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SOME COMMON FIXED POINT THEOREMS FOR NON CONTRACTION MAPPINGS IN BANACH SPACES

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ABSTRACT

In this paper, we prove some fixed point and common fixed point theorems for non contraction rational mappings in Banach space. Also we generalized many more known results.

Key words: *Self mapping, Continuous mappings, Non contraction mappings, Banach Space.*

AMS subject classification: *47H10, 54H25,*

1. INTRODUCTION

It is well known that the differential and integral equations that arise in many physical problems are mostly non linear and fixed point technique provides a powerful tool for obtaining the solutions of such equations which otherwise are difficult to solve by ordinary methods. While stating this, however, we mention that some qualitative properties of the solution of related equations are lost by functional analysis approach. Many attempts have been made to formulate fixed point theorems in this direction and the well known Schauder's fixed point principle formulated by J. Schauder in 1930.

Brouwder [1], Gohde [5], and Kirk [12], have independently proved the fixed point theorem for non–expansive mappings defined on a closed bounded and convex subset of a uniformly convex Banach space and in spaces with richer structure. Many other mathematicians gave a number of generalizations of non- expansive mappings, like Dotson [3], Emmouele[4], Goebel[6], Goebel and Zlotkiewicz [7],Goebel, Kirk and Shimi [8], Massa and Roux [13], Rhodes [14], are of special significance. A comprehensive survey concerning fixed point theorem for non-expansive and related mappings can be found in Kirk [10].

Let X be a Banach space and C a closed subset of X. then the well known Banach Contraction Principle states that a contraction mapping of C into itself has a unique fixed point. The same holds good if we assume that only some positive power of a mapping is a contraction (For example, Bryant [2]). But this result is no longer true for non expansive mappings. Many mathematicians have studied the existence of fixed points of non expansive maps defined on a closed bounded and convex subset of a uniformly convex Banach space, and in a space with a normal structure. For the results of this kind one is referred to Browder [1], Goebel[6], and kirk [10], it is natural with a non expansive iteration. The answer, in general, is negative However Goebel and Zlokiewicz [7] have answered this problem in affirmative with some restriction, and thus generalizing a result of Browder [1].

MAIN RESULTS

Theorem :2.1 Let F be mappings of a Banach space X into itself. If F satisfies the following conditions;

$F^2 = I$, where I is identity mapping (2.1.1)

$$\|F(X)-F(Y)\| \le \alpha \frac{\|X-F(X)\|\|Y-F(Y)\|}{\|X-F(X)\|+\|X-F(Y)\|} + \beta \frac{\|X-F(Y)\|\|Y-F(X)\|}{\|X-F(Y)\|+\|Y-F(X)\|} + \gamma\,[\|X-F(X)\|+\|Y-F(Y)\|] + \delta\,[\|X-F(Y)\|+\|Y-F(X)\|] + \eta\,\|X-Y\| \quad (2.1.2)$$

For every x,y $\in$ X where $\alpha, \beta, \gamma, \delta, \eta \geq 0$ and $7\alpha + 8\beta + 4\gamma < 8$ then F has a fixed point.

If $\frac{\beta}{2} + 2\delta + \eta < 1$.Then F has a unique fixed point

Proof: Suppose X is a point in the Banach space X. Taking Y= $\frac{1}{2}$ (F+ I) X, Z = F(Y), U = 2Y-Z, We have

$$\|z-x\| = \|F(Y) - F^2(X)\|$$

$$= \|F(Y) - F(F(X)\|$$

$$\leq \alpha \frac{\|Y-F(Y)\|\|F^2(X)-F(X)\|}{\|Y-F^2(X)\|+\|Y-F(Y)\|} + \beta \frac{\|Y-F^2(X)\|\|F(Y)-F(X)\|}{\|Y-F^2(X)\|+\|F(Y)-F(X)\|}$$
$$+ \gamma [\|F(X)-F^2(X)\| + \|Y-F(Y)\|] + \delta [\|F(X)-F(Y)\| + \|Y-F^2(X)\|] + \eta\|F(X)-Y\|$$

$$= \alpha \frac{\|Y-F(Y)\|\|X-F(X)\|}{\|Y-X\|+\|Y-F(Y)\|} + \beta \frac{\|Y-X\|\|F(Y)-F(X)\|}{\|Y-X\|+\|F(Y)-F(X)\|} + \gamma [\|X-F(X)\|$$
$$+ \|Y-F(Y)\|] + \delta [\|X-Y\| + \|F(Y)-F(X)\|] + \eta \|F(X)-Y\|$$

$$= \alpha \frac{\|Y-F(Y)\|\|X-F(X)\|}{\|X-F(Y)\|} + \beta \frac{\frac{1}{2}\|X-F(X)\|\|F(Y)-X+X-F(X)\|}{\frac{1}{2}\|X-F(X)\| + \|F(Y)-X+X-F(X)\|}$$
$$+ \gamma [\|X-F(X)\| + \|Y-F(Y)\|] + \delta [\frac{1}{2}\|X-F(X)\| + \|F(Y)-X+X-F(X)\|]$$
$$+ \eta \frac{1}{2}\|F(X)-X\|$$

$$\leq \alpha \frac{\|Y-F(Y)\|\|X-F(X)\|}{\frac{1}{2}\|X-F(X)\|} + \beta \frac{\frac{1}{2}\|X-F(X)\|[\|F(Y)-X\|+\|X-F(X)\|]}{\frac{1}{2}\|X-F(X)\| + \|F(Y)-X\|+\|X-F(X)\|}$$
$$+ \gamma [\|X-F(X)\| + \|Y-F(Y)\|] + \delta [\frac{1}{2}\|X-F(X)\| + \|F(Y)-X\|+\|X-F(X)\|]$$
$$+ \eta \frac{1}{2}\|F(X)-X\|$$

$$= 2\alpha \|Y-F(Y)\| + \beta \frac{\frac{1}{2}\|X-F(X)\|[\frac{1}{2}\|F(X)-X\|+\|X-F(X)\|]}{\frac{1}{2}\|X-F(X)\| + \frac{1}{2}\|F(X)-X\|+\|X-F(X)\|} + \gamma [\|X-F(X)\|$$
$$+ \|Y-F(Y)\|] + \delta [\frac{1}{2}\|X-F(X)\| + \frac{1}{2}\|F(X)-X\|+\|X-F(X)\| + \eta \frac{1}{2}\|F(X)-X\|$$

$$= (2\alpha + \gamma)\|Y-F(Y)\| + (\frac{3}{8}\beta + \gamma + 2\delta + \frac{\eta}{2})\|X-F(X)\|$$

Therefore

$$\|z-x\| \le (2\alpha+\gamma)\|Y-F(Y)\| + (\frac{3}{8}\beta+\gamma+2\delta+\frac{\eta}{2})\|X-F(X)\|$$

and

$$\|u-x\| = \|2Y-Z-X\| = \|(F+I)X-F(Y)-X\| = \|F(X)-F(Y)\|$$

$$\le \alpha\,\frac{\|X-F(X)\|\|Y-F(Y)\|}{\|X-F(X)\|+\|X-F(Y)\|} + \beta\,\frac{\|X-F(Y)\|\|Y-F(X)\|}{\|X-F(Y)\|+\|Y-F(X)\|}$$
$$+\gamma[\|X-F(X)\| + \|Y-F(Y)\|] + \delta[\|X-F(Y)\| + \|Y-F(X)\|] + \eta\|X-Y\|$$

$$\le \alpha\,\frac{\|X-F(X)\|\|Y-F(Y)\|}{\|X-F(X)\|+\frac{1}{2}\|X-F(X)\|} + \beta\,\frac{\frac{1}{4}\|X-F(X)\|\|X-F(X)\|}{\frac{1}{2}\|X-F(X)\|+\frac{1}{2}\|X-F(X)\|}$$
$$+\ \gamma[\|X-F(X)\|\ \|Y-F(Y)\|] + \delta[\frac{1}{2}\|X-F(X)\| + \frac{1}{2}\|X-F(X)\|] + \frac{1}{2}\eta\|X-F(X)\|$$

$$= \alpha + \frac{\|X-F(X)\|\|Y-F(Y)\|}{\frac{3}{2}\|X-F(X)\|} + \beta\,\frac{\frac{1}{4}\|X-F(X)\|\|X-F(X)\|}{\|X-F(X)\|}$$
$$+\ \gamma[\|X-F(X)\| + \|Y-F(Y)\|] + \delta\|X-F(X)\| + \frac{1}{2}\eta\|X-F(X)\|$$

$$= (2\alpha+\gamma)\|Y-F(Y)\| + (\frac{3}{8}\beta+\gamma+2\delta+\frac{\eta}{2})\|X-F(X)\|$$

Therefore

$$\|u-x\| \le (2\alpha+\gamma)\|Y-F(Y)\| + (\frac{3}{8}\beta+\gamma+2\delta+\frac{\eta}{2})\|X-F(X)\| \tag{2.1(a)}$$

Now

$$\|z-u\| \le \|z-x\| + \|x-u\|$$
$$\le (2\alpha+\gamma)\|Y-F(Y)\| + (\frac{3}{8}\beta+\gamma+2\delta+\frac{\eta}{2})\|X-F(X)\| + (2\alpha+\gamma)\|Y-F(Y)\|$$
$$+(\frac{3}{8}\beta+\gamma+2\delta+\frac{\eta}{2})\|X-F(X)\|$$

$$= (4\alpha+2\gamma)\|Y-F(Y)\| + (\frac{3}{4}\beta+2\gamma+4\delta+\eta)\|X-F(X)\|$$

Thus

$$\|z-u\| \le (4\alpha+2\gamma)\|Y-F(Y)\| + (\frac{3}{4}\beta+2\gamma+4\delta+\eta)\|X-F(X)\| \tag{2.1(b)}$$

Also

$$\|z-u\| = \|F(Y)-(2Y-Z)\|$$

$= \|F(Y) - 2Y + Z)\|$

$= 2\|Y - F(Y)\|$

Combining 2.1(a) and 2.1(b) , we have

$$\|Y - F(Y)\| \leq (4\alpha + 2\gamma)\|Y - F(Y)\| + (\frac{3}{4}\beta + 2\gamma + 4\delta + \eta)\|X - F(X)\|$$

Therefore,

$$\|Y - F(Y)\| \leq q\|X - F(X)\|$$

Where $q = \dfrac{\frac{3}{4}\beta + 2\gamma + 4\delta + \eta}{2 - 4\alpha - 2\gamma} < 1$

Since $7\alpha + 8\beta + 4\gamma < 8$

Let $G = \frac{1}{2}\|F + I\|$ then for every xϵX

$$\begin{aligned} \|G^2(X) - G(X)\| &= \|G(Y) - Y\| \\ &= \left\|\frac{1}{2}(F + I)Y - Y\right\| \\ &= \frac{1}{2}\|Y - F(Y)\| \\ &< \frac{q}{2}\|X - F(X)\| \end{aligned}$$

By the definition of q, we claim that {Gn(X)} is a Cauchy sequence in X.

By the completeness, {Gn(X)} converges to some element X0 in X.

i.e. $\lim_{n\to\infty} G^n(X) = X_0$

Which implies that G(X0) = X0.

Hence F(X0) = X0

i.e. X0 is a fixed point of F.

For the uniqueness, if possible let Y0($\neq$ X0) be another fixed point of F then

$$\|X_0 - Y_0\| = \|F(X_0) - F(Y_0)\|$$

$$\leq \alpha \frac{\|X_0 - F(X_0)\|\|Y_0 - F(Y_0)\|}{\|X_0 - F(X_0)\| + \|X_0 - F(Y_0)\|} + \beta \frac{\|X_0 - F(Y_0)\|\|Y_0 - F(X_0)\|}{\|X_0 - F(Y_0)\| + \|Y_0 - F(X_0)\|} + \gamma[\|X_0 - F(X_0)\| + \|Y_0 - F(Y_0)\|] + \delta[\|X_0 - F(Y_0)\| + \|Y_0 - F(X_0)\|] + \eta\|X_0 - Y_0\|$$

$$= \beta \frac{\|X_0 - Y_0\|\|Y_0 - X_0\|}{\|X_0 - Y_0\| + \|Y_0 - X_0\|} + \gamma [\|X_0 - Y_0\| + \|Y_0 - X_0\|] + \eta \|X_0 - Y_0\|$$

$$= (\frac{\beta}{2} + 2\delta + \eta) \|X_0 - Y_0\|$$

Therefore

$$\|X_0 - Y_0\| \le (\frac{\beta}{2} + 2\delta + \eta) \|X_0 - Y_0\|$$

Therefore

$$(1 - \frac{\beta}{2} - 2\delta - \eta) \|X_0 - Y_0\| \le 0$$

Since $\frac{\beta}{2} + 2\delta + \eta < 1$

Therefore $\|X_0 - Y_0\| = 0$

Hence $X_0 = Y_0$

2.2 Now we prove the following theorem which generalises the theorem-2.1

Theorem: 2.2 Let K be closed and convex subject of a Banach space X. Let F: K →K, G: K →K satisfy the following conditions

F and G commute (2.2.1)

$F^2 = I$ and $G^2 = I$ where I denotes the indentity mapping (2.2.2)

$$\|F(X) - F(Y)\| \le \alpha \frac{\|F(X) - G(X)\|\|F(Y) - G(Y)\|}{\|G(X) - F(X)\| + \|G(X) - F(Y)\|} + \beta \frac{\|F(Y) - G(X)\|\|F(X) - G(Y)\|}{\|G(X) - F(Y)\| + \|G(Y) - F(X)\|}$$
$$+ \gamma [\|G(X) - F(X)\| + \|G(Y) - F(Y)\|] + \delta [\|G(X) - F(Y)\| + \|G(Y) - F(X)\|]$$
$$+ \eta \|G(X) - G(Y)\| \tag{2.2.3}$$

For every x,y $\in$ K where $\alpha, \beta, \gamma, \delta, \eta \ge 0$ and $7\alpha + 8\beta + 4\gamma < 8$ then there exists at least one fixed point, $X_0 \in$ K such that $F(X_0)=G(X_0)=X_0$.Further if $\frac{\beta}{2} + 2\delta + \eta < 1$.Then X_0 is the unique fixed point of F and G.

Proof: From (2, 2, 1) and (2.2.2) it follows that $(FG)^2=I$ and (2.2.2) and (2.2.3) implies

$$\|FG.G(X) - FG.G(Y)\| \le \alpha \frac{\|FG^2(X) - GG^2(X)\|\|FG^2(Y) - GG^2(Y)\|}{\|GG^2(X) - FG^2(X)\| + \|GG^2(X) - FG^2(Y)\|}$$
$$+ \beta \frac{\|FG^2(Y) - GG^2(X)\|\|FG^2(X) - GG^2(Y)\|}{\|GG^2(X) - FG^2(Y)\| + \|GG^2(Y) - FG^2(X)\|} + \gamma [\|GG^2(X) - FG^2(X)\|$$
$$+ \|GG^2(Y) - FG^2(Y)\|] + \delta [\|GG^2(X) - FG^2(Y)\| + \|GG^2(Y) - FG^2(X)\|]$$
$$+ \eta \|GG^2(X) - GG^2(Y)\|$$

$$= \alpha \frac{\|FG.G(X)-G(X)\|\|FG.G(Y)-G(Y)\|}{\|G(X)-FG.G(X)\|+\|G(X)-FG.G(Y)\|}$$
$$+\beta \frac{\|FG.G(Y)-G(X)\|\|FG.G(X)-G(Y)\|}{\|G(X)-FG.G(Y)\|+\|G(Y)-FG.G(X)\|}+\gamma\,[\|G(X)-FG.G(X)\|$$
$$+\|G(Y)-FG.G(Y)\|]+\delta\,[\|G(X)-FG.G(Y)\|+\|G(Y)-FG.G(X)\|]$$
$$+\eta\,\|G(X)-G(Y)\|$$

Now we put G(X) =Z and G(Y)=W then we get

$$\|FG(Z)-FG(W)\| \le \alpha \frac{\|FG(Z)-Z\|\|FG(W)-W\|}{\|Z-FG(Z)\|+\|Z-FG(W)\|}+\beta \frac{\|FG(W)-Z\|\|FG(Z)-W\|}{\|Z-FG(W)\|+\|W-FG(Z)\|}$$
$$+\gamma\,[\|Z-FG(Z)\|+\|W-FG(W)\|]+\delta\,[\|Z-FG(W)\|+\|W-FG(Z)\|]+\eta\,\|Z-W\|$$

Where $(FG)^2 = I$ and so by theorem-1, FG has at least one fixed point , say X_0 in K

i.e. $FG(X_0) = X_0$.(A)

and so $FFG(X_0) = F(X_0)$

or $G(X_0) = F(X_0)$ (B)

Now,

$$\|F(X_0)-X_0\| = \|F(X_0)-F^2(X_0)\| = \|F(X_0)-F(FX_0)\|$$

$$\le \alpha \frac{\|F(X_0)-G(X_0)\|\|FF(X_0)-GF(Y_0)\|}{\|G(X_0)-F(X_0)\|+\|G(X_0)-FF(X_0)\|}$$
$$+\beta \frac{\|FF(X_0)-G(X_0)\|\|F(X_0)-GF(X_0)\|}{\|G(X_0)-FF(X_0)\|+\|GF(X_0)-F(X_0)\|}$$
$$+\gamma\,[\|G(X_0)-F(X_0)\|+\|GF(X_0)-FF(X_0)\|]+\delta\,[\|G(X_0)-FF(X_0)\|$$
$$+\|GF(X_0)-F(X_0)\|]+\eta\,\|G(X_0)-GF(X_0)\|$$

$$= \alpha \frac{\|F(X_0)-F(X_0)\|\|X_0-X_0\|}{\|F(X_0)-F(X_0)\|+\|F(X_0)-X_0\|}+\beta \frac{\|X_0-F(X_0)\|\|F(X_0)-X_0\|}{\|F(X_0)-X_0\|+\|X_0-F(X_0)\|}$$
$$+\gamma\,[\|F(X_0)-F(X_0)\|+\|X_0-X_0\|]+\delta\,[\|F(X_0)-X_0\|+\|X_0-F(X_0)\|]$$
$$+\eta\,\|F(X_0)-X_0\|$$

$$= (2\delta+\eta)\,\|F(X_0)-X_0\|+\frac{\beta}{2}\,\|F(X_0)-X_0\|$$

$$= (\frac{\beta}{2}+2\delta+\eta)\,\|X_0-Y_0\|$$

Therefore

$$\|F(X_0)-X_0\| \le (\frac{\beta}{2}+2\delta+\eta)\,\|X_0-Y_0\|$$

This is contradiction

Since $(\frac{\beta}{2}+2\delta+\eta)<1$

Therefore $F(X_0) = X_0$

I.e. X0 is fixed point of F, but $F(X_0) = G(X_0)$ and therefore we have $G(X_0) = X_0$

I.e. X0 is the common fixed point of F and G.

Now, we shall prove that X0 is the unique common fixed point of F and G. If possible let Y0 be another fixed point of F and G.

Now by (2.2.1), (2.2.2), (2.2.3), (A) and (B) we have

$$\|X_0 - Y_0\| = \|F^2(X_0) - F^2(Y_0)\| = \|FF(X_0) - FF(Y_0)\|$$

$$\leq \alpha \frac{\|FF(X_0)-GF(X_0)\|\|FF(Y_0)-GF(Y_0)\|}{\|GF(X_0)-FF(X_0)\|+\|GF(X_0)-FF(Y_0)\|} + \beta \frac{\|FF(Y_0)-GF(X_0)\|\|FF(X_0)-GF(Y_0)\|}{\|GF(X_0)-FF(Y_0)\|+\|GF(Y_0)-FF(X_0)\|} + \gamma [\|GF(X_0)-FF(X_0)\| + \|GF(Y_0)-FF(Y_0)\|] + \delta [\|GF(X_0)-FF(Y_0)\| + \|GF(Y_0)-FF(X_0)\|] + \eta \|GF(X_0)-GF(Y_0)\|$$

$$= \beta \frac{\|X_0-Y_0\|\|Y_0-X_0\|}{\|X_0-Y_0\|+\|Y_0-X_0\|} + \gamma [\|X_0-Y_0\| + \|Y_0-X_0\|] + \eta \|X_0-Y_0\|$$

$$= (\frac{\beta}{2}+2\delta+\eta)\|X_0-Y_0\|$$

Therefore $\|X_0-Y_0\| \leq (\frac{\beta}{2}+2\delta+\eta)\|X_0-Y_0\|$

Since $(\frac{\beta}{2}+2\delta+\eta)<1$ it follows that $X_0 = Y_0$ Proving the uniqueness of X0, the proof of theorem 2 is complete.

Theorem: 2.3 Let K be a closed and convex subset of a Banach space X. Let F, G and H be three mappings of X into itself such that

FG = GF, GH = HG and FH = HF (2.3.1)

$F^2 = I$, $G^2 = I$, $H^2 = I$, where I denotes the identify mapping (2.3.2)

$$\|F(X)-F(Y)\| \leq \alpha \frac{\|F(X)-GH(X)\|\|F(Y)-GH(Y)\|}{\|GH(X)-F(X)\|+\|GH(X)-F(Y)\|} + \beta \frac{\|F(Y)-GH(X)\|\|F(X)-GH(Y)\|}{\|GH(X)-F(Y)\|+\|GH(Y)-F(X)\|} + \gamma [\|GH(X)-F(X)\| + \|GH(Y)-F(Y)\|] + \delta [\|GH(X)-F(Y)\| + \|GH(Y)-F(X)\|] + \eta \|GH(X)-GH(Y)\|$$

For every X, Y ϵ K and $\alpha,\beta,\gamma,\delta,\eta \ge 0$ such that $7\alpha+8\beta+4\gamma<8$ then there exist at least onefixed point X_0 ϵX such that $F(X_0) = GH(X_0)$ and $FG(X_0) = H(X_0)$. Further if $(\frac{\beta}{2}+2\delta+\eta)<1$ then X_0 is the common fixed point of F, G and H.

Proof: From (2.3.1) and (2.3.2) it follows that $(FGH)^2=I$ where I is the identity mapping. And by (2.3.2) and (2.3.3) we have,

$$\begin{aligned}\|FGH.G(X)-FGH.G(Y)\| \le{}& \alpha\,\frac{\|FGH.G(X)-(GH)^2.G(X)\|\|FGH.G(Y)-(GH)^2.G(Y)\|}{\|FGH.G(X)-(GH)^2.G(X)\|+\|FGH.G(Y)-(GH)^2G(X)\|}\\ &+\beta\,\frac{\|FGH.G(Y)-(GH)^2.G(X)\|\|FGH.G(X)-(GH)^2G(Y)\|}{\|(GH)^2.G(X)-FGH.G(Y)\|+\|(GH)^2.G(Y)-FGH.G(X)\|}\\ &+\gamma\,[\|(GH)^2G(X)-FGH.G(X)\|+\|(GH)^2.G(Y)-FGH.G(Y)\|]\\ &+\delta\,[\|(GH)^2.G(X)-FGH.G(Y)\|+\|(GH)^2.G(Y)-FGH.G(X)\|]\\ &+\eta\,\|(GH)^2.G(X)-(GH)^2.G(Y)\|\end{aligned}$$

$$\begin{aligned}={}&\alpha\,\frac{\|G(X)-FGH.G(X)\|\|FGH.G(Y)-G(Y)\|}{\|FGH.G(X)-G(X)\|+\|FGH.G(Y)-G(X)\|}\\ &+\beta\,\frac{\|FGH.G(Y)-G(X)\|\|FGH.G(X)-G(Y)\|}{\|G(X)-FGH.G(Y)\|+\|G(Y)-FGH.G(X)\|}\\ &+\gamma\,[\|G(X)-FGH.G(X)\|+\|G(Y)-FGH.G(Y)\|]\\ &+\delta\,[\|G(X)-FGH.G(Y)\|+\|G(Y)-FGH.G(X)\|]+\eta\|G(X)-G(Y)\|\end{aligned}$$

Now if we put G(x) = Z and G(x) = W, we get

$$\begin{aligned}\|FGH(Z)-FGH(W)\| \le{}& \alpha\,\frac{\|FGH(Z)-Z\|\|FGH(W)-W\|}{\|FGH(Z)-Z\|+\|FGH(W)-Z\|}+\beta\,\frac{\|FGH(W)-Z\|\|FGH(Z)-W\|}{\|Z-FGH(W)\|+\|W-FGH(Z)\|}\\ &+\gamma\,[\|Z-FGH(Z)\|+\|W-FGH(W)\|]+\delta\,[\|Z-FGH(W)\|+\|W-FGH(Z)\|\\ &+\eta\,\|Z-W\|\end{aligned}$$

Where $(FGH)^2=I$ and $7\alpha+8\beta+4\gamma<8$

Then by theorem 1 , we write that FGH has at least one fixed point , say X_0 in K

i.e. $FGH(X_0)= X_0$ 3.(A)

and so $GH(FGH)(X_0)=GH(X_0)$ or $.F(X_0)=GH(X_0)$ 3.(B)

Also $H(FGH)(X_0)=H(X_0)$ or $.FG(X_0)=H(X_0)$ 3 (C)

Now by using (2.3.1), (2.3.2), (2, 3, 3) and 3(A), 3(B) and 3(C), we have

$$\|H(X_0)-X_0\| = \|FG(X_0)-F^2(X_0)\| = \|FG(X_0)-F(F(X_0))\|$$

$$\leq \alpha \frac{\|FG(X_0)-GHG(X_0)\|\|FF(X_0)-GHF(X_0)\|}{\|FG(X_0)-GHG(X_0)\|+\|FF(X_0)-GHG(Y_0)\|}$$
$$+\beta \frac{\|FF(X_0)-GHG(X_0)\|\|FG(X_0)-GHF(X_0)\|}{\|GHG(X_0)-FF(Y_0)\|+\|GHF(X_0)-FG(X_0)\|}$$
$$+\gamma [\|GHG(X_0)-FG(X_0)\|+\|GHF(X_0)-FF(X_0)\|]$$
$$+\delta [\|GHG(X_0)-FF(X_0)\|+\|GHF(X_0)-FG(X_0)\|]$$
$$+\eta \|GHG(X_0)-GHF(X_0)\|$$

$$= \alpha \frac{\|H(X_0)-H(X_0)\|\|X_0-X_0\|}{\|H(X_0)-H(X_0)\|+\|X_0-H(X_0)\|} + \beta \frac{\|X_0-H(X_0)\|\|H(X_0)-X_0\|}{\|H(X_0)-X_0\|+\|X_0-H(X_0)\|}$$
$$+\gamma [\|H(X_0)-H(X_0)\|+\|X_0-X_0\|]+\delta [\|H(X_0)-X_0\|+\|X_0-H(X_0)\|]$$
$$+\eta \|H(X_0)-X_0\|$$

$$= (\frac{\beta}{2}+2\delta+\eta)\ \|H(X_0)-X_0\|$$

Therefore,

$$\|H(X_0)-X_0\| \leq (\frac{\beta}{2}+2\delta+\eta)\ \|H(X_0)-X_0\|$$

Since $=(\frac{\beta}{2}+2\delta+\eta)<1$ it follows that $H(X_0)=X_0$

i.e. X_0 is the fixed point of H. Thus we have from 2.(b), $G(X_0)=F(X_0)$

Again,

$$\|F(X_0)-X_0\| = \|F(X_0)-F^2(X_0)\| = \|F(X_0)-F(F(X_0))\|$$
$$\leq \alpha \frac{\|F(X_0)-GH(X_0)\|\|FF(X_0)-GHF(X_0)\|}{\|F(X_0)-GH(X_0)\|+\|FF(X_0)-GH(X_0)\|}$$
$$+\beta \frac{\|FF(X_0)-GH(X_0)\|\|F(X_0)-GHF(X_0)\|}{\|GH(X_0)-FF(X_0)\|+\|GHF(X_0)-F(X_0)\|}$$
$$+\gamma [\|GH(X_0)-F(X_0)\|+\|GHF(X_0)-FF(X_0)\|]$$
$$+\delta [\|GH(X_0)-FF(X_0)\|+\|GHF(X_0)-F(X_0)\|]$$
$$+\eta \|GH(X_0)-GHF(X_0)\|$$

$$= \alpha \frac{\|F(X_0)-F(X_0)\|\|X_0-X_0\|}{\|F(X_0)-F(X_0)\|+\|X_0-F(X_0)\|} + \beta \frac{\|X_0-F(X_0)\|\|F(X_0)-X_0\|}{\|F(X_0)-X_0\|+\|X_0-F(X_0)\|}$$
$$+\gamma [\|F(X_0)-F(X_0)\|+\|X_0-X_0\|]+\delta [\|F(X_0)-X_0\|+\|X_0-F(X_0)\|]$$
$$+\eta \|F(X_0)-X_0\|$$

$$= (\frac{\beta}{2}+2\delta+\eta)\ \|F(X_0)-X_0\|$$

Therefore

$$\|F(X_0)-X_0\| \le (\frac{\beta}{2}+2\delta+\eta)\ \|F(X_0)-X_0\|$$

Which is a contradiction ,Since $(\frac{\beta}{2}+2\delta+\eta)<1$ Hence, it follows that $F(X_0)=X_0$ But $F(X_0)=G(X_0)$

Therefore, $F(X_0)=G(X_0)=H(X_0)=X_0$

i.e. X_0 is the common fixed point of F,G and H.

Now to show the uniqueness of X_0 , we let Y_0 be another common fixed point of F, G and H.

Using (2.3.1),(2.3.2),(2.3.3) and 3.(a),3.(b),3.(c), we get

$$\|X_0-Y_0\| = \|F^2(X_0)-F^2(Y_0)\| = \|FF(X_0)-FF(Y_0)\|$$

$$\le \alpha\ \frac{\|FF(X_0)-GHF(X_0)\|\|FF(Y_0)-GHF(Y_0)\|}{\|FF(X_0)-GHF(X_0)\|+\|FF(Y_0)-GHF(X_0)\|}$$

$$+\ \beta\ \frac{\|FF(Y_0)-GHF(X_0)\|\|FF(X_0)-GHF(Y_0)\|}{\|GHF(X_0)-FF(Y_0)\|+\|GHF(Y_0)-FF(X_0)\|}$$

$$+\ \gamma\,[\|GHF(X_0)-FF(X_0)\|+\|GHF(Y_0)-FF(Y_0)\|\,]$$

$$+\delta\,[\|GHF(X_0)-FF(Y_0)\|+\|GHF(Y_0)-FF(X_0)\|\,]+\eta\|GHF(X_0)-GHF(Y_0)\|$$

$$=\frac{\beta}{2}\|X_0-Y_0\|+2\delta\ \|X_0-Y_0\|+\eta$$

$$=(\frac{\beta}{2}+2\delta+\eta)\ \|X_0-Y_0\|$$

Therefore

$$\|X_0-Y_0\| \le (\frac{\beta}{2}+2\delta+\eta)\|X_0-Y_0\|$$

Which is a contradiction, since $(\frac{\beta}{2}+2\delta+\eta)<1$

Hence, it follows that $X_0=Y_{0,}$

Proving the uniqueness of $X_{0.}$

This completes the proof of the theorem.

REFERENCES

[1] Browder, F. E. "Non expansive non linear operators in Banach Spaces," proc. Nat. Acad. Sci. U. S. A., Vol- 54, 1965, 1041-1044.

[2] Bryant V.W. "A remark on a fixed point theorem for iterated Mapping," Amer. Math. Soc. Mont. 75 (1968) 399-400.

[3] Dotson W. G. Jr.," Fixed points of quasi non- expansive mappings." Jour. Austral. Math. Soc. Vol-13, 1972, 167-170.

[4] Emmanuele G. "Fixed point theorems in complete metric space" Non – linear Anal. Vol- 5, 1981, 287-292.

[5] Gohde, D. "Zum prinzip dev kontraktiven abbildung" Math. Nachr. Vol- 30 1965, 251-258.

[6] Goebel K. " An elementary proof of the fixed point theorem of Browder and Kirk", Michigan Math. J. Vol- 16, 1969, 381-383.

[7] Goebel K. & Zlitkiewicz E. "Some fixed point theorem is Banach Spaces" Colloq Math. Vol. 23, 1971, 103-106

[8] Goebel K., Kirk W.A., & Shimi T. N., " A fixed point theorem in uniformly convex spaces." Boll. Un. Math. Italy, Vol.- 4 1973, 67-75.

[9] Iseki, K. "Fixed point theorem in Banach Spaces", Math. Semi. Notes. Kobe. Univ. Vol.- 2(1),1977, paper No. 3, 4pp. MR 136421.

[10] Kirk, W.A., "A fixed point theorem for non expansive mappings II" Contemp. Math. Vol.- 18, 1983, 121-140.

[11] Khan M. S. "Fixed point and their approximation in Banach Spaces for certain mappings." Glasgow Math. Jour. Vol. - 23, 1982, 1-6.

[12] Kirk, W.A., "A fixed point theorem for non expansive mappings", Lecture notes in Math. Springer- Verlag, Berlin and New York, Vol.- 886, 1981, 484- 505

[13] Massa S. & Roux. D., "A fixed point theorem for generalized non expansive mappings." Bull. Univ. Math. Italy. Vol. - 5, 15 A, 1978, 654- 634.

[14] Rhoades B.E.,"Some fixed point theorems for generalized non- expansive mappings in Non- linear Analysis and Application." Lecture notes in pure Appl. Math. Vol. - 80, 1982, Marcel- Dekkar, 223- 228.

[15] Sharma P.L. & Rajput, S.S.' "Fixed point theorem in Banach Spaces." Vikram Math. Jour. Vol. - 4, 1983, 35.

[16] Singh M. R & Chaterjee, "Fixed point theorems in Banach space."Pure Math. Manu. Vol. - 6, 1987, 53- 61.

[17] Yadav R.N, Rajput S.S, Bhardwaj R," Some fixed point theorems in banach space" Acta Ciencia Indica, Vol. XXXIII 2, 2007, 453.

[18] Jain R. K. & Jain R., "A result on fixed points in Banach Space." Acta Indica, Vol. - XV M. No. 3, 297, 1989.

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