DUAL STONE ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

In this paper, the dual Stonity of a dual pseudo-complemented ADL is defined and proved that it is a sub ADL. The concept of a dual Stone Almost Distributive Lattice is introduced. Several necessary and sufficient conditions for an Almost Distributive Lattice (ADL) to become a dual Stone ADL are obtained.

Keywords: Almost Distributive Lattice (ADL); Birkhoff center; Principal ideal; Dual pseudo-complementation; Dual Stonity; Dual Stone ADL.

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1. INTRODUCTION

Boolean algebra is a homomorphic image of the algebra \( C(X, 2) \) of all continuous functions mappings of a compact Hausdorff and totally disconnected topological space \( X \) onto the discrete two element Boolean algebra \( 2 \). In place of the algebra \( 2 \), if we take a dense bounded distributive lattice \( D \), then \( C(X, D) \) and its homomorphic images represent a special class of pseudo-complemented distributive lattices, namely Stone lattices. A pseudo-complemented distributive lattice \( A \) is said to be a Stone lattice if \( x^* \lor x^{**} = 0^* \) for all \( x \in A \). The concept of Almost Distributive Lattice(ADL) was introduced in [5] as a common abstraction of the most of the existing ring theoretic generalization of a Boolean algebra on one hand and the class of distributive lattice on the other. Later, in [6], pseudo-complementation in an ADL was introduced and the properties of Stone ADL were studied in [7]. Unlike in lattices, the dual of an ADL in not an ADL in general. For this reason, we introduced the concept of dual pseudo-complemented ADL (Dual PCADL) in [3] and studied its properties. In this paper, we define the concept of dual Stonity is a dual PCADL and derive the inter-relation between different dual pseudo-complementations in an ADL \( A \) and observe that the dual Stonity of \( A \) under different pseudo-complementations is same. We derive necessary and sufficient conditions for an ADL become a dual Stone ADL.

2. PRELIMINARIES

In this section, we give the necessary definitions and important properties of an ADL taken from [5] for ready reference.

**Definition 2.1:** [5] An algebra \( (A, V, \wedge, 0) \) of type \( (2, 2, 0) \) is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:

(i) \( x \lor 0 = x \)
(ii) \( 0 \wedge x = 0 \)
(iii) \( (x \lor y) \wedge z = (x \wedge z) \lor (y \wedge z) \)
(iv) \( x \wedge (y \lor z) = (x \wedge y) \lor (x \wedge z) \)
(v) \( x \lor (y \wedge z) = (x \lor y) \wedge (x \lor z) \)
(vi) \( (x \lor y) \wedge y = y \) for all \( x, y, z \in A \).
A non-empty subset \( I \) of an ADL \( A \) is called an ideal of \( A \) if \( x \lor y \in I \) and \( x \land a \in I \) for any \( x, y \in I \) and \( a \in A \). The principal ideal of \( A \) generated by \( x \) is denoted by \( (x) \) and it is given by \( (x) = \{ x \land a : a \in A \} \). The set \( PI(A) \) of all principal ideals of \( A \) forms a distributive lattice under the operations \( \lor, \land \) defined by \( (x) \lor (y) = (x \lor y) \) and \( (x) \land (y) = (x \land y) \) in which \( 0 \) is the least element. If \( A \) has a maximal element \( m \), then \( (m) \) is the greatest element of \( PI(A) \). We extended many existing concept in the class of distributive lattices to the class of ADL, through the principal ideal lattice \( PI(A) \). For other properties of ADLs we refer to [5]. In [6], the concept of Pseudo-Complementation in an ADL was defined as follows.

**Definition 2.2:** [6] Let \((A, V, \Lambda, 0)\) be an ADL. Then a unary operation \( x \rightarrow x^* \) on \( A \) is called a pseudo-complementation on \( A \) if, for any \( x, y \in A \), it satisfies the following conditions:

(i) \( x \land y = 0 \Rightarrow x^* \land y = y \).

(ii) \( x \land x^* = 0 \).

(iii) \( (x \lor y)^* = x^* \land y^* \).

An ADL \((A, V, \Lambda, 0)\) together with a pseudo-complement is called a pseudo-complementation on \( A \). The concept of a Stone ADL is defined in [7] as follows.

**Definition 2.3:** [7] Let \((A, V, \Lambda, 0)\) be an ADL with a pseudo-complement \( \ast \). Then \( A \) is called a Stone Almost Distributive Lattice (Stone ADL) if, for any \( x \in A \), \( x^* \lor x^{**} = 0^* \).

For other properties of PCADL, Stone ADL we refer to [6], [7].

**3. DUAL STONITY DS (A)**

The concept of pseudo-complementation in an ADL \( A \) was introduced in [6] and it was observed that, the set \( A^* \) of all pseudo-complementation of elements of \( A \) forms a Boolean algebra with respect to the same partial ordering of \( A \). A pseudo-complemented ADL \((A, \lor, \land, *, 0)\) is called a Stone ADL if \( x^* \lor x^{**} = 0^* \) for all \( x \in A \). We begin with the following.

**Definition 3.1:** [3] Let \((A, V, \Lambda, 0)\) be an ADL. Then a unary operation \( * \) on \( A \) is called a dual pseudo-complementation on \( A \) if, for any \( x, y \in A \), it satisfies the following conditions:

\( d_1 \): if \( x \lor y = m \), then \( (x \lor y)^* \lor m = y \lor m \).

\( d_2 \): \( x \lor x^* \) is a maximal element of \( A \).

\( d_3 \): \( (x \land y)^* = x \land y^* \).

An ADL \( A \) with a dual pseudo-complementation is called a dual Pseudo-Complemented Almost Distributive Lattice (or simply dual PCADL). We state the important properties of dual PCADL, taken from [3] in the following theorem.

**Theorem 3.2:** [3] Let \( A \) be an ADL and \( * \) be a dual pseudo-complementation on \( A \). For any \( x, y \in A \), we have the following:

(i) \( x_* = x_{**} \).

(ii) \( (x \lor y)^{**} = x_{**} \lor y_{**} \).

(iii) \( (x \land m)^{**} = x_* \).

(iv) If \( x \leq y \), then \( y_* \leq x_* \).

Unlike in distributive lattices, an ADL can have more than one dual pseudo-complementation. The following theorem is taken from [3] will be used in this paper.
Theorem 3.3: [3] Let \( A \) be an ADL and \(*\) and \( \bot\) be a dual pseudo-complementation on \( A \). For any \( x, y \in A \), we have the following:

(i) \( x_\bot = x_\bot \).
(ii) \( x_\ast = y_\ast \Leftrightarrow x_\bot = y_\bot \).
(iii) \( x_\ast \land x_\ast = 0 \Leftrightarrow x_\bot \land x_\bot = 0 \).

For other properties of a dual PCADL, we refer to [3]. The following definition of dual annihilator of a subset of \( A \) is taken from [4].

Definition 3.4: [4] Let \( A \) be an ADL with a maximal element \( m \) and \( S \) is any non-empty subset of \( A \). Write \( S^+ = \{a \in A \mid s \land a \text{ is maximal for all } s \in S\} \). Then \( S^+ \) is called the dual annihilator of \( S \) in \( A \) and it can be verified that \( S^+ \) is a filter of \( A \). We write \( S^\ast \) for \( S^+ \) if \( S = \{ \} \).

Now we prove the following theorem.

Theorem 3.5: Let \( A \) be a dual PCADL. Then, for any \( x \in A, x^\ast = [x_\ast] \).

Proof: Since, for any \( x \in A, x \lor x_\ast \) is maximal, we get \( x_\ast \in x^\ast \) and hence \( [x_\ast] \subseteq x^\ast \). On the other hand, if \( y \in x^\ast \), then \( x \lor y \) is maximal and hence \( (x_\lor y) \land m = y \land m \).

Now \( y \land x_\ast = y \land m \land x_\ast = (x_\lor y) \land m \land x_\ast = x_\ast \) and hence \( y \lor x_\ast = y \). Thus \( y \in [x_\ast] \). Hence \( x^\ast \subseteq [x_\ast] \).

Corollary 3.6: Let \( A \) be an ADL with a maximal element \( m \) and \(*, \bot\) be dual pseudo-complementations on \( A \). Then, for any \( x \in A, x^\ast = [x_\bot] \).

If \(*\) is a dual pseudo-complementation on \( A \), then it was proved in [3], that \( A_* = \{x_\ast \land m \mid x \in A\} \) is a Boolean algebra under operations \( \lor, \land \) where for any \( x_\ast \land m, y_\ast \land m \in A_*, (x_\ast \land m) \lor (y_\ast \land m) = (x_\ast \lor y_\ast) \land m \).

Now we prove the following.

Theorem 3.7: Let \( A \) be an ADL with a dual pseudo-complementation \(*\). Then the map \( f : A \rightarrow A_* \) defined by \( f(x) = x_\ast \land m \) is an epimorphism.

Proof: Let \( x, y \in A \). Then \( f(x \land y) = [x \land y]_\ast \land m = [x_\ast \lor y_\ast]_\ast \land m = (x_\ast \lor y_\ast) \land m = f(x) \lor f(y) \) and \( f(x \lor y) = (x \lor y)_\ast \land m = (x_\ast \land m) \lor (y_\ast \land m) = f(x) \lor f(y) \).

Also \( f(x_\ast) = x_\ast \land m = x_\ast \land m \). Hence \( f \) is an epimorphism.

Theorem 3.8: Let \( A \) be an ADL with two dual pseudo-complementations \(*\) and \( \bot\). Then the map \( f : A \rightarrow A_\bot \) defined by \( f(x \land m) = x_\bot \land m \) is an isomorphism of the Boolean algebras \( A_\ast \) and \( A_\bot \).
Proof: By Theorem 3.3, the map \( f : A \rightarrow A \) defined by \( f(x \wedge m) = x \wedge m \) for all \( x \wedge m \in A \), is a bijection. Let \( x, y \in A \). Then
\[
\begin{align*}
  f\left((x \wedge m) \wedge (y \wedge m)\right) &= f\left((x \wedge y) \wedge m\right) \\
  &= (x \wedge (y \wedge m)) \wedge m \\
  &= (x \wedge y) \wedge m \\
  &= (x \wedge y) \wedge (y \wedge m) \\
  &= f(x \wedge m) \wedge f(y \wedge m)
\end{align*}
\]
and
\[
\begin{align*}
  f\left((x \wedge m) \vee (y \wedge m)\right) &= f\left((x \wedge y) \wedge m\right) \\
  &= (x \wedge (y \wedge m)) \vee m \\
  &= (x \wedge y) \wedge m \\
  &= (x \wedge y) \wedge (y \wedge m) \\
  &= f(x \wedge m) \vee f(y \wedge m)
\end{align*}
\]

Definition 3.9: Let \( A \) be a dual PCADL with a maximal element \( m \). Then the set \( \{x \in A \mid x \wedge x_\ast = 0\} \) is called the Dual Stonity of \( A \) and it is denoted by \( DS(A) \).

In view of Theorem 3.3, we get that the set \( DS(A) \) is independent of the dual pseudo-complementation on \( A \). Now we prove the following.

Theorem 3.10: Let \( A \) be a dual PCADL. Then \( DS(A) \) is a sub ADL of \( A \), closed under dual pseudo-complementation.

Proof: If \( x \in DS(A) \), then \( x_\ast \wedge x_\ast = x_\ast \wedge x_\ast = 0 \) and hence \( x \in DS(A) \). Let \( x, y \in A \). Then
\[
\begin{align*}
  (x \wedge y)_\ast \wedge (x \wedge y)_\ast &= (x \wedge y)_\ast \wedge (x \wedge y)_\ast \\
  &= \left((x \wedge y)_\ast \wedge (y \wedge x)_\ast \right) \\
  &\leq (x_\ast \wedge x_\ast) \vee (y_\ast \wedge y_\ast) = 0
\end{align*}
\]
and
\[
\begin{align*}
  (x \vee y)_\ast \wedge (x \vee y)_\ast &= (x \vee y)_\ast \wedge (x \vee y)_\ast \\
  &= \left((x \vee y)_\ast \wedge (y \vee x)_\ast \right) \\
  &\leq (x_\ast \wedge x_\ast) \vee (y_\ast \wedge y_\ast) = 0
\end{align*}
\]
Therefore \( x \wedge y, x \vee y \in DS(A) \). Also, since \( 0, 0_\ast = 0_\ast \wedge 0 = 0 \), we get \( 0 \in DS(A) \). Hence \( DS(A) \) is a sub ADL of \( A \).

4. DUAL STONE ADL

The concept of Stone lattice was given by T.P. Speed. Later, this concept was extended to the class of ADLs in [7]. In this section, we introduce the concept of a dual Stone ADL and study its properties. We derive necessary and sufficient conditions for an Dual PCADL to become a Dual Stone ADL. We begin with the following definition.

Definition 4.1: An ADL \( A \) with a dual pseudo-complementation \( * \) is called a dual Stone ADL if \( DS(A) = A \), or equivalently, \( x_\ast \wedge x_\ast = 0 \) for all \( x \in A \). Here afterwards \( A \) stands for an ADL \( (A, V, \Lambda, 0) \) with a dual pseudo-complementation \( * \) and with a maximal element \( m \).
Theorem 4.2: \( A \) is a dual Stone ADL if and only if \( (DS(A))^* = A \).

Proof: Suppose \( (DS(A))^* = A \). Let \( x \in A \). Then \( x^* \in A \) and hence \( x^* \land x^* = x^* \). Thus \( x \in DS(A) \). Therefore \( A \) is a dual Stone ADL. Conversely, it is trivial.

Theorem 4.3: The following are equivalent:
(i) \( A \) is a dual Stone ADL.
(ii) \( x \lor y = A \) whenever \( x, y \in A \) and \( x \lor y \) is maximal.
(iii) \( x \land y = 0 \) whenever \( x, y \in A \) and \( x \lor y \) is maximal.

Proof: (i) \( \Rightarrow \) (ii): Suppose \( A \) is a dual Stone ADL and let \( x, y \in A \) such that \( x \lor y \) is maximal. Then \( (x \lor y) \land m = m \) and hence \( (x \lor y) \land m = y \land m \). Thus \( y \land x^* = x^* \).

Now \( (x^* \lor y) \land m = (y \land x^*) \lor y \land m = (x^* \lor y) \land m = m \). So that \( x^* \lor y \) is maximal and hence \( x^* \in y^+ \). Since \( x \lor x^* \) is maximal, we get \( x^* \in x^+ \). Thus \( 0 = x^* \land x^* \in x^+ \lor y^+ \).

Hence \( x^* \lor y^+ = A \). (ii) \( \Rightarrow \) (iii): Let \( x, y \in A \) such that \( x \lor y \) is maximal. Then \( 0 = s \land t \) for some \( s \in x^+ \) and \( t \in y^+ \). So that \( s \lor x \) is maximal and hence \( s \land x^* = x^* \). Similarly, \( t \land y^* = y^* \). Hence \( x^* \land y^* = s \land x^* \land t \land y^* = 0 \). (iii) \( \Rightarrow \) (i): If \( x \in A \), then \( x \lor x^* \) is maximal, and hence \( x \land x^* = 0 \). Thus \( A \) is a dual Stone ADL.

Theorem 4.4: Let \( A \) be a dual Stone ADL. Then, for any \( x, y \in A \), \( (x \lor y)^* \land m = x^* \land y^* \land m \).

Proof: Since \( x^* \land m \leq (x \lor y)^* \land m \), we get \( (x \lor y)^* \land m \leq x^* \land m \). Similarly, we get \( (x \lor y)^* \land m \leq y^* \land m \).

Hence \( (x \lor y)^* \land m \leq x^* \land y^* \land m \). On the other hand, we have
\[
\begin{align*}
x^* \land y^* \land (x \lor y)^* & = (x^* \land y^* \land x^*) \lor (x^* \land y^* \land y^*) = 0, \\
\Rightarrow (x^* \land y^* \land (x \lor y)^*) & = 0 \lor (x^* \land y^*), \\
\Rightarrow [x^* \land y^* \lor (x \lor y^*)] \land [(x \lor y)^* \lor (x \lor y^*),] \land m & = (x \lor y)^* \land m, \\
\Rightarrow (x^* \land y^* \lor (x \lor y^*)) \land m & = (x \lor y^* \land m.
\end{align*}
\]

Hence \( x^* \land y^* \land m \leq (x \lor y^*) \land m \). Thus we get the result.

Theorem 4.5: \( A \) is a dual Stone ADL if and only if \( (x \lor y)^* \land m = x^* \land y^* \land m \) for all \( x, y \in A \).

Proof: Suppose \( A \) is a dual Stone ADL and \( x, y \in A \). Then
\[
\begin{align*}
(x \lor y)^* \land m & = (x \lor y)^* \land m, \\
& = (x^* \lor y^*) \land m, \\
& = x^* \land y^* \land m (\text{by theorem 4.4}) \\
& = x^* \land y^* \land m.
\end{align*}
\]

Conversely, suppose, for any \( x, y \in A \), \( (x \lor y)^* \land m = x^* \land y^* \land m \). Since \( x \lor x^* \) is maximal, we get \( (x \lor x^*) \land m = m \). Then \( m \land m = \left((x \lor x^*) \land m \right) \land m \)
\[ = \left( (x \lor x_*) \lor m_ \right) \land m \]
\[ = (x_ \land x_*) \land m. \]

Therefore \( x_ \land x_* = 0 \) and hence \( A \) is a dual Stone ADL.

**Theorem 4.6:** Let \( A \) be a dual Stone ADL. Write \( A_* = \{ x_ \land m \mid x \in A \} \). Then \((A_*, \lor, \land, *, 0, 0_ \land m)\) is a Boolean algebra.

**Proof:** Clearly, \( A_* \) is a bounded distributive lattice. Let \( x_ \land m \in A_* \). Now \( x_ \land m \land x_* \land m = x_ \land x_* \land m = 0 \)
and \((x_ \land m) \lor (x_* \land m) = (x_ \land x_*) \land m = m \). Hence \( A_* \) is a Boolean algebra.

Now we prove the following.

**Note 4.7:** Since the dual of a distributive lattice is again a distributive lattice, we say that a distributive lattice \((A, V, \Lambda)\) is a dual Stone lattice if its dual \((A_*, \Lambda, V)\) is a Stone lattice. Now we prove the following.

**Theorem 4.8:** Let \( A \) be a dual PCADL with a maximal element \( m \). Then the following are equivalent:

(i) \( A \) is a dual Stone ADL.

(ii) \([a, a \lor m]\) is a dual Stone lattice for all \( a \in A \).

(iii) \([0, m]\) is a dual Stone lattice.

**Proof:** (i) \( \Rightarrow \) (ii): Suppose \( A \) is a dual Stone ADL and \( a \in A \). For any \( x \in A \), define \( x_ = a \lor (x_ \land m) \). Then by Theorem 3.11 [3], we get \( x_ \land x_ = [a \lor (x_ \land m)] \land [a \lor (x_* \land m)] = a \lor (x_ \land x_* \land m) = a \lor 0 = a \). Hence \([a, a \lor m]\) is a dual Stone lattice. (ii) \( \Rightarrow \) (iii) is trivial. (iii) \( \Rightarrow \) (i): Suppose \([0, m]\) is a dual Stone lattice under the unary operation \( \bot \). For \( x \in A \), define \( x_* = (x \land m)_ \). Then by Theorem 3.11 [3], we get \( A \) is a dual PCADL. Now, for any \( x \in A \),
\[ x_ \land x_* = (x \land m)_ \land ((x_ \land m)_ \land m)_ = x_ \land (x_ \land m)_ = x_ \land x_ = 0 \]. Hence \( A \) is a dual Stone ADL.

Finally, we conclude this paper with the following.

**Theorem 4.9:** Let \( A \) be an ADL. Then \( A \) is a dual Stone ADL if and only if \( PI(A) \) is a dual Stone lattice.

**Proof:** Suppose \( A \) is a dual Stone ADL. For any \( x \in A \), define \( x_+ = (x_ \land m)_ \). Then by Theorem 3.12 [3], we get \( PI(A) \) is a dual pseudo-complemented lattice. Now \( (x)_+ \cap (x_\land m)_+ = (x_ \land m)_+ = 0 \). Hence \( PI(A) \) is a dual Stone lattice. Conversely, suppose \( PI(A), \lor, \land, + \) is a dual Stone lattice. For \( x \in A \), define \( x_ = a \land m \) where \( (x)_+ = (a)_ \). Then by Theorem 3.12 [3], we get that \( A \) is a dual PCADL. Let \( x \in A \). Suppose \( (x)_+ = (a)_ \) and \( (x_\land m)_+ = (b)_+ \). Since \( (x)_+ \cap (x_\land m)_+ = 0 \), we get \( a \land b = 0 \). Since \( (a \land m)_+ = (a)_ \), we get that \( x_ \land x_* = a \land m \land b \land m = 0 \). Hence \( A \) is a dual Stone ADL.

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