1}} F_i^*$$
$$-[p_i^3 f_{i-1}^{\{1\}} \Im_{i-1}(5,1) + \{p_i^3 \Im_{i-1}(9,1) + p_{i-1}^3 \Im_i(7,-1)\} f_i^{\{1\}} + p_{i-1}^3 f_{i+1}^{\{1\}} \Im_i(3,-1)] = R_i(f) \quad \text{(say)} \tag{3.9}$$

Writing equation (3.9) is of the form of divided difference and using Lemma 3.1, given by Lyche [5,6].

Since $$\sum_{i=1}^{3} a_i = \left[p_{i-1}^3 \Im_i(14,2) + p_i^3 \Im_{i-1}(14,2)\right] = \sum_{j=1}^{4} b_i$$

We get

$$|(R_i(f))| = |\sum a_i [x_{i0}, x_{i1}]_f - \sum b_j [y_{j0}, y_{j1}]_f \tag{3.10}$$
$$\leq w(f^{\{1\}}, p) \sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

where
$$a_1 = p_i^3 (10 p_{i-1}^2 - 2h^2) = p_i^3 \Im_{i-1}(10,-2),$$
$$a_2 = p_{i-1}^3 (14 p_i^2 + 2h^2) = p_i^3 \Im_i(14,2),$$
$$a_3 = 4 p_i^3 (h^2 + p_{i-1}^2) = p_i^3 \Im_{i-1}(4,4),$$
$$b_1 = p_i^3 (5 p_{i-1}^2 + h^2) = p_i^3 \Im_{i-1}(5,1),$$
$$b_2 = p_i^3 \Im_{i-1}(9,1) + p_{i-1}^3 \Im_i(7,-1)$$
$$b_3 = p_{i-1}^3 \Im_i(3,1) = p_{i-1}^3 \Im_i(3,-1)$$
$$b_4 = p_{i-1}^3 \Im_i(4,4) = p_{i-1}^3 \Im_i(4,4)$$

and
$$x_{10} = x_{i-1}, x_{11} = x_i = x_{20} = y_{11} = y_{20} = y_{30} = y_{41}$$
$$x_{21} = x_{i+1} = y_{21} = x_{31}, \quad y_{31} = \alpha_{i+1}, y_{51} = \beta_{i+1}$$
$$x_{30} = z_{i+1}, y_{40} = z_i,$$

Now using the equtions (3.9) and (3.10).

$$\|e^{\{1\}}(x)\| \leq y(h) k \ (p,h) w(f^{\{1\}}, p) \tag{3.11}$$

$$\|e^{\{1\}}(x)\| \leq k_1(p,h) w(f^{\{1\}}, p)$$

This complete proof of inequality (3.5) of Theorem 3.1.

Next, we have to get $e(x)$. Writing equation (2.5) in form of error function as follows.

$$e(x) = p_i e^{\{1\}} P_4(t) + p_i e_{i+1}^{\{1\}} P_5(t) + L_i(f)$$

where
$$L_i(f) = f_i P_1(t) + p_i f^{\{1\}}(z_{i+1}) P_2(t) + f_{i+1} P_3(t) + p_i f_i^{\{1\}} P_4(t) + p_i f_{i+1}^{\{1\}} P_5(t) - f(x) \tag{3.12}$$

We write $L_i(f)$ in form of divided difference and using Lemma 3.1, we get

$$|L_i(f)| \leq w(f^{\{1\}}, p) \sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j$$
$$= p_i (z + z^2 - 2z^3 + z^4)$$
$$= p_i k(z,h) \quad (say) \tag{3.13}$$

where

$$a_1 = \frac{1}{2}(7z^2 + 6z^3 - z^4)$$
$$a_2 = \frac{1}{4}(4z + 2z^3 - 7z^2 + z^4)$$
$$a_3 = \frac{1}{4}(-3z^2 + 2z^3 + z^4)$$
$$b_1 = z$$
$$b_2 = z^2 + z^2 - 2z^3$$

and

$$x_{10} = x_{20} = x_{30} = x_i$$
$$x_{11} = x_{21} = x_{31} = x_{i+1}$$
$$y_{10} = x \ , \ y_{11} = x_i$$
$$y_{20} = z_{i+1}, \quad y_{21} = x_{i+1}$$

From equations (3.11) and (3.12) gives inequality (3.4) of theorem (3.1).

This complete proof of inquality of theorem 3.1.

Again we have to find error bound of second difference as follows.

i.e. $e^{\{2\}}(x\ ) = [S_i^{\{2\}}(x) - f_i^{\{2\}}(x)]$

Now

$$P_i^2 e^{\{2\}}(x) = [f_i A_i^{\{2\}}(t) + P_i f^{\{1\}}(z_{i+1}) A_2^{\{2\}}(t) + f_{i+1} A_3^{\{2\}}(t) + P_i\{e_i^{\{1\}} A_4^{\{2\}}(t) + e_{i+1}^{\{1\}} A_5^{\{2\}}(t)\} + p_i\,[f_i^{\{1\}} A_4^{\{2\}}(t) + f_{i+1}^{\{1\}} A_5^{\{2\}}(t)\} - P_i^2 f_i^{\{2\}}(x)] \tag{3.14}$$

For convience we write (3.14) equation

$$P_i^2 e^{\{2\}}(x) = P_i[e_i^{\{1\}} A_4^{\{2\}}(t) + e_i^{\{2\}} A_5^{\{2\}}(t)] + Q_i(f) \tag{3.15}$$

where

$$Q_i(f) = [f_i A_i^{\{2\}}(t) + P_i f(z_{i+1}) A_2^{\{2\}}(t) + f_{\{i\}} A_3^{\{2\}}(t) + P_i\{f_1^{\{1\}} A_4^{\{2\}} + f_{i+1}^{\{1\}} A_5^{\{2\}}(t)\} - P_i^2 f_i^{\{2\}}(x)]$$

We write $Q_i(f)$ in form of divided difference and using Lemma 3.1, we get

$$\begin{aligned} |\,Q_i(f)\,| &= |\sum_{i=1}^{5} a_i [x_{i0},\ x_{i1}]_f - \sum_{j=1}^{1} b_j [y_{j0}\ , y_{j1}]_f\ | \\ &\le (\sum_{i=1}^{5} a_i \ = \ \sum_{j=1}^{1} b_j) w(f^{\{1\}}, P) \end{aligned} \tag{3.16}$$

where

$$a_1 = P_i[7 - 18z + (6z^2 + h^2)]$$
$$a_2 = P_i[2 + 12z - 2(6z^2 + h^2)]$$
$$a_3 = P_i\ [-7/2 + 3z + 1/2(6z^2 + h^2)]$$
$$a_4 = P_i\ [-3/2 + 3z + 1/2(6z^2 + h^2)]$$
$$a_5 = P_i$$
$$b_1 = P_i$$

Since $a_1 + a_2 + a_3 + a_4 + a_5 = P_i \;\; = b_1$

$$\text{i.e. } \sum_{i=1}^{5} a_i = b_1 = P_i$$

and

$$x_{10} = y_{10} = x_{40} = x_i,$$
$$x_{11} = x_{i+1} = x_{21} = x_{41} = x_{51}$$
$$x_{20} = z_{i+1},\;\; x_{30} = x_{i-1},\;\; x_{31} = x_i$$
$$x_{50} = x \;\; = y_{11},$$

Hence

$$| Q_i(f) | \le P_i \;\; w \;\; (f^{\{1\}}, P) \tag{3.17}$$

By using (3.5) and (3.15) in (3.14) we get inequality (3.6) of theorem (3.1).

This complete proof of theorem 3.1.

REFERENCE:

[1] Astor, P.H and Duris, C.S. Discrete splines, Numer. Math., 22 (1974), 393-402.

[2] Dikshit, H.P. and Rana, S.S., Discrete cubic spline Interpolation with a non uniform mesh, Rocy Mount, Journal Math's., 17(1987), 700-718.

[3] Dikshit, H.P. and Powar, P.L. Area Matching Interpolation by Discrete cubic splines, Approximation Theory and Application Proc. Int. Cont. St. John's New Foundland (1984), 35-45.

[4] Jia, Rong Qing, Total positivity of the Discrete spline collection Matrix, J. approx. Theory, 39 (1983), 11-23.

[5] Lyche, T., Discrete cubic spline Interpolation, Report RRS, University of cyto, 1975.

[6] Lyche, T., Discrete Cubic Spline interpolation, BIT 16(1976), 281-290.

[7] Mangasarian, O.L. and Schumaker, L.L. Discrete Spline Via Mathematical Programming SIAMJ. Cantrola (1971), 174-183.

[8] Rana S.S., Local behavior of the first Difference of Discrete Cubic Spline Interpolation, Approx. Theory and Appl. 6, No. (1990), 58-64.

[9] Rana, S.S. and Dubey, Y.P. Local Behavior of the Deficient Discrete Cubic Spline Interpolation, J. Approx. Theory, 86 (1996), 120-127.

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