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PERFORMANCE ANALYSIS OF REPAIRABLE M/G/1 RETRIAL QUEUE WITH BERNOULLI VACATION AND ORBITAL SEARCH

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ABSTRACT

In this paper we consider a repairable M/G/1 retrial queue with Bernoulli vacation and orbital search. By supplementary variable technique, the probability generating functions of the system size distribution and the orbit size distribution under steady state are obtained. Queueing as well as reliability indices to predict the behavior of the system are derived. Various models studied earlier are deduced as special cases by appropriate choice of parameter values.

Key words: *Retrial queue, Breakdown, Bernoulli Vacation, Orbital Search.*

INTRODUCTION

The theory of retrial queues have been extensively applied in the study of communication and computer networks. The special characteristic of retrial queues is that, a customer who finds a busy server does not leave the system or joins a queue. He joins an orbit (retrial group) from where he makes repeated attempts to obtain service. Several survey articles, bibliographic articles and monographs have been published on retrial queues, see Artalejo (1999a, 1999b), Falin and Templeton (1997).

Queueing systems with vacation time and server breakdowns have been found to be useful in modeling the systems in which the server has additional tasks. Single server queueing systems with server breakdowns and Bernoulli vacation have been studied by many researchers including Li et al. (1997), Wang et al. (2008), Zhou Zong-hao et al. (2009) and Peishu Chen et al. (2010).

In the retrial setup, each service is preceded and followed by the server's idle time because of the ignorance of the status of the server and orbital customers by each other. Server's idle time is reduced by the introduction of search of orbital customers immediately after a service completion. Search for orbital customers was introduced by Neuts et al. (1984) where the authors examined classical queue with search for customers immediately on termination of a service. Orbital search after service have been investigated by Artalejo et al. (2002), Dudin et al. (2004), Krishnamoorthy et al. (2005) and Chakravarthy et al. (2006).

The present investigation is concerned with the analysis of retrial queuing system with Bernoulli vacation, server breakdown and orbital search.

THE MATHEMATICAL MODEL

A repairable M/G/1 retrial queue, where the server applies a single vacation policy and orbital search is considered. New customers arrive from outside according to Poisson process with rate λ. Once an arriving customer finds the server free, it begins the service. If the server is busy or repair or on vacation, the arriving customer must leave the service area and join the orbit. The customers in the orbit try to require the service later and the inter retrial times have a general distribution A(x) with corresponding Laplace-Stieltjes transform A*(s). The service times follow a general distribution B(x) with corresponding Laplace-Stieltjes transform B*(s) and the first two moments μ_1 and μ_2.

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It is assumed that when the server is busy it fails at an exponential rate α. When the server fails, the repair starts immediately and the customer just under service waits for the server until repair completion in order to accomplish its remaining service. The repair time is a random variable with probability distribution function R(x), Laplace-Stieltjes transform R*(s) and the first two moments r_1 and r_2.

At the completion epoch of the service the server may take a single vacation with probability β or waits for the next customer with probability $\delta(=1-\beta)$. After completing vacation the server searches for the customers in the orbit (if any) with probability θ or remains idle with its complimentary probability. The search time is assumed to be negligible. The vacation times are arbitrarily distributed with distribution function V(x), Laplace-Stieltjes transform V*(s) and the first two moments v_1 and v_2.

STABILITY CONDITION

The system is said to be stable if and only if the inequality $\lambda(\mu_1+\alpha\mu_1 r_1+\beta v_1) < A^*(\lambda)+\theta\beta(1-A^*(\lambda))$ is satisfied.

Proof: Let Q_n be the orbit length at the time of departure of n[th] customer $n \geq 1$. Then $\{Q_n, n\epsilon N\}$ is irreducible and aperiodic. We now prove that the embedded Markov chain $\{Q_n, n\epsilon N\}$ is ergodic if and only if $\lambda(\mu_1+\alpha\mu_1 r_1+\beta v_1) < A^*(\lambda)+\theta\beta(1-A^*(\lambda))$.

Foster's criterion states that an irreducible and aperiodic Markov chain is ergodic if there exists a nonnegative function f(j), j ϵ N, and $\xi > 0$, such that the mean drift

$X_j = E\,[f(Q_{n+1})-f(Q_n) \,/\, Q_n = j]$ is finite for all j ϵ N and $X_j < -\xi$, except for a finite number of j.

In our case $\{Q_n, n\epsilon N\}$ is irreducible and aperiodic. Now considering the function f(j) = j, we have

$X_j = \lambda(\mu_1+\alpha\mu_1 r_1+\beta v_1) - A^*(\lambda) - \theta\beta(1-A^*(\lambda))$, $j \geq 1$.

Hence if $\lambda(\mu_1+\alpha\mu_1 r_1+\beta v_1) < A^*(\lambda)+\theta\beta(1-A^*(\lambda))$ then $\{Q_n\}$ is ergodic.

The necessary condition follows from Kaplan's condition namely $X_j < \infty$ for all $j \geq 0$ and there exists j_0 ϵ N such that $X_j \geq 0$ for $j \geq j_0$. Since the arrival stream is Poisson, it can be shown from Burke's theorem that $\lambda(\mu_1+\alpha\mu_1 r_1+\beta v_1) < A^*(\lambda)+\theta\beta(1-A^*(\lambda))$ is also a necessary condition of our Queueing system.

SYSTEM ANALYSIS

The state $\{N(t); t \geq 0\}$ of the system at time t can be described by the Markov process

$$\{N(t); t\geq 0\}=\{C(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t); t\geq 0\}$$

where C(t) denote the server state at time t given by

$$C(t) = \begin{cases} 0, \text{if the server is free} \\ 1, \text{if the server is busy} \\ 2, \text{if the server is on repair} \\ 3, \text{if the server is on vacation} \end{cases}$$

and X (t) represents the number of customers in the retrial group at time t. If C(t) = i, i=0, 1, 2, 3, then $\xi_i(t)$ denotes the elapsed retrial time, server times, repair times and the server vacation times at time t respectively.

Assume that $\eta(x)$, $\mu(x)$, $\gamma(x)$ and $\omega(x)$ are the conditional completion rates (at time t) for repeated attempts, for service, for repair and for vacation respectively. Then

$$\eta(x)=\frac{a(x)}{1-A(x)},\ \mu(x)=\frac{b(x)}{1-B(x)},\ \gamma(x)=\frac{r(x)}{1-R(x)} \text{ and } \omega(x)=\frac{v(x)}{1-V(x)}.$$

For the process {N (t); t≥0}, define the probabilities

$$\mathrm{I}_0(\mathrm{t}) = \mathrm{P}\{\mathrm{C}(\mathrm{t}) = 0, \mathrm{X}(\mathrm{t}) = 0\}$$

$$\mathrm{I}_\mathrm{n}(\mathrm{x}, \mathrm{t})\mathrm{dx} = \mathrm{P}\{\mathrm{C}(\mathrm{t}) = 0, \mathrm{X}(\mathrm{t}) = \mathrm{n},\ \mathrm{x} \le \xi_0(t) < x + dx, t \ge 0, x \ge 0, n \ge 1\}$$

$$\mathrm{P}_\mathrm{n}(\mathrm{x}, \mathrm{t})\mathrm{dx} = \mathrm{P}\{\mathrm{C}(\mathrm{t}) = 1, \mathrm{X}(\mathrm{t}) = \mathrm{n},\ \mathrm{x} \le \xi_1(t) < x + dx, t \ge 0, x \ge 0, n \ge 0\}$$

$$\mathrm{Q}_\mathrm{n}(\mathrm{x}, \mathrm{y}, \mathrm{t})\mathrm{dxdy} = \mathrm{P}\{\mathrm{C}(\mathrm{t}) = 2, \mathrm{X}(\mathrm{t}) = \mathrm{n},\ \mathrm{x} \le \xi_1(t) < x + dx, y \le \xi_2(t) \le y + dy, t \ge 0, x \ge 0, n \ge 0\}$$

$$\mathrm{V}_\mathrm{n}(\mathrm{x}, \mathrm{t})\mathrm{dx} = \mathrm{P}\{\mathrm{C}(\mathrm{t}) = 3, \mathrm{X}(\mathrm{t}) = \mathrm{n},\ \mathrm{x} \le \xi_3(t) < x + dx, t \ge 0, x \ge 0, n \ge 0\}$$

By the method of supplementary variable technique, the Kolmogorov forward equations which govern the system under steady state conditions are obtained as follows.

$$\lambda I_0 = \delta\int_0^\infty P_0(x)\mu(x)dx + \int_0^\infty V_0(x)\omega(x)dx \tag{1}$$

$$\left(\frac{\partial}{\partial x} + \lambda + \eta(x)\right)I_n(x) = 0, n \ge 1 \tag{2}$$

$$\left(\frac{\partial}{\partial x} + \lambda + \alpha + \mu(x)\right)P_n(x) = \lambda P_{n-1}(x) + \int_0^\infty Q_n(x, y)\gamma(y)dy, n \ge 0 \tag{3}$$

$$\left(\frac{\partial}{\partial y} + \lambda + \gamma(y)\right)Q_n(x, y) = \lambda Q_{n-1}(x, y), n \ge 0 \tag{4}$$

$$\left(\frac{\partial}{\partial x} + \lambda + \omega(x)\right)V_n(x) = \lambda V_{n-1}(x), n \ge 0 \tag{5}$$

with the boundary conditions

$$I_n(0) = \delta\int_0^\infty P_n(x)\mu(x)dx + (1-\theta)\int_0^\infty V_n(x)\omega(x)dx, n \ge 1 \tag{6}$$

$$P_0(0) = \int_0^\infty I_1(x)\eta(x)dx + \lambda I_0 + \theta\int_0^\infty V_1(x)\omega(x)dx \tag{7}$$

$$P_n(0) = \int_0^\infty I_{n+1}(x)\eta(x)dx + \theta\int_0^\infty V_{n+1}(x)\omega(x)dx + \lambda\int_0^\infty I_n(x)dx, n \ge 1 \tag{8}$$

$$Q_n(x,0) = \alpha P_n(x), n \ge 0 \tag{9}$$

$$V_n(0) = \beta\int_0^\infty P_n(x)\mu(x)dx, n \ge 0 \tag{10}$$

Define the probability generating functions

$$I(x,z) = \sum_{n=1}^\infty I_n(x)z^n\ ;\ P(x,z) = \sum_{n=0}^\infty P_n(x)z^n\ ;\ Q(x,y,z) = \sum_{n=0}^\infty Q(x,y)z^n\ ;\ V(x,z) = \sum_{n=1}^\infty V_n(x)z^n$$

Multiplying equations (1) to (10) by z^n and summing for all possible values of n, we have

$$\left(\frac{\partial}{\partial x}+\lambda+\eta(x)\right)I(x,z)=0 \tag{11}$$

$$\left(\frac{\partial}{\partial x}+\lambda(1-z)+\alpha+\mu(x)\right)P(x,z)=\int_0^\infty Q(x,y,z)\gamma(y)dy \tag{12}$$

$$\left(\frac{\partial}{\partial y}+\lambda(1-z)+\gamma(y)\right)Q(x,y,z)=0 \tag{13}$$

$$\left(\frac{\partial}{\partial x}+\lambda(1-z)+\omega(x)\right)V(x,z)=0 \tag{14}$$

$$I(0,z)=\delta\int_0^\infty P(x,z)\mu(x)dx+(1-\theta)\int_0^\infty V(x,z)\omega(x)dx-\lambda I_0 \tag{15}$$

$$P(0,z)=\frac{1}{z}\int_0^\infty I(x,z)\eta(x)dx+\frac{\theta}{z}\int_0^\infty V(x,z)\omega(x)dx+\lambda I_0+\lambda\int_0^\infty I(x,z)dx \tag{16}$$

$$Q(z,x,0)=\alpha P(x,z) \tag{17}$$

$$V(0,z)=\beta\int_0^\infty P(x,z)\mu(x)dx \tag{18}$$

Solving the partial differential equations (11) to (14), we have

$$I(x,z)=I(0,z)e^{-\lambda x}(1-A(x)) \tag{19}$$

$$P(x,z)=P(0,z)e^{-(\lambda(1-z)+\alpha-\alpha R^*(\lambda-\lambda z))x}(1-B(x)) \tag{20}$$

$$Q(x,y,z)=Q(x,0,z)e^{-\lambda(1-z)y}(1-R(y)) \tag{21}$$

$$V(x,z)=V(0,z)e^{-\lambda(1-z)x}(1-V(x)) \tag{22}$$

Using equations (19) to (22) in (15) to (18) and solving we obtain

$$I(x,z)=\lambda A^*(\lambda)(z-1)B^*(G(\lambda-\lambda z))[\delta+(1-\theta)\beta V^*(\lambda-\lambda z)-\lambda I_0]e^{-\lambda x}(1-A(x))I_0/D(z) \tag{23}$$

$$P(x,z)=\lambda A^*(\lambda)(z-1)I_0e^{-G(\lambda-\lambda z)x}(1-B(x))/D(z) \tag{24}$$

$$Q(x,y,z)=\alpha\lambda A^*(\lambda)(z-1)I_0e^{-G(\lambda-\lambda z)x}(1-B(x))e^{-(\lambda-\lambda z)y}(1-R(y))/D(z) \tag{25}$$

$$V(x,z)=\beta\lambda A^*(\lambda)(z-1)B^*(G(\lambda-\lambda z))I_0e^{-(\lambda-\lambda z)x}(1-V(x))/D(z) \tag{26}$$

where $G(\lambda-\lambda z)x=(\lambda-\lambda z)x+\alpha-\alpha R^*(\lambda-\lambda z)x$ and

$$D(z)=z-(z+(1-z)A^*(\lambda))B^*(G(\lambda-\lambda z))[\delta+(1-\theta)\beta V^*(\lambda-\lambda z)]-\theta\beta B^*(G(\lambda-\lambda z))V^*(\lambda-\lambda z)$$

The partial probability generating function of the orbit size when the server is idle is

$$I(z) = \int_0^\infty I(x,z)dx$$
$$= [(1 - A^*(\lambda))((z-1)A^*(\lambda)B^*(G(\lambda - \lambda z))(\delta + (1-\theta)\beta V^*(\lambda - \lambda z)) - D(z))]I_0 / D(z)$$

The partial probability generating function of the orbit size when the server is busy is

$$P(z) = \int_0^\infty P(x,z)dx$$
$$= \lambda(z-1)A^*(\lambda)(1 - B^*(G(\lambda - \lambda z)))I_0 / D(z)G(\lambda - \lambda z)$$

The partial probability generating function of the orbit size when the server is on repair is

$$Q(z) = \int_0^\infty \int_0^\infty Q(z,x,y)dxdy$$
$$= \alpha A^*(\lambda)(1 - B^*(G(\lambda - \lambda z)))(R^*(\lambda - \lambda z) - 1)I_0 / D(z)G(\lambda - \lambda z)$$

The partial probability generating function of the orbit size when the server is on vacation is

$$V(z) = \int_0^\infty V(x,z)dx$$
$$= \beta A^*(\lambda)B^*(G(\lambda - \lambda z))(V^*(\lambda - \lambda z) - 1)I_0 / D(z)$$

Using the normalizing condition $I_0 + \lim_{z \to 1}\big(I(z) + P(z) + Q(z) + V(z)\big) = 1$ and applying L'Hospital's rule we get I_0, the steady state probability that the system is empty as $I_0 = (\theta\beta(1 - A^*(\lambda)) + A^*(\lambda) - \lambda\mu_1(1 + \alpha r_1) - \beta\lambda v_1) / A^*(\lambda)$

The probability generating function of the number of customers in the orbit is

$$\mathrm{P_q(z) = I_0 + I(z) + P(z) + Q(z) + V(z)}$$
$$= A^*(\lambda)(z-1)I_0 / D(z)$$

The probability generating function of the number of customers in the system is

$$\mathrm{P_s(z) = I_0 + I(z) + zP(z) + zQ(z) + V(z)}$$
$$= P_q(z)B^*(G(\lambda - \lambda z))$$

PERFORMANCE CHARACTERISTICS

The probability that the server is idle during the retrial time is given by

$$I = \lim_{z \to 1} I(z) = (1 - A^*(\lambda))(\lambda\mu_1(1 + \alpha r_1) + \beta\lambda v_1 - \theta\beta) / A^*(\lambda)$$

The probability that the server is busy is given by

$$P = \lim_{z \to 1} P(z) = \lambda\mu_1$$

The probability that the server is on repair is given by

$$Q = \lim_{z \to 1} Q(z) = \lambda\alpha r_1\mu_1$$

The probability that the server is on vacation is given by

$$V = \lim_{z \to 1} V(z) = \beta\lambda v_1$$

The mean number of customers in the orbit under steady state conditions is

$$L_q = \lim_{z \to 1} P_q(z)$$

$$= \frac{\lambda^2/2(\beta v_2 + \mu_2(1+\alpha r_1)^2 + \alpha\mu_1 r_2) - \lambda\mu_1(1+\alpha r_1) + \beta\lambda v_1(1-\theta) - A^*(\lambda)((1-\theta)\beta\lambda v_1 + (1-\theta\beta)\lambda\mu_1(1+\alpha r_1))}{\theta\beta(1-A^*(\lambda)) + A^*(\lambda) - \lambda\mu_1(1+\alpha r_1) - \beta\lambda v_1}$$

The mean number of customers in the system under steady state conditions is

$$L_s = \lim_{z\to 1} P_s(z) = L_q + \lambda\mu_1(1+\alpha r_1)$$

RELIABILITY INDEXES OF THE SERVER

The steady state availability of the server is

$$\begin{aligned} A &= I_0 + I + P \\ &= 1 - \lambda(\alpha\mu_1 r_1 + \beta v_1) \end{aligned}$$

The steady state failure frequency of the server is

$$\begin{aligned} F &= \alpha P \\ &= \alpha\lambda\mu_1 \end{aligned}$$

Special Cases

- If $A^*(\lambda) \to 1$, then the model reduces to M/G/1 queue with server breakdown and vacation.
- If $\theta = 0$, the model reduces to an M/G/1 retrial queueing system with server breakdown and vacation.
- If $\alpha = 0$, the system reduces to retrial queues with vacation and orbital search.
- If $A^*(\lambda) \to 1$ and $\theta = 0$, the model reduces to an M/G/1 queue with server breakdown and vacation.

Numerical Results:

The effect of various parameters λ (arrival rate), η (retrial rate), μ (service rate), γ (repair completion rate), ω (vacation completion rate), α (repair rate) and θ (orbital search probability) on availability of the server, failure frequency and system size are studied numerically.

The calculations are carried out by considering the distributions of retrial times, service times, vacation times and repair times as exponential. We set the default parameters $\lambda = 1$, $\mu = 10$, $\beta = 0.5$, $\theta = 0.5$, $\gamma = 10$, $\omega = 10$, $\eta = 6$, $\alpha = 5$.

It is noticed that

- Increase in α and λ decreases availability and increases failure frequency.
- Increase in μ increases availability and decreases failure frequency.
- Increase in μ, η and γ decreases system size.
- Increase in α, ω and λ increases system size.

Figure 1: Availability and Failure frequency against μ

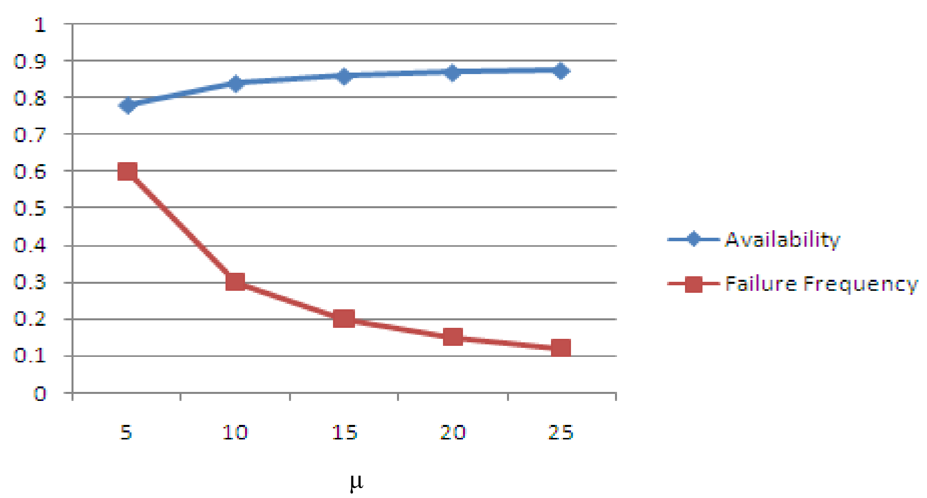


Figure 2: Availability and Failure frequency against λ

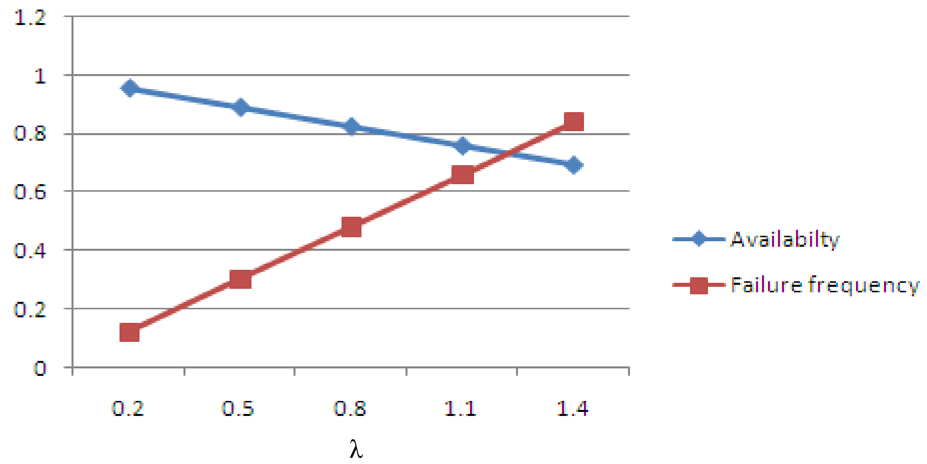


Figure 3: Availability and Failure frequency against α

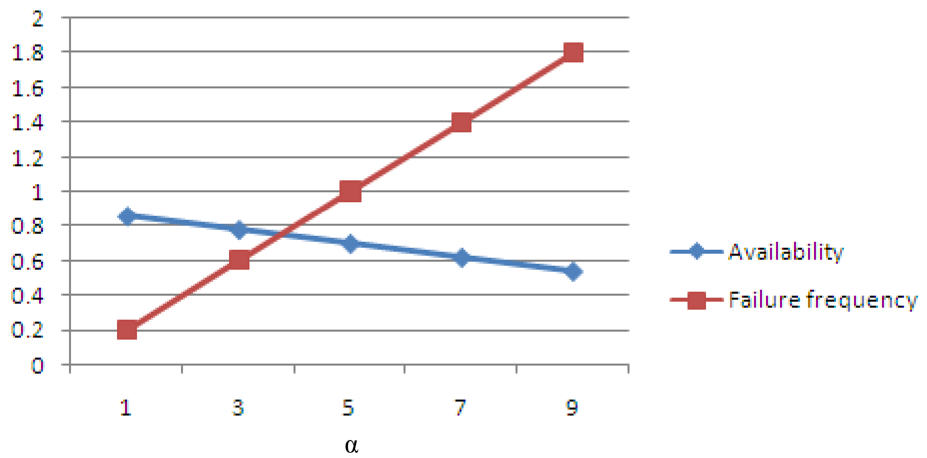
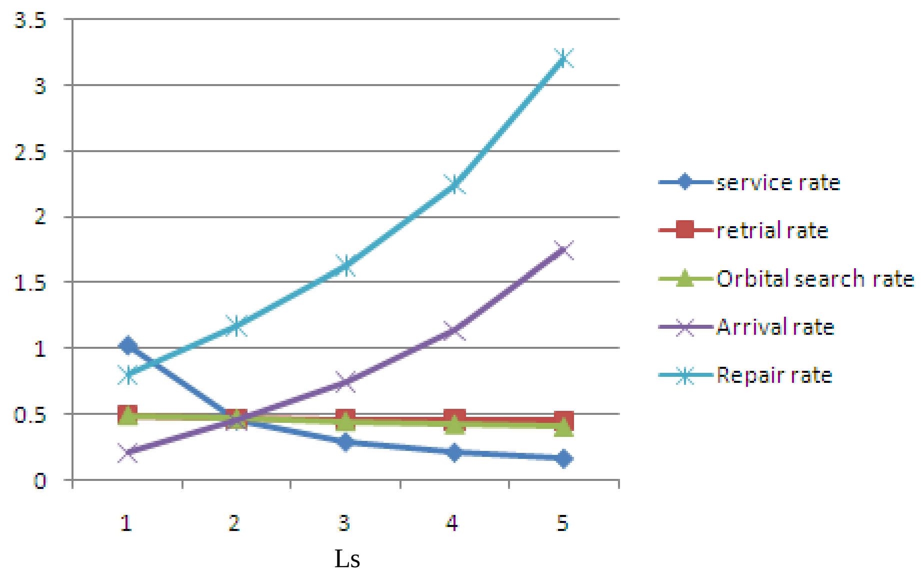


Figure 4: System size against various parameters



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