gα-separation axioms

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ABSTRACT

In this paper we discuss new separation axioms using gα-open sets.

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1. Introduction:

Norman Levine introduced generalized closed sets in 1970. After him various Authors studied different versions of generalized sets and related topological properties. Recently V.K. Sharma and the author of the present paper defined separation axioms for g-open; gs-open; sg-open; rg-open sets and studied their basic properties.

Definition 1.1: A ⊆ X is called generalized closed [resp: regular generalized; generalized regular] [briefly: g-closed; rg-closed; gα-closed] if cl{A} ⊆ U whenever A ⊆ U and U is open [resp: regular open, open] and generalized [resp: regular generalized; generalized regular] open if its complement is generalized [resp: regular generalized; generalized regular] closed.

Note 1: The class of regular open sets, open sets, g-open sets, gα-open sets are denoted by RO(X), τ(X), αO(X), and gαO(X) respectively. Clearly RO(X) ⊂ τ(X) ⊂ αO(X) ⊂ gαO(X).

Note 2: A ∈ gαO(X, x) means A is generalized α-open neighborhood of X containing x.

Definition 1.2: A ⊆ X is called clopen [resp: gα-clopen] if it is both open [resp: gα-open] and closed [resp: gα-closed].

Definition 1.3: A function f: X → Y is said to be
(i) g-continuous [resp: gα-continuous] if inverse image of closed set is g-closed [resp: gα-closed] and g-irresolute [resp: gα-irresolute] if inverse image of g-closed [resp: gα-closed] set is g-closed [resp: gα-closed]
(ii) gα-open if the image of open set gα-open
(iv) gα-homeomorphism [resp: gα-homeomorphism] if f is bijective, gα-continuous [resp: gα-irresolute] and f⁻¹ is gα-continuous [resp: gα-irresolute]

Definition 1.4: X is said to be
(i) compact [resp: nearly compact, semi-compact, g-compact, gα-compact] if every open [resp: regular-open, semi-open, g-open, gα-open] cover has a finite sub cover.
(ii) T0 [resp: rT0, sT0, g0] space if for each x ≠ y ∈ X  U ∈ τ(X) [resp: RO(X); SO(X); GO(X)] containing either x or y.
(iii) T1 [resp: rT1, g1, T2 [resp: rT2, g2]] space if for each x ≠ y ∈ X  {disjoint} U, V ∈ τ(X) [resp: RO(X); GO(X)], G and H containing x and y respectively.

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2. $g\alpha$-Continuity and product spaces:

**Theorem 2.1:** Let $Y$ and $\{X_\alpha: \alpha \in I\}$ be Topological Spaces. Let $f: Y \to \Pi X_\alpha$ be a function. If $f$ is $g\alpha$-continuous, then $\pi_\alpha \circ f: Y \to X_\alpha$ is $g\alpha$-continuous.

Converse of the above theorem is not true in general as shown by the following Example:

**Example 2.1:** Let $X = \{p, q, r, s\}$; $\tau_X = \{\emptyset, \{p\}, \{q\}, \{s\}, \{p, q\}, \{p, s\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, X\}$, $Y_1 = Y_2 = \{a, b\}$; $\tau_{Y_1} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$; $\tau_{Y_2} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$; $Y = Y_1 \times Y_2 = \{(a, a), (a, b), (b, a), (b, b)\}$ and $\tau_Y = \{\emptyset, \{(a, a)\}, \{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(a, a), (b, b)\}, \{(a, a), (a, b), (b, a)\}, \{(a, a), (a, b), (b, b)\}, \{(a, a), (a, b), (b, a), (b, b)\}\}$. Let $X = \{a, b, c, d\}$ and $\tau_X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$; $Y = Y_1 \times Y_2 = \{(a, a), (a, b), (b, a), (b, b)\}$ and $\tau_Y = \{\emptyset, \{(a, a)\}, \{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(a, a), (b, b)\}, \{(a, a), (a, b), (b, a)\}, \{(a, a), (a, b), (b, b)\}, \{(a, a), (a, b), (b, a), (b, b)\}\}$. Define $f: X \to Y$ by $f(a) = (a, a)$, $f(b) = (b, b)$, $f(c) = (a, b)$, $f(d) = (b, a)$. It is easy to see that $\pi_1 \circ f$ and $\pi_2 \circ f$ are $g\alpha$-continuous. However $\{(b, b)\}$ is closed in $Y$ but $f^{-1}\{(b, b)\} = \{q\}$ is not $g\alpha$-closed in $X$. Therefore $f$ is not $g\alpha$-continuous.

**Theorem 2.2:** If $Y$ is $rT_{1/2}$ and $\{X_\alpha: \alpha \in I\}$ be Topological Spaces. Let $f: Y \to \Pi X_\alpha$ be a function, then $f$ is $g\alpha$-continuous iff $\pi_\alpha \circ f: Y \to X_\alpha$ is $g\alpha$-continuous.

**Corollary 2.3:** (i) Let $f_\alpha: X_\alpha \to Y_\alpha$ be a function and let $f: \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f(x_\alpha)_{\alpha \in I} = (f_\alpha(x_\alpha))_{\alpha \in I}$. If $f$ is $g\alpha$-continuous then each $f_\alpha$ is $g\alpha$-continuous.

(ii) For each $\alpha$, let $X_\alpha$ be $rT_{1/2}$ and let $f_\alpha: X_\alpha \to Y_\alpha$ be a function and let $f: \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f(x_\alpha)_{\alpha \in I} = (f_\alpha(x_\alpha))_{\alpha \in I}$; then $f$ is $g\alpha$-continuous iff each $f_\alpha$ is $g\alpha$-continuous.

3. $g\alpha\alpha$ spaces $i = 0, 1, 2$:

**Definition 3.1:** $X$ is said to be

(i) a $g\alpha_0$ space if for each pair of distinct points $x, y$ of $X$, there exists a $g\alpha$-open set $G$ containing either of the point $x$ or $y$.

(ii) a $g\alpha_i$ [resp: $g\alpha_2$] space if for each pair of distinct points $x, y$ of $X$ there exists [resp: disjoint] $g\alpha$-open sets $G$ and $H$ containing $x$ and $y$ respectively.

**Note 3:**

(i) $rT_1 \to rT_i \to c_\alpha \to g\alpha_i$, $i = 0, 1, 2$, but the converse is not true in general.

(ii) $X$ is $g\alpha_0 \to X$ is $g\alpha_0$.

**Example 3.1:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, X\}\}$, then $X$ is $g\alpha_0$ but not $rT_0$ and $T_0$, $i = 0, 1, 2$ for $i = 0, 1, 2$.

**Example 3.2:** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ then $X$ is not $g\alpha_i$ for $i = 0, 1, 2$.

**Remark 3.1:** If $X$ is $rT_{1/2}$ then $rT_i$ and $g\alpha_i$ are one and the same for $i = 0, 1, 2$.

**Theorem 3.1:** The following are true

(i) Every [resp: regular open] open subspace of $g\alpha_i$ space is $g\alpha_i$ for $i = 0, 1, 2$.

(ii) The product of $g\alpha_i$ spaces is again $g\alpha_i$ for $i = 0, 1, 2$.

(iii) $g\alpha$-continuous image of $T_i$ [resp: $rT_i$] spaces is $g\alpha_i$ for $i = 0, 1, 2$.

(iv) $X$ is $g\alpha_0$ iff $\forall x \in X, \exists U \in \text{gO}(X)$ containing $x$ such that the subspace $U$ is $g\alpha_0$.

(v) $X$ is $g\alpha_i$ iff distinct points of $X$ have disjoint $g\alpha$-closures.

(vi) If $X$ is $g\alpha_i$ then distinct points of $X$ have disjoint $g\alpha$-closures.

**Theorem 3.2:** The following are equivalent:

(i) $X$ is $g\alpha_i$.

(ii) Each one point set is $g\alpha$-closed.

(iii) Each subset of $X$ is the intersection of all $g\alpha$-open sets containing it.

(iv) For any $x \in X$, the intersection of all $g\alpha$-open sets containing the point is the set \{x\}.

**Theorem 3.3:** Suppose $x$ is a $g\alpha$-limit point of a subset of $A$ of a $g\alpha_i$ space $X$. Then every neighborhood of $x$ contains infinitely many distinct points of $A$.

**Theorem 3.4:** The following are true

(i) $X$ is $g\alpha_i$ iff the intersection of all $g\alpha$-closed, $g\alpha$-neighborhoods of each point of the space is reduced to that point.
Theorem 3.5: The following are true
(i) If \( f: X \rightarrow Y \) is injective, \( g\alpha \)-irresolute and \( Y \) is \( g\alpha \), then \( X \) is \( g\alpha \), \( i = 0, 1, 2 \).
(ii) If \( f: X \rightarrow Y \) is injective, \( g\alpha \)-continuous and \( Y \) is \( T_1 \) then \( X \) is \( g\alpha \), \( i = 0, 1, 2 \).
(iii) If \( f: X \rightarrow Y \) is injective, \( r\)-irresolute-[\( r\)-continuous] and \( Y \) is \( rT_1 \) then \( X \) is \( g\alpha \), \( i = 0, 1, 2 \).
(iv) The property of being a space is \( g\alpha \). It is a \( g\alpha \)-Topological property.
(v) Let \( f: X \rightarrow Y \) be a \( g\alpha \)-homeomorphism, then \( X \) is \( g\alpha \) if \( Y \) is \( g\alpha \), \( i = 0, 1, 2 \).
(vi) Let \( X \) be \( T_1 \) and \( f: X \rightarrow Y \) be \( g\alpha \)-closed surjection. Then \( X \) is \( g\alpha_1 \).
(vii) Every \( g\alpha \)-irresolute map from a \( g\alpha \)-compact space into a \( g\alpha \)-space is \( g\alpha \)-closed.
(viii) Any \( g\alpha \)-irresolute bijection from a \( g\alpha \)-compact space onto a \( g\alpha \)-space is a \( g\alpha \)-homeomorphism.
(ix) Any \( g\alpha \)-continuous bijection from a \( g\alpha \)-compact space onto a \( g\alpha \)-space is a \( g\alpha \)-homeomorphism.

Theorem 3.6: The following are equivalent:
(i) \( X \) is \( g\alpha_2 \).
(ii) For each pair \( x \neq y \in X \exists \) a \( g\alpha \)-open, \( g\alpha \)-closed set \( V \) such that \( x \in V \) and \( y \notin V \), and
(iii) For each pair \( x \neq y \in X \exists \) \( f: X \rightarrow [0,1] \) such that \( f(x) = 0 \) and \( f(y) = 1 \) and \( f \) is \( g\alpha \)-continuous.

Theorem 3.7: If \( f: X \rightarrow Y \) is \( g\alpha \)-irresolute and \( Y \) is \( g\alpha \) then
(i) the set \( A = \{(x_1, x_2); f(x_1) = f(x_2)\}\) is \( g\alpha \)-closed in \( X \times X \).
(ii) Graph of \( f \), is \( g\alpha \)-closed in \( X \times Y \).

Theorem 3.8: If \( f: X \rightarrow Y \) is \( g\alpha \)-open and \( A = \{(x_1, x_2); f(x_1) = f(x_2)\}\) is closed in \( X \times X \). Then \( Y \) is \( g\alpha_2 \).

Theorem 3.9: Let \( Y \) and \( \{X_i; i \in I\} \) be Topological Spaces. If \( f: Y \rightarrow \Pi X_i \) be a \( g\alpha \)-continuous function and \( Y \) is \( rT_1/2 \), then \( \Pi X_i \) and each \( X_i \) are \( g\alpha \), \( i = 0, 1, 2 \).

Theorem 3.10: Let \( X \) be an arbitrary space, \( R \) an equivalence relation in \( X \) and \( p: X \rightarrow X/R \) the identification map. If \( R \subset X \times X \) is \( g\alpha \)-closed in \( X \times X \) and \( p \) is \( g\alpha \)-open map, then \( X/R \) is \( g\alpha \).

Proof: Let \( p(U), p(V) \) be distinct members of \( X/R \). Since \( x \neq y \) and \( R \) are not related, \( R \subset X \times X \) is \( g\alpha \)-closed in \( X \times X \). There are \( g\alpha \)-open sets \( U \) and \( V \) such that \( x \in U \), \( y \notin V \) and \( U \times V \subset R^c \). Thus \( \{p(U), p(V)\} \) are disjoint and also \( g\alpha \)-open in \( X/R \) since \( p \) is \( g\alpha \)-open.

Theorem 3.11: The following four properties are equivalent:
(i) \( X \) is \( g\alpha_2 \).
(ii) For each \( x \neq x \in U \in g\alpha O(X) \) such that \( x \in U \) and \( y \notin \alpha \) \( g\alpha cl(U) \).
(iii) For each \( x \in x \in X, \cap\{ g\alpha cl(U) \cup g\alpha O(X) \) and \( x \in U \} = \{x\} \).
(iv) \( \Delta = \{(x, x) | \in X \times X \} \) is \( g\alpha \)-closed in \( X \times X \).

Proof: (i) \( \Rightarrow \) (ii) Let \( x \in X \) and \( y \neq x \). Then there are disjoint \( g\alpha \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \notin V \). Clearly \( V^c \) is \( g\alpha \)-closed, \( g\alpha cl(U) \subset V^c \), \( y \notin V^c \) and therefore \( y \notin g\alpha cl(U) \).

(ii) \( \Rightarrow \) (iii) If \( y \neq x \), then \( \exists \) \( U \in g\alpha O(X) \) s.t. \( x \in U \) and \( y \notin g\alpha cl(U) \). So \( \exists y \in \{ g\alpha cl(U) \cup g\alpha O(X) \) and \( x \in U \} \).

(iii) \( \Rightarrow \) (iv) We prove \( g\alpha^* \) is \( g\alpha \)-open. Let \( (x, y) \notin A \). Then \( y \neq x \) and \( \cap\{ g\alpha cl(U) \cup g\alpha O(X) \) and \( x \in U \} = \{x\} \) there is some \( U \in \{ g\alpha O(X) \) with \( x \in U \) and \( y \notin g\alpha cl(U) \). Since \( U \cap (g\alpha cl(U))^c = \emptyset \), \( U \times (g\alpha cl(U))^c = \emptyset \) is a \( g\alpha \)-open set such that \( x \neq y \in U \times (g\alpha cl(U))^c \subset \Delta \).
(iv) \( \Rightarrow \) (i) \( y \neq x \), then \( (x, y) \in \Delta \) and thus there exist \( g\alpha \)-open sets \( U \) and \( V \) such that \( (x, y) \in U \cap V \) and \( (U \cap V) \cap \Delta = \emptyset \). Clearly, for the \( g\alpha \)-open sets \( U \) and \( V \) we have; \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

4. \( g\alpha\)-\( R_i \) spaces; \( i = 0, 1 \):

Definition 4.1: Let \( x \in X \). Then
(i) \( g\alpha \)-kernel of \( x \) is defined and denoted by \( Ker_{g\alpha}(x) = \cap \{ U : U \in g\alpha O(X) \text{ and } x \in U \} \)
(ii) \( Ker_{g\alpha}(F) = \cap \{ U : U \in g\alpha O(X) \text{ and } F \subset U \} \)

Lemma 4.1: Let \( A \subseteq X \), then \( Ker_{g\alpha}(A) = \{ x \in X : g\alpha cl(x) \cap A \neq \emptyset \} \).

Lemma 4.2: Let \( x \in X \). Then \( x \in Ker_{g\alpha}(x) \) iff \( x \in g\alpha cl[y] \).

Proof: Suppose that \( y \in Ker_{g\alpha}(x) \). Then \( \exists \ V \subseteq g\alpha O(X) \) containing \( x \) such that \( y \notin V \). Therefore we have \( x \notin g\alpha cl[y] \). The proof of converse part can be done similarly.

Lemma 4.3: For any points \( x \neq y \in X \), the following are equivalent:
(i) \( Ker_{g\alpha}(x) \neq Ker_{g\alpha}(y) \);
(ii) \( g\alpha cl(x) \neq g\alpha cl(y) \).

Proof: (i) \( \Rightarrow \) (ii): Let \( Ker_{g\alpha}(x) \neq Ker_{g\alpha}(y) \), then \( \exists z \in X \) such that \( z \in Ker_{g\alpha}(x) \) and \( z \notin Ker_{g\alpha}(y) \). From \( Ker_{g\alpha}(x) \) it follows that \( \{ x \} \cap g\alpha cl(z) \neq \emptyset \) \( \Rightarrow \exists y \in g\alpha cl(z) \). By \( z \notin Ker_{g\alpha}(y) \), we have \( \{ y \} \cap g\alpha cl(z) = \emptyset \). Since \( x \in g\alpha cl(z) \), \( g\alpha cl(x) \subseteq g\alpha cl(z) \) and \( \{ y \} \cap g\alpha cl(x) = \emptyset \). Therefore \( g\alpha cl(x) \neq g\alpha cl(y) \). Now \( Ker_{g\alpha}(x) \neq Ker_{g\alpha}(y) \) \( \Rightarrow \) \( g\alpha cl(x) \neq g\alpha cl(y) \).

(ii) \( \Rightarrow \) (i): If \( g\alpha cl(x) \neq g\alpha cl(y) \), then \( \exists z \in X \) such that \( z \neq g\alpha cl(x) \) and \( z \notin g\alpha cl(y) \). Then \( \exists a \subseteq g\alpha -open\) set containing \( z \) and therefore containing \( x \) but not \( y \), namely, \( y \notin Ker_{g\alpha}(x) \). Hence \( Ker_{g\alpha}(x) \neq Ker_{g\alpha}(y) \).

Definition 4.2: \( X \) is said to be
(i) \( g\alpha -R_0 \) iff \( g\alpha cl(x) \subseteq G \) whenever \( x \in G \subseteq g\alpha O(X) \).
(ii) weakly \( g\alpha -R_0 \) iff \( g\alpha cl(x) = \emptyset \).
(iii) \( g\alpha -R_1 \) iff for \( x, y \in X \) such that \( g\alpha cl(x) \neq g\alpha cl(y) \), \( \exists \) disjoint \( U \subseteq g\alpha O(X) \) such that \( g\alpha cl(x) \subseteq U \) and \( g\alpha cl(y) \subseteq V \).

Example 4.1: Let \( X = \{ a, b, c, d \} \) and \( \tau = \{ \emptyset, \{ b \}, \{ a, b \}, \{ b, c \}, \{ a, b, c \}, X \} \), then \( X \) is not weakly \( g\alpha \)-\( R_0 \) and not \( g\alpha R_i \), \( i = 0, 1 \).

Remark 4.1:
(i) \( R_i \Rightarrow R_i \Rightarrow \alpha R_i \Rightarrow g\alpha R_i \), \( i = 0, 1 \).
(ii) Every weakly-\( R_i \) space is weakly \( g\alpha R_i \).

Lemma 4.1: Every \( g\alpha R_0 \) space is weakly \( g\alpha R_0 \).

Converse of the above Theorem is not true in general by the following Examples.

Example 4.2: Let \( X = \{ a, b, c, d \} \) and \( \tau = \{ \emptyset, \{ a \}, \{ b \}, \{ a, b \}, \{ a, d \}, \{ b, d \}, \{ a, b, c \}, \{ a, b, d \}, X \} \). Clearly, \( X \) is weakly \( g\alpha R_0 \), since \( \cap g\alpha cl(x) = \emptyset \). But it is not \( g\alpha R_0 \), for \( \{ a \} \subseteq X \) is \( g\alpha \)-open and \( g\alpha cl[a] = \{ a \} \subseteq \{ a \} \).

Theorem 4.1: Every \( g\alpha \)-regular space \( X \) is \( g\alpha_2 \) and \( g\alpha -R_0 \).

Proof: Let \( X \) be \( g\alpha \)-regular and let \( x \neq y \in X \). By Lemma 4.1, \( \{ x \} \) is either \( g\alpha \)-open or \( g\alpha \)-closed. If \( \{ x \} \) is \( g\alpha \)-open, \( \{ x \} \) is \( g\alpha \)-open and hence \( g\alpha \)-clopenn. Thus \( \{ x \} \) and \( X - \{ x \} \) are separating \( g\alpha \)-open sets. Similarly for \( \{ x \} \) is \( g\alpha \)-closed, \( \{ x \} \) and \( X - \{ x \} \) are separating \( g\alpha \)-closed sets. Thus \( X \) is \( g\alpha_2 \) and \( g\alpha -R_0 \).

Theorem 4.2: \( X \) is \( g\alpha -R_0 \) iff given \( x \neq y \in X \); \( g\alpha cl(x) \neq g\alpha cl(y) \).

Proof: Let \( X \) be \( g\alpha R_0 \) and let \( x \neq y \in X \). Suppose \( U \) is a \( g\alpha \)-open set containing \( x \) but not \( y \), then \( y \in g\alpha cl[y] \subseteq X \) \( \setminus U \) and so \( x \notin g\alpha cl[y] \). Hence \( g\alpha cl(x) \neq g\alpha cl(y) \).
Conversely, let $x \neq y \in X$ such that $g \text{ac}(x) \neq g \text{ac}(y)$. Then by (iii), $g \text{ac}(x) \subseteq X$ and $g \text{ac}(y) = U$ (say) is a $g \alpha$-open set in $X$. This is true for every $g \text{ac}(x)$. Thus $g \text{ac}(x) \subseteq U$ where $x \in g \text{ac}(x) \subseteq U \subseteq g \mathcal{O}(X)$, which in turn implies $\cap g \text{ac}(x) \subseteq U$ where $x \in U \in g \mathcal{O}(X)$. Hence $X$ is $g \alpha R_0$.

**Theorem 4.3:** X is weakly $g \alpha R_0$ iff $\text{Ker}_{g \alpha}(x) \neq X$ for any $x \in X$.

**Proof:** Let $x_0 \in X$ such that $\text{Ker}_{g \alpha}(x_0) = X$. This means that $x_0$ is not contained in any proper $g \alpha$-open subset of $X$.

Thus $x_0$ belongs to the $g \alpha$-closure of every singleton set. Hence $x_0 \in \cap g \text{ac}(x)$, a contradiction.

Conversely assume $\text{Ker}_{g \alpha}(x) \neq X$ for any $x \in X$. If there is an $x_0 \in X$ such that $x_0 \in \cap g \text{ac} \{g \text{ac}(x)\}$, then every $g \alpha$-open set containing $x_0$ must contain every point of $X$. Therefore, the unique $g \alpha$-open set containing $x_0$ is $X$. Hence $\text{Ker}_{g \alpha}(x_0) = X$, which is a contradiction. Thus $X$ is weakly $g \alpha R_0$.

**Theorem 4.4:** The following are equivalent:

(i) $X$ is $g \alpha R_0$ space.

(ii) For each $x \in X$, $g \text{ac}(x) \subseteq \text{Ker}_{g \alpha}(x)$

(iii) For any $g \alpha$-closed set $F$ and a point $x \notin F$, $\exists U \subseteq g \mathcal{O}(X)$ such that $x \notin U$ and $F \subseteq U$.

(iv) Each $g \alpha$-closed set $F$ can be expressed as $F = \cap \{G: G$ is $g \alpha$-open and $F \subseteq G\}$.

(v) Each $g \alpha$-open set $G$ can be expressed as $G = \cup \{A: A$ is $g \alpha$-closed and $A \subseteq G\}$.

(vi) For each $g \alpha$-closed set $F$, $x \notin F$ implies $g \text{ac}(x) \cap F = \emptyset$.

**Proof:** (i) $\Rightarrow$ (ii) For any $x \in X$, we have $\text{Ker}_{g \alpha}(x) = \cap \{U: U \subseteq g \mathcal{O}(X)$ and $x \in U\}$. Since $X$ is $g \alpha R_0$, each $g \alpha$-open set containing $x$ contains $g \text{ac}(x)$. Hence $g \text{ac}(x) \subseteq \text{Ker}_{g \alpha}(x)$.

(ii) $\Rightarrow$ (iii) Let $x \notin F \subseteq g \mathcal{C}(X)$. Then for any $y \in F$, $g \text{ac}(y) \subseteq F$ and so $x \notin g \text{ac}(y)$ implies $y \notin g \text{ac}(x)$ that is $\exists U_y \subseteq g \mathcal{O}(X)$ such that $y \in U_y$ and $x \notin U_y \forall y \in F$. Let $U = \cup \{U_y: U_y$ is $g \alpha$-open, $y \in U_y$ and $x \notin U_y\}$. Then $U$ is $g \alpha$-open such that $x \notin U$ and $F \subseteq U$.

(iii) $\Rightarrow$ (iv) Let $F$ be any $g \alpha$-open and $N = \cap \{G: G$ is $g \alpha$-open and $F \subseteq G\}$. Then $F \subseteq N \rightarrow (1)$.

Let $x \notin F$, then by (iii), $\exists G \subseteq g \mathcal{O}(X)$ such that $x \notin G$ and $F \subseteq G$.

Hence $x \notin N$ which implies $x \in N \Rightarrow x \in F$. Hence $N \subseteq F \rightarrow (2)$.

Therefore from (1) and (2), each $g \alpha$-closed set $F = \cap \{G: G$ is $g \alpha$-open and $F \subseteq G\}$

(iv) $\Rightarrow$ (v) obvious.

(v) $\Rightarrow$ (vi) Let $x \notin F \subseteq g \mathcal{C}(X)$. Then $X - F = G$ is a $g \alpha$-open set containing $x$. Then by (v), $G$ can be expressed as the union of $g \alpha$-open sets $A$ contained in $G$, and so there is an $M \subseteq g \mathcal{C}(X)$ such that $x \in M \subseteq g \mathcal{C}(X)$ and hence $g \text{ac}(x) \subseteq G$ which implies $g \text{ac}(x) \cap F = \emptyset$.

(vi) $\Rightarrow$ (i) Let $x \in G \subseteq g \mathcal{O}(X)$. Then $x \in (X - G)$, which is a $g \alpha$-closed set. Therefore by (vi) $g \text{ac}(x) \cap (X - G) = \emptyset$, which implies that $g \text{ac}(x) \subseteq G$. Thus $X$ is $g \alpha R_0$ space.

**Theorem 4.5:** Let $f: X \rightarrow Y$ be a $g \alpha$-closed one-one function. If $X$ is weakly $g \alpha R_0$, then so is $Y$.

**Theorem 4.6:** If $X$ is weakly $g \alpha R_0$, then for every space $Y$, $X \times Y$ is weakly $g \alpha R_0$.

**Proof:** $\cap g \text{ac}(\{(x,y)\}) \subseteq \cap \{g \text{ac}(x) \times g \text{ac}(y)\} = \cap \{g \text{ac}(x) \times g \text{ac}(y)\} \subseteq \phi \times Y = \emptyset$. Hence $X \times Y$ is $g \alpha R_0$.

**Corollary 4.1:**

(i) If $X$ and $Y$ are weakly $g \alpha R_0$, then $X \times Y$ is weakly $g \alpha R_0$.

(ii) If $X$ and $Y$ are weakly $g \alpha R_0$, then $X \times Y$ is weakly $g \alpha R_0$.

(iii) If $X \times Y$ are weakly $g \alpha R_0$, then $X \times Y$ is weakly $g \alpha R_0$.

(iv) If $Y$ is $g \alpha R_0$ and $Y$ are weakly $g \alpha R_0$, then $X \times Y$ is weakly $g \alpha R_0$.

**Theorem 4.7:** X is $g \alpha R_0$ iff for any $x, y \in X$, $g \text{ac}(x) \neq g \text{ac}(y)$ implies $g \text{ac}(x) \cap g \text{ac}(y) = \emptyset$. 

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Proof: Let $X$ is a $\alpha R_0$ space and $x, y \in X$ such that $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. Then $\exists z \in g\alpha cl\{x\}$ such that $ze g\alpha cl\{y\}$ (or $ze g\alpha cl\{y\}$) such that $z \notin g\alpha cl\{x\}$. There exists $V \subseteq g\alpha cl\{y\}$ such that $y \notin V$ and $z \notin V$; hence $x \in V$. Therefore, $x \notin g\alpha cl\{y\}$.

Thus $x \in [g\alpha cl\{y\}] \setminus g\alpha cl\{y\}$, which implies $g\alpha cl\{x\} \subseteq [g\alpha cl\{y\}] \setminus g\alpha cl\{y\}$ and $g\alpha cl\{x\} \cap g\alpha cl\{y\} = \emptyset$. The proof for otherwise is similar.

**Sufficiency:** Let $x \in V \subseteq g\alpha cl\{y\}$. We show that $g\alpha cl\{x\} \subseteq V$. Let $y \in V$, i.e., $y \in V$. Then $x \notin y$ and $x \notin g\alpha cl\{y\}$. Hence $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. But $g\alpha cl\{x\} \cap g\alpha cl\{y\} = \emptyset$. Hence $y \notin g\alpha cl\{x\}$. Therefore $g\alpha cl\{x\} \subseteq V$.

**Theorem 4.8:** $X$ is a $\alpha R_0$ iff for any points $x, y \in X, Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\} \Rightarrow Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\} = \emptyset$.

Proof: Suppose $X$ is a $\alpha R_0$. Thus by Lemma 4.3 for any $x, y \in X$ if $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$ then $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. Assume that $z \in Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\}$. By $z \in Ker_{\{g\alpha\}}\{x\}$ and Lemma 4.2, it follows that $x \in g\alpha cl\{z\}$. Since $x \in g\alpha cl\{z\}, g\alpha cl\{x\} = g\alpha cl\{z\}$. Similarly, we have $g\alpha cl\{y\} = g\alpha cl\{z\} = g\alpha cl\{x\}$. This is a contradiction. Therefore, we have $Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\} = \emptyset$.

Conversely, let $x, y \in X$, s.t. $g\alpha cl\{x\} \neq g\alpha cl\{y\}$, then by Lemma 4.3, $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$. Hence by hypothesis $Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\} = \emptyset$ which implies $g\alpha cl\{x\} \cap g\alpha cl\{y\} = \emptyset$. Because $z \notin g\alpha cl\{x\}$ implies that $x \in Ker_{\{g\alpha\}}\{z\}$ and therefore $Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{z\} = \emptyset$. Therefore by Theorem 4.7 $X$ is a $\alpha R_0$ space.

**Theorem 4.9:** The following are equivalent:

(i) $X$ is a $\alpha R_0$ space.

(ii) For any $A \neq \emptyset$ and $G \subseteq g\alpha cl\{x\}$ such that $A \cap G = \emptyset \not\exists F \subseteq g\alpha cl\{x\}$ such that $A \cap F = \emptyset$ and $F \subseteq G$.

**Proof:** (i) $\Rightarrow$ (ii): Let $A \neq \emptyset$ and $G \subseteq g\alpha cl\{x\}$ such that $A \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \subseteq g\alpha cl\{x\}$, $g\alpha cl\{x\} \subseteq G$. Set $F = g\alpha cl\{x\}$, then $F \subseteq g\alpha cl\{x\}, F \subseteq G$ and $A \cap F = \emptyset$.

(ii) $\Rightarrow$ (i): Let $G \subseteq g\alpha cl\{x\}$ and $x \in G$. By (2), $g\alpha cl\{x\} \subseteq G$. Hence $X$ is a $\alpha R_0$.

**Theorem 4.10:** The following are equivalent:

(i) $X$ is a $\alpha R_0$ space.

(ii) $x \in g\alpha cl\{y\}$ iff $y \in g\alpha cl\{x\}$, for any points $x$ and $y$ in $X$.

**Proof:** (i) $\Rightarrow$ (ii): Assume $X$ is a $\alpha R_0$. Let $x \in g\alpha cl\{y\}$ and $D$ be any $\alpha$-open set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every $\alpha$-open set which contain $y$ contains $x$. Hence $y \in g\alpha cl\{x\}$.

(ii) $\Rightarrow$ (i): Let $U$ be a $\alpha$-open set and $x \in U$. If $y \in U$, then $x \in g\alpha cl\{y\}$ and hence $y \in g\alpha cl\{x\}$. This implies that $g\alpha cl\{x\} \subseteq U$. Hence $X$ is a $\alpha R_0$.

**Theorem 4.11:** The following are equivalent:

(i) $X$ is a $\alpha R_0$ space.

(ii) If $F$ is $\alpha$-closed, then $F = Ker_{\{g\alpha\}}\{F\}$.

(iii) If $F$ is $\alpha$-closed and $x \notin F$, then $Ker_{\{g\alpha\}}\{x\} \subseteq F$.

(iv) If $x \in X$, then $Ker_{\{g\alpha\}}\{x\} \subseteq g\alpha cl\{x\}$.

**Proof:** (i) $\Rightarrow$ (ii): Let $x \notin F \subseteq g\alpha cl\{x\} \Rightarrow (X-F) \subseteq g\alpha cl\{x\}$ and contains $x$. For $X$ is a $\alpha R_0$, $g\alpha cl\{x\} \subseteq (X-F)$. Thus $g\alpha cl\{x\} \subseteq F$ and $x \in Ker_{\{g\alpha\}}\{F\}$. Hence $Ker_{\{g\alpha\}}\{F\} = F$.

(ii) $\Rightarrow$ (iii): $A \subseteq B \Rightarrow Ker_{\{g\alpha\}}\{A\} \subseteq Ker_{\{g\alpha\}}\{B\}$. Therefore, by (2) $Ker_{\{g\alpha\}}\{x\} \subseteq Ker_{\{g\alpha\}}\{F\} = F$.

(iii) $\Rightarrow$ (iv): Since $x \in g\alpha cl\{x\}$ and $g\alpha cl\{x\}$ is $\alpha$-closed, by (3) $Ker_{\{g\alpha\}}\{x\} \subseteq g\alpha cl\{x\}$.

(iv) $\Rightarrow$ (i): Let $x \in g\alpha cl\{y\}$. Then by Lemma 4.2 $y \in Ker_{\{g\alpha\}}\{x\}$. Since $x \in g\alpha cl\{x\}$ and $g\alpha cl\{x\}$ is $\alpha$-closed, by (iv) we obtain $y \in Ker_{\{g\alpha\}}\{x\} \subseteq g\alpha cl\{x\}$. Therefore $x \in g\alpha cl\{y\}$ implies $y \in g\alpha cl\{x\}$. The converse is obvious and $X$ is a $\alpha R_0$.

**Corollary 4.2:** The following are equivalent:

(i) $X$ is a $\alpha R_0$.

(ii) $g\alpha cl\{x\} = Ker_{\{g\alpha\}}\{x\} \forall \ x \in X$.}

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Proof: Follows from Theorems 4.4 and 4.11.

Recall that a filterbase $F$ is called $g\alpha$-convergent to a point $x$ in $X$, if for any $g\alpha$-open set $U$ of $X$ containing $x$, there exists $B \in F$ such that $B \subseteq U$.

Lemma 4.4: Let $x$ and $y$ be any two points in $X$ such that every net in $X$ $g\alpha$-converging to $y$ $g\alpha$-converges to $x$. Then $x \in g\alpha cl\{y\}$.

Theorem 4.12: The following are equivalent:
(i) $X$ is a $g\alpha R_{0}$ space;
(ii) If $x, y \in X$, then $y \in g\alpha cl\{x\}$ iff every net in $X$ $g\alpha$-converging to $y$ $g\alpha$-converges to $x$.

Proof:
(i) $\Rightarrow$ (ii): Let $x, y \in X$ such that $y \in g\alpha cl\{x\}$. Suppose that $\{x_{\alpha}\}_{\alpha \in I}$ is a net in $X$ such that $\{x_{\alpha}\}_{\alpha \in I}$ $g\alpha$-converges to $y$. Since $y \in g\alpha cl\{x\}$, by Thm. 4.7 we have $g\alpha cl\{x\} = g\alpha cl\{y\}$. Therefore $x \in g\alpha cl\{y\}$. This means that $\{x_{\alpha}\}_{\alpha \in I}$ $g\alpha$-converges to $x$.

Conversely, let $x, y \in X$ such that every net in $X$ $g\alpha$-converging to $y$ $g\alpha$-converges to $x$. Then $x \in g\alpha cl\{y\}$ [by 4.4]. By Thm. 4.7, we have $g\alpha cl\{x\} = g\alpha cl\{y\}$. Therefore $y \in g\alpha cl\{x\}$.

(ii) $\Rightarrow$ (i): Let $x, y \in X$ such that $g\alpha cl\{x\} \cap g\alpha cl\{y\} \neq \emptyset$. Let $z \in g\alpha cl\{x\} \cap g\alpha cl\{y\}$. So $\exists$ a net $\{x_{\alpha}\}_{\alpha \in I}$ in $g\alpha cl\{x\}$ such that $\{x_{\alpha}\}_{\alpha \in I}$ $g\alpha$-converges to $z$. Since $z \in g\alpha cl\{y\}$, then $\{x_{\alpha}\}_{\alpha \in I}$ $g\alpha$-converges to $y$. It follows that $y \in g\alpha cl\{x\}$. Similarly we obtain $x \in g\alpha cl\{y\}$. Therefore $g\alpha cl\{x\} = g\alpha cl\{y\}$. Hence $X$ is $g\alpha R_{0}$.

Theorem 4.13:
(i) Every subspace of $g\alpha R_{1}$ space is again $g\alpha R_{1}$,
(ii) Product of any two $g\alpha R_{1}$ spaces is again $g\alpha R_{1}$,
(iii) $X$ is $g\alpha R_{1}$ iff given $x \neq y \in X$, $g\alpha cl\{x\} \neq g\alpha cl\{y\}$.
(iv) Every $g\alpha_{2}$ space is $g\alpha R_{1}$.

The converse of 4.13(iv) is not true. However, we have the following result.

Theorem 4.14: Every $g\alpha_{1}$ and $g\alpha R_{1}$ space is $g\alpha_{2}$.

Proof: Let $x \neq y \in X$. Since $X$ is $g\alpha_{1}$, $\{x\}$ and $\{y\}$ are $g\alpha$-closed sets such that $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. Since $X$ is $g\alpha R_{1}$, there exists disjoint $g\alpha$-open sets $U$ and $V$ such that $x \in U$, $y \in V$. Hence $X$ is $g\alpha_{2}$.

Corollary 4.3: $X$ is $g\alpha_{2}$ iff it is $g\alpha R_{1}$ and $g\alpha_{1}$.

Theorem 4.15: The following are equivalent
(i) $X$ is $g\alpha R_{1}$
(ii) $\cap g\alpha cl\{x\} = \{x\}$.
(iii) For any $x \in X$, intersection of all $g\alpha$-neighborhoods of $x$ is $\{x\}$.

Proof:
(i) $\Rightarrow$ (ii) Let $y \neq x \in X$ such that $y \in g\alpha cl\{x\}$. Since $X$ is $g\alpha R_{1}$, $\exists U \subseteq g\alpha O(X)$ such that $y \in U$, $x \notin U$ or $x \in U$, $y \notin U$. In either case $y \not\in g\alpha cl\{x\}$. Hence $\cap g\alpha cl\{x\} = \{x\}$.

(ii) $\Rightarrow$ (iii) If $y \neq x \in X$, then $x \notin \cap g\alpha cl\{y\}$, so there is a $g\alpha$-open set containing $x$ but not $y$. Therefore $y$ does not belong to the intersection of all $g\alpha$-neighborhoods of $x$. Hence intersection of all $g\alpha$-neighborhoods of $x$ is $\{x\}$.

(iii) $\Rightarrow$ (i) Let $x \neq y \in X$, by hypothesis, $y$ does not belong to the intersection of all $g\alpha$-neighborhoods of $x$ and $x$ does not belong to the intersection of all $g\alpha$-neighborhoods of $y$, which implies $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. Hence $X$ is $g\alpha R_{1}$.

Theorem 4.16: The following are equivalent:
(i) $X$ is $g\alpha R_{1}$
(ii) For each pair $x, y \in X$ with $g\alpha cl\{x\} \neq g\alpha cl\{y\}$, $\exists$ a $g\alpha$-open, $g\alpha$-closed set $V$ s.t. $x \in V$ and $y \notin V$, and
(iii) For each pair $x, y \in X$ with $g\alpha cl\{x\} \neq g\alpha cl\{y\}$, $\exists f: X \rightarrow [0, 1]$ s.t. $f(x) = 0$ and $f(y) = 1$ and $f$ is $g\alpha$-continuous.
Theorem 4.17:
(i) If $X$ is $g\alpha$-$R_1$, then $X$ is $g\alpha$-$R_0$.
(ii) $X$ is $g\alpha$-$R_1$ iff for $x, y \in X$, $Ker_{(g\alpha)}\{x\} \neq Ker_{(g\alpha)}\{y\}$, $\exists$ disjoint $U; V \in g\alpha O(X)$ such that $g\alpha cl\{x\} \subset U$ and $g\alpha cl\{y\} \subset V$.

5. $g\alpha$-$C_i$ and $g\alpha$-$D_i$ spaces, $i = 0, 1, 2$:

Definition 5.1:
(i) $X$ is said to be a $g\alpha$-$C_0$ space if for each pair of distinct points $x, y$ of $X$ there exists a $g\alpha$-open set $G$ whose closure contains either of the point $x$ or $y$.
(ii) $g\alpha$-$C_1$ [resp: $g\alpha$-$C_2$] space if for each pair of distinct points $x, y$ of $X$ there exists disjoint $g\alpha$-open sets $G$ and $H$ such that closure of $G$ containing $x$ but not $y$ and closure of $H$ containing $y$ but not $x$.

Note 4: $g\alpha$-$C_2 \Rightarrow g\alpha$-$C_1 \Rightarrow g\alpha$-$C_0$. Converse need not be true in general:

Example 5.1:
(i) Let $X = \{a, b, c\}$ and $\tau = \{\phi, X\}$, then $X$ is $g\alpha$-$C_i$ for $i = 0, 1, 2$.
(ii) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ then $X$ is not $g\alpha$-$C_i$ for $i = 0, 1, 2$.

Theorem 5.1:
(i) Every subspace of $g\alpha$-$C_i$ space is $g\alpha$-$C_i$.
(ii) Every $g\alpha$ space is $g\alpha$-$C_i$.
(iii) Product of $g\alpha$-$C_i$ spaces are $g\alpha$-$C_i$.
(iv) Let $X$ be any $g\alpha$-$C_i$ space and $A \subset X$ then $A$ is $g\alpha$-$C_i$ iff $(A, \tau_A)$ is $g\alpha$-$C_i$.
(v) If $X$ is $g\alpha$-$C_1$ then each singleton set is $g\alpha$-closed.
(vi) In an $g\alpha$-$C_1$ space disjoint points of $X$ has disjoint $g\alpha$- closures.

Definition 5.2: $A \subset X$ is called a $g\alpha$-Difference(Shortly $g\alpha$D-set) set if there are two $U, V \in g\alpha O(X)$ such that $U \neq X$ and $A = U - V$.

Clearly every $g\alpha$-open set different from $X$ is a $g\alpha$D-set if $A = U$ and $V = \phi$.

Definition 5.3: $X$ is said to be a $g\alpha$D-
(i) $g\alpha$-$D_0$ if for any pair of distinct points $x$ and $y$ of $X$ there exist a $g\alpha$D-set in $X$ containing $x$ but not $y$ or a $g\alpha$D-set in $X$ containing $y$ but not $x$.
(ii) $g\alpha$-$D_1$ [resp: $g\alpha$-$D_2$] if for any pair of distinct points $x$ and $y$ in $X$ there exists disjoint $g\alpha$D-sets $G$ and $H$ in $X$ containing $x$ and $y$ respectively.

Remark 5.2: (i) If $X$ is $rT_i$, then it is $g\alpha_i$, $i = 0, 1, 2$ and converse is false.
(ii) If $X$ is $g\alpha_i$, then it is $g\alpha_{i+1}$, $i = 1, 2$.
(iii) If $X$ is $g\alpha_i$, then it is $g\alpha$-$D_i$, $i = 0, 1, 2$.
(iv) If $X$ is $g\alpha$-$D_i$, then it is $g\alpha$-$D_{i+1}$, $i = 1, 2$.

Theorem 5.2: The following statements are true:
(i) $X$ is $g\alpha$-$D_0$ iff it is $g\alpha_0$.
(ii) $X$ is $g\alpha$-$D_1$ iff it is $g\alpha$-$D_2$.

Corollary 5.1: If $X$ is $g\alpha$-$D_1$, then it is $g\alpha_0$.

Proof: Remark 5.1(iv) and Theorem 5.1(vi)

Definition 5.4: A point $x \in X$ which has $X$ as the unique $g\alpha$-neighborhood is called $g\alpha$.c.c point.

Theorem 5.3: For an $g\alpha_0$ space $X$ the following are equivalent:
(i) $X$ is $g\alpha$-$D_1$;
(ii) $X$ has no $g\alpha$.c.c point.

Proof: (i) $\Rightarrow$ (ii) Since $X$ is $g\alpha$-$D_1$, then each point $x$ of $X$ is contained in a $g\alpha$D-set $O = U - V$ and thus in $U$. By Definition $U \neq X$. This implies that $x$ is not a $g\alpha$.c.c point.
(ii) $\Rightarrow$ (i) If $X$ is $g\alpha_0$, then for each $x \neq y \in X$, at least one of them, $x$ (say) has a $g\alpha$-neighborhood $U$ containing $x$ and not $y$. Thus $U$ which is different from $X$ is a $g\alpha D$-set. If $X$ has no $g\alpha$-point, then $y$ is not a $g\alpha$-point. This means that there exists a $g\alpha$-neighborhood $V$ of $y$ such that $V \neq X$. Thus $y \in V - U$ but not $x$ and $V - U$ is a $g\alpha D$-set. Hence $X$ is $g\alpha D_1$.

**Definition 5.5:** $X$ is $g\alpha$-symmetric if for $x$ and $y$ in $X$, $x \in g\alpha cl\{y\}$ implies $y \in g\alpha cl\{x\}$.

**Theorem 5.4:** $X$ is $g\alpha$-symmetric iff $\{x\}$ is $g\alpha$-closed for each $x \in X$.

**Proof:** Assume that $x \in g\alpha cl\{y\}$ but $y \notin g\alpha cl\{x\}$. This means that $[g\alpha cl\{x\}]^c$ contains $y$. This implies that $g\alpha cl\{y\} \subset [g\alpha cl\{x\}]^c$. Now $[g\alpha cl\{x\}]^c$ contains $x$ which is a contradiction.

Conversely, suppose that $\{x\} \subset E \in g\alpha O(X)$ but $g\alpha cl\{x\} \notin E$. This means that $g\alpha cl\{x\}$ and $E^c$ are not disjoint. Let $y$ belongs to their intersection. Now we have $x \in g\alpha cl\{y\} \subset E^c$ and $x \notin E$. But this is a contradiction.

**Corollary 5.2:** If $X$ is a $g\alpha_1$, then it is $g\alpha$-symmetric.

**Proof:** Follows from Theorem 2.2(ii) and Theorem 5.4.

**Corollary 5.3:** The following are equivalent:
(i) $X$ is $g\alpha$-symmetric and $g\alpha_0$
(ii) $X$ is $g\alpha_1$.

**Proof:** By Corollary 5.2 and Remark 5.1 it suffices to prove only (i) $\Rightarrow$ (ii). Let $x \neq y$ and by $g\alpha_0$, we may assume that $x \in G \cap \{y\}^c$ for some $G \in g\alpha O(X)$. Then $x \notin g\alpha cl\{y\}$ and hence $y \notin g\alpha cl\{x\}$. There exists a $G \in g\alpha O(X)$ such that $y \in G \subset \{x\}^c$ and $X$ is a $g\alpha_1$ space.

**Theorem 5.5:** For a $g\alpha$-symmetric space $X$ the following are equivalent:
(i) $X$ is $g\alpha_0$; (ii) $X$ is $g\alpha D_1$; (iii) $X$ is $g\alpha_1$.

**Proof:** (i) $\Rightarrow$ (iii) Corollary 5.3 and (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) Remark 5.1.

**Theorem 5.6:** If $f: X \rightarrow Y$ is a $g\alpha$-irresolute surjective function and $E$ is a $g\alpha D$-set in $Y$, then the inverse image of $E$ is a $g\alpha D$-set in $X$.

**Proof:** Let $E$ be a $g\alpha D$-set in $Y$. Then there are $g\alpha$-open sets $U_1$ and $U_2$ in $Y$ such that $E = U_1 + U_2$ and $U_1 \neq Y$. By the $g\alpha$-irresoluteness of $f$, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $g\alpha$-open in $X$. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$.

Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a $g\alpha D$-set.

**Theorem 5.7:** (i) If $Y$ is $g\alpha D_1$ and $f: X \rightarrow Y$ is $g\alpha$-irresolute and bijective, then $X$ is $g\alpha D_1$.
(ii) $X$ is $g\alpha D_1$ iff for each pair of $x \neq y$ in $X$ there exist a $g\alpha$-irresolute surjective function $f: X \rightarrow Y$, where $Y$ is a $g\alpha D_1$ space such that $f(x)$ and $f(y)$ are distinct.

**Corollary 5.4:** Let $\{X_\alpha: \alpha \in I\}$ be any family of spaces. If $X_\alpha$ is $g\alpha D_1$ for each $\alpha \in I$, then $\Pi X_\alpha$ is $g\alpha D_1$.

**Proof:** Let $(x_\alpha) \neq (y_\alpha)$ in $\Pi X_\alpha$. Then there exists an index $\beta \in I$ s. t. $x_\beta \neq y_\beta$. The natural projection $P_\beta: \Pi X_\alpha \rightarrow X_\beta$ is almost continuous and almost open and $P_\beta ((x_\alpha)) = P_\beta((y_\alpha))$. Since $X_\beta$ is $g\alpha D_1$, $\Pi X_\alpha$ is $g\alpha D_1$.

**References:**


[9] Jiling Cao, M. Ganster and Ivan Reily, on sg-closed sets and $g\alpha$-closed sets.


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