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FORMATION OF SOME SUMMATION FORMULAE INVOLVING HYPERGEOMETRIC FUNCTION

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ABSTRACT

The main object of this paper is to establish some summation formuale involving Gauss second summation theorem .The results derived in this paper are of general character.

Key words and Phrases: *Contiguous relation,Recurrence relation, Gauss second summation theorem .*

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A. Introduction:

The Pochhammer's symbol is defined by

$$(\alpha,\ k) = (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 ; & \text{if } k = 0 \\ k! ; & \text{if } \alpha = 1 \end{cases} \tag{1}$$

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Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lc} a_1, a_2, \cdots, a_A \;; & \\ & z \\ b_1, b_2, \cdots, b_B \;; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{lc} (a_A) \;; & \\ & z \\ (b_B) \;; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lc} (a_j)_{j=1}^{A} \;; & \\ & z \\ (b_j)_{j=1}^{B} \;; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{2} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation is defined by

[Andrews p.363(9.16), E. D. p.51(10), H.T. F. I p.103(32)]

$$ (a-b)\ {}_2F_1\left[\begin{array}{cc} a,\ b\ ; & z \\ c\ ; & \end{array}\right] = a\ {}_2F_1\left[\begin{array}{cc} a+1,\ b\ ; & z \\ c\ ; & \end{array}\right] - b\ {}_2F_1\left[\begin{array}{cc} a,\ b+1\ ; & z \\ c\ ; & \end{array}\right] \tag{3} $$

Legendre's duplication formula

$$ \sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z + \frac{1}{2}\right) \tag{4} $$

$$ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{5} $$

$$ = \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{6} $$

Recurrence relation is defined by

$$ \Gamma(z+1) = z\ \Gamma(z) \tag{7} $$

Gauss second summation theorem is defined by [Prud., 491(7.3.7.5)]

$$ {}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b+1}{2} & ; \end{array} \quad \frac{1}{2}\right] = \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \tag{8} $$

$$ = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})} \tag{9} $$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prud., p.491(7.3.7.3)]

$$ {}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b-1}{2} & ; \end{array} \quad \frac{1}{2}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\ \Gamma(b)}\right] \tag{10} $$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b-1}{2} & ; \end{array} \quad \frac{1}{2}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{11} $$

B. MAIN RESULTS OF SUMMATION FORMULAE

$$ {}_2F_1\left[\begin{array}{cc} a,\ \ b & ; \\ \frac{a+b+6}{2} & ; \end{array} \quad \frac{1}{2}\right] = \frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b)^2\ \Gamma(b)} \times $$

$$ \times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(-8a-4a^2+4a^3+8b+40ab+20a^2b-4b^2-20ab^2-4b^3)}{(a-b+4)(a-b+2)(a-b+1)(a-b-2)} + \right.\right. $$

$$ \left. + \frac{(-16ab+16a^2b-16b^2-16b^3)}{(a-b+2)(a-b-1)(a-b-2)(a-b-4)}\right\} - $$

$$ -\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(16a^2+16a^3+16ab-16ab^2)}{(a-b+4)(a-b+2)(a-b+1)(a-b-2)} + \right. $$

$$ \left.\left. + \frac{(-8a+4a^2+4a^3+8b-40ab+20a^2b+4b^2-20ab^2-4b^3)}{(a-b+2)(a-b-1)(a-b-2)(a-b-4)}\right\}\right] \tag{12} $$

$$
{}_2F_1\left[\begin{matrix} a, \quad b & ; & \frac{1}{2} \\ \frac{a+b+7}{2} & ; & \end{matrix}\right] = \frac{2^b\ \Gamma(\frac{a+b+7}{2})}{(a-b)\ \Gamma(b)}\times
$$

$$
\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(4a(a^2+5b^2+10ab-4a+3))}{(a-b+5)(a-b+3)(a-b+1)(a-b-1)(a-b-3)}+\right.\right.
$$

$$
\left.+\frac{4b(b^2+5a^2+10ab-4b+3)}{(a-b+3)(a-b+1)(a-b-1)(a-b-3)(a-b-5)}\right\}-
$$

$$
-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{8(5a^2+b^2+10ab+4b+3)}{(a-b+5)(a-b+3)(a-b+1)(a-b-1)(a-b-3)}+\right.
$$

$$
\left.\left.+\frac{8(5b^2+a^2+10ab+4a+3)}{(a-b+3)(a-b+1)(a-b-1)(a-b-3)(a-b-5)}\right\}\right] \tag{13}
$$

$$
{}_2F_1\left[\begin{matrix} a, \quad b & ; & \frac{1}{2} \\ \frac{a+b+8}{2} & ; & \end{matrix}\right] = \frac{2^b\ \Gamma(\frac{a+b+8}{2})}{(a-b)\ \Gamma(b)}\times
$$

$$
\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(8(8a-6a^2+a^3+8b+15a^2b+6b^2+15ab^2+b^3)}{(a-b+6)(a-b+4)(a-b+2)(a-b)(a-b-2)(a-b-4)}+\right.\right.
$$

$$
\left.+\frac{16b(8+2a+3a^2-2b+10ab+3b^2)}{(a-b+4)(a-b+2)(a-b)(a-b-2)(a-b-4)(a-b-6)}\right\}-
$$

$$
-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{16a(8-2a+3a^2+2b+10ab+3b^2)}{(a-b+6)(a-b+4)(a-b+2)(a-b)(a-b-2)(a-b-4)}+\right.
$$

$$
\left.\left.+\frac{8(8a+6a^2+a^3+8b+15a^2b-6b^2+15ab^2+b^3)}{(a-b+4)(a-b+2)(a-b)(a-b-2)(a-b-4)(a-b-6)}\right\}\right] \tag{14}
$$

C. DERIVATIONS OF SUMMATION FORMULAE (12) TO (14):

Derivation of (12): Substituting $c=\frac{a+b+6}{2}$ and $z=\frac{1}{2}$ in equation (2), we get

$$
(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b & ; & \frac{1}{2} \\ \frac{a+b+6}{2} & ; & \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b & ; & \frac{1}{2} \\ \frac{a+b+6}{2} & ; & \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1 & ; & \frac{1}{2} \\ \frac{a+b+6}{2} & ; & \end{matrix}\right]
$$

Now with the help of Gauss second summation theorem, we get

$$L.H.S = a\ \frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b+1)\Gamma(b)}\left[\frac{4\ \Gamma(\frac{b}{2})}{a\ \Gamma(\frac{a}{2})}\left\{\frac{(a^2+3ab+a+3b)}{(a-b+4)(a-b+2)(a-b)}+\right.\right.$$

$$\left.+\frac{(b^2+3ab+2b)}{(a-b+2)(a-b)(a-b-2)}\right\}-\frac{2\ \Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(6a+2b+8)}{(a-b+4)(a-b+2)(a-b)}+\right.$$

$$\left.\left.+\frac{(2a+6b+4)}{(a-b+2)(a-b)(a-b-2)}\right\}\right]-$$

$$-\frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b-1)\Gamma(b)}\left[\frac{4\ \Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(a^2+3ab+2a)}{(a-b+2)(a-b)(a-b-2)}+\frac{(b^2+3ab+b+3a)}{(a-b)(a-b-2)(a-b-4)}\right\}-\right.$$

$$\left.-\frac{2b\ \Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(6a+2b+4)}{(a-b+2)(a-b)(a-b-2)}+\frac{(2a+6b+8)}{(a-b)(a-b-2)(a-b-4)}\right\}\right]$$

$$=\frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b+1)\Gamma(b)}\left[\frac{4\ \Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(a^2+3ab+a+3b)}{(a-b+4)(a-b+2)(a-b)}+\right.\right.$$

$$\left.+\frac{(b^2+3ab+2b)}{(a-b+2)(a-b)(a-b-2)}\right\}-\frac{2\ \Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(6a^2+2ab+8a)}{(a-b+4)(a-b+2)(a-b)}+\right.$$

$$\left.\left.+\frac{(2a^2+6ab+4a)}{(a-b+2)(a-b)(a-b-2)}\right\}\right]-$$

$$-\frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b-1)\Gamma(b)}\left[\frac{4\ \Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(a^2+3ab+2a)}{(a-b+2)(a-b)(a-b-2)}+\frac{(b^2+3ab+b+3a)}{(a-b)(a-b-2)(a-b-4)}\right\}-\right.$$

$$\left.-\frac{2\ \Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(6ab+2b^2+4b)}{(a-b+2)(a-b)(a-b-2)}+\frac{(2ab+6b^2+8b)}{(a-b)(a-b-2)(a-b-4)}\right\}\right]$$

$$=\frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b+1)\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(4a^2+12ab+4a+12b)}{(a-b+4)(a-b+2)(a-b)}+\right.\right.$$

$$\left.+\frac{(4b^2+12ab+8b)}{(a-b+2)(a-b)(a-b-2)}\right\}-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(12a^2+4ab+16a)}{(a-b+4)(a-b+2)(a-b)}+\right.$$

$$+\frac{(4a^2+12ab+8a)}{(a-b+2)(a-b)(a-b-2)}\Bigg\}\Bigg]-$$

$$-\frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b-1)\Gamma(b)}\left[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(4a^2+12ab+8a)}{(a-b+2)(a-b)(a-b-2)}+\frac{(4b^2+12ab+4b+12a)}{(a-b)(a-b-2)(a-b-4)}\right\}-\right.$$

$$\left.-\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(12ab+4b^2+8b)}{(a-b+2)(a-b)(a-b-2)}+\frac{(4ab+12b^2+16b)}{(a-b)(a-b-2)(a-b-4)}\right\}\right]$$

On simplification ,we get

$${}_2F_1\left[\begin{array}{cc} a,\ \ b & ; \quad 1 \\ \frac{a+b+6}{2} & ; \quad 2 \end{array}\right]=\frac{2^b\ \Gamma(\frac{a+b+6}{2})}{(a-b)^2\ \Gamma(b)}\times$$

$$\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{(-8a-4a^2+4a^3+8b+40ab+20a^2b-4b^2-20ab^2-4b^3)}{(a-b+4)(a-b+2)(a-b+1)(a-b-2)}+\right.\right.$$

$$\left.+\frac{(-16ab+16a^2b-16b^2-16b^3)}{(a-b+2)(a-b-1)(a-b-2)(a-b-4)}\right\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{(16a^2+16a^3+16ab-16ab^2)}{(a-b+4)(a-b+2)(a-b+1)(a-b-2)}+\right.$$

$$\left.\left.+\frac{(-8a+4a^2+4a^3+8b-40ab+20a^2b+4b^2-20ab^2-4b^3)}{(a-b+2)(a-b-1)(a-b-2)(a-b-4)}\right\}\right]$$

Thus , we prove the result (12).

Similarly, we can prove the other results.

REFERENCES

1. Arora, Asish, Singh, Rahul , Salahuddin. ; Development of a family of summation formulae of half argument using Gauss and Bailey theorems *Journal of Rajasthan Academy of Physical Sciences.*, 7(2008), 335-342.

2. Lavoie, J. L.; Some summation formulae for the series ${}_3F_2$, *Math. Comput.*, 49(1987), 269-274.

3. Lavoie, J. L., Grondin, F. and Rathie, A.K.; Generalizations of Watson's theorem on the sum of a ${}_3F_2$, *Indian J. Math.*, 34(1992), 23-32.

4. Lavoie, J. L., Grondin, F. and Rathie, A.K.; Generalizations of Whipple's theorem on the sum of a ${}_3F_2$, *J. Comput. Appl. Math.*, 72(1996), 293-300.

5. Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O.I.; *Integrals and Series Vol. 3: More Special Functions.* Nauka, Moscow, 1986. Translated from the Russian by G.G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.

6. Rainville, E. D.; The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n\ (\zeta, p, \nu)$, *Bull. Amer. Math. Soc.*, 51(1945), 714-723.

7. Salahuddin, Chaudhary, M.P ; Development of some summation formulae using Hypergeometric function, *Global journal of Science Frontier Research*, 10(2010),36-48.

8. Salahuddin, Chaudhary, M.P ; Certain summation formulae associated to Gauss second summation theorem, *Global journal of Science Frontier Research*, 10(2010),30-35.