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POSITIVITY-NEGATIVITY AND EMBEDDING THEOREMS FOR ELLIPTIC SYSTEMS

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ABSTRACT

In this paper we study embedding of a non co-operative elliptic system into a cooperative elliptic system and positivity of a solution. Using the results of Figueiredo and Mitidieri, [1], we slightly modify the Theorem and obtain positivity-negativity Theorem. In section 3, we obtain a Theorem for positivity of a solution of 3×3 *non co-operative elliptic system by embedding it into a* 4×4 *co-operative elliptic system. Section 4 deals with positivity of solution of higher order elliptic equations. This work is the extension of the work of Figueiredo and Mitidieri,[1].*

***Key words**: Boundary value problems for elliptic systems. General theory of elliptic systems of PDE, BVP for higher order elliptic equations.*

***AMS Subject Clafication**: 35J55, 35J45, 35J40.*

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1. INTRODUCTION:

Let Ω be a bounded domain in R^N with boundary $\partial\Omega$, $\overline{\Omega}=\Omega+\partial\Omega$ be the closure of Ω. The points of R^N are denoted by $x=(x_1,x_2,..,x_N)$. For $U=(u_1,u_2,..,u_n)$, with $u_k(x)\in C^2(\Omega)\cap C^1(\overline{\Omega})$, let $D_iu_k(x)=\dfrac{\partial u_k(x)}{\partial x_i}$, for $i=1,2,..,N,\ k=1,2,..,n.$

$$D_iD_ju_k(x)=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_k(x),\quad i=1,2,..,N,\ j=1,2,..,N,$$

for $k=1,2,..,n.$

We consider the following system.

$$L_k(D)u_k(x)=\sum_{j=1}^{n}a_{kj}(x)u_j(x)+f_k(x), \tag{1}$$

where

$$L_k(D)u_k(x)\equiv-\sum_{i,j=1}^{N}b_{ij}^{\,k}(x)D_iD_ju_k+\sum_{i=1}^{N}b_i^{\,k}(x)D_iu_k(x),k=1,2,\cdots n$$

$b_{ij}^{\,k}(x),\ b_i^{\,k}(x),\ a_{kj}(x),\ f_k(x)$ are real valued functions on Ω. The system (1) can be denoted by

$$L(D)U=AU+F, \tag{2}$$

where, $L(D)\equiv[L_1(D),L_2(D),..,L_n(D)]$ is a diagonal operator matrix of second order elliptic operators; $F=(f_1(x),f_2(x),..,f_n(x))$ are functions in $C(\Omega)$, and $A=[a_{ij}(x)]$ is a $n\times n$ matrix of real valued functions defined on Ω.

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Consider a boundary value problem

$$L(D)U = AU + F, \text{ in } \Omega \tag{3}$$

$$U = 0 \text{ on } \partial\Omega . \tag{4}$$

We state the following conditions:

(α_1) : $a_{kj}(x) \geq 0, \ x \in \Omega, \ k \neq j.$

(α_2) : (α_2): $\sum_{i,j=1}^{N} b_{ij}{}^{k}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2, \ \lambda(x) > 0, \ \forall x \in \Omega, \ \xi \in R^N, \ \forall \ k.$

(α_3) : $\dfrac{b_i{}^k(x)}{\lambda(x)} \leq M, \ x \in \Omega, \ \forall i, k$ where $M > 0$ is a constant.

Definition: 1.1 The system (3) is said to be elliptic for $x \in \Omega$ if condition (α_2) is satisfied. It is elliptic in $\Omega,$ if it is elliptic for all $x \in \Omega.$

Definition: 1.2 The system (2) is said to be co-operative elliptic system if conditions (α_1) and (α_2) are satisfied, [5], Hiwarekar, Kasture.

Definition: 1.3 Positivity Theorem: A positivity theorem is said to hold if $F \geq 0$ in Ω implies $U \geq 0$ in $\Omega,$ where U is a solution of a elliptic system (3), (4), [5], Hiwarekar and Kasture.

By a solution U of a boundary value problem (3), (4), we mean a classical solution. Here solution U is defined in a given domain $\overline{\Omega},$ which is continuous in $\overline{\Omega},$ and belongs to $C^2(\Omega).$ We are assuming that solution of a problem exists.

Next section deals with embedding and positivity-negativity theorems. We are considering a particular case of system (3), (4) by taking $L_k = L = -\Delta, \ k = 1, 2, \cdots, n,$

where $\Delta \equiv \sum_{i=1}^{N} \dfrac{\partial^2}{\partial x_i{}^2}$, which is the Laplace operator.

2. EMBEDDING AND POSITIVITY-NEGATIVITY THEOREMS:

We consider 2×2 non-cooperative elliptic system for u_1, u_2, u_3 under some conditions, if we can construct a 3×3 co-operative elliptic system in u_1, u_2, u_3 such that if (u_1, u_2, u_3) is its solution, then (u_1, u_2) is a solution of the corresponding 2×2 non-cooperative elliptic system, then this is called an embedding of the 2×2 system into the 3×3 system,[1], Figueiredo and Mitidieri.

Consider the following system:

$$\begin{aligned} -\Delta u_1 &= a_{11}u_1 + a_{12}u_2 + f_1, \\ -\Delta u_2 &= a_{21}u_1 + a_{22}u_2 + f_2, \end{aligned} \tag{5}$$

with $a_{12}(x) \leq 0, a_{12}(x) \not\equiv 0, a_{21}(x) \geq 0.$ (6)

It follows from the definition (1.2) that the system (5) is a non-cooperative elliptic system. Let

$u_3(x) = u_1(x) + \delta u_2(x),$ where $\delta \neq 0.$

Also consider the following elliptic system

$$\begin{aligned} -\Delta u_1 &= (a_{11} - r)u_1 + (a_{12} - r\delta)u_2 + ru_3 + f_1, \\ -\Delta u_2 &= a_{21}u_1 + a_{22}u_2 + f_2, \\ -\Delta u_3 &= (a_{11} - s + a_{21}\delta)u_1 + (a_{12} + a_{22}\delta - s\delta)u_2 + su_3 + f_1 + \delta f_2. \end{aligned} \tag{7}$$

Here functions $r(x)$ and $s(x)$ are real valued functions to be determined such that system (7) will be co-operative elliptic system.

Now we state the following Theorem from [1], Figueiredo and Mitidieri.

Theorem: 2.1 Part-I: The non-cooperative elliptic system (5) can be embedded into a co-operative elliptic system (7) if there exist a $\delta(x) < 0$ such that the condition

(α_4) : $a_{21}\delta^2 + (a_{11} - a_{12})\delta - a_{12} \le 0, \;\; x \in \Omega,$ is satisfied.
Further

Part-II: If

$$\begin{aligned} &(\alpha_5): a_{11} < \lambda_1, \\ &(\alpha_6): a_{21} + a_{22} < \lambda_1, \\ &(\alpha_7): a_{11} + a_{21}\delta_- < \lambda_1, \end{aligned}$$

where λ_1 is a first eigen value of $-\Delta,$ then $f_1 \ge 0,\ f_2 \ge 0$ and $f_1 + f_2\delta_- \ge 0$ in $\Omega,$ imply $u_1 \ge 0, u_2 \ge 0$ in $\Omega.$

Here $\delta_-(x)$ and $\delta_+(x)$ are the roots of the equation

$$a_{21}\delta^2 + (a_{11} - a_{12})\delta - a_{12} = 0, \tag{8}$$

with

$$\sup_{\Omega} \delta_-(x) \le \sup_{\Omega} \delta_+(x), \tag{9}$$

[1], Figueiredo and Mitidieri.

A further generalization of this theorem is in Fleckinger and Serag,[2], they considered following system

$$\begin{aligned} -\Delta u &= a\rho(x)u + b\rho(x)v + f(x,u,v), \\ -\Delta v &= c\rho(x)u + d\rho(x)v + g(x,u,v) \quad in \quad \Omega, \end{aligned} \tag{10}$$

where u and v tends to zero as $|x|$ tends to zero, and a,b,c,d are constants. They proved the positivity of a solution above co-operative system with $b \ge 0, c \ge 0$ subject to

$$a < \lambda_1, d < \lambda_1, \tag{11}$$

$$(\lambda_1 - a)(\lambda_1 - d) > bc, \tag{12}$$

where λ_1 is a first eigen value of $-\Delta.$ Further they generalized the results for $n \times n$ co-operative elliptic system.

The given system in (5) is non-cooperative elliptic system due to $a_{12}(x) < 0,$ we take $a_{12}(x) \ge 0$ and obtain the following slightly modified form of Theorem 2.1.

Theorem: 2.2 (Positivity-Negativity Theorem): Part-I: The non-cooperative elliptic system (5) with $a_{12}(x) \geq 0,\ a_{12}(x) \not\equiv 0, a_{21}(x) \leq 0,\ f_1(x) \geq 0,\ f_2(x) \leq 0,$ can be embedded into a co-operative elliptic system if, there exists $\delta < 0$ such that

$$(\alpha_8): -a_{21}\delta^2 + (a_{11}(x) - a_{22}(x))\delta - a_{12}(x) \leq 0, x \in \Omega,$$

Part-II: If

$$(\alpha_9): a_{11}(x) < \lambda_1,$$
$$(\alpha_{10}): -a_{21}(x) + a_{22}(x) < \lambda_1,$$
$$(\alpha_{11}): a_{11}(x) - a_{21}(x)\delta_- < \lambda_1,$$
$$(\alpha_{12}):\ f_1(x) - f_2(x)\delta_- \geq 0, \quad x \in \Omega,$$

then $u_1 \geq 0, u_2 \leq 0$ in $\Omega.$

Proof: Define

$$u_2^* = -u_2, a_{12}^* = -a_{12},$$
$$a_{21}^* = -a_{21},\ f_2^* = -f_2,$$

with this notations system (5) can be written as

$$\begin{aligned} -\Delta u_1 &= a_{11}u_1 + a_{12}^* u_2^* + f_1, \\ -\Delta u_2^* &= a_{21}^* u_1 + a_{22}u_2^* + f_2^*, \end{aligned} \quad \text{in } \Omega. \tag{13}$$

The above system (13) is a non-cooperative as $a_{12}^* \leq 0.$ Using Theorem 2.1 part-I, it can be embedded into a co-operative elliptic system

$$\begin{aligned} -\Delta u_1 &= (a_{11} - r)u_1 + (a_{12}^* - r\delta)u_2^* + ru_3 + f_1, \\ -\Delta u_2^* &= a_{21}^* u_1 + a_{22}u_2^* + f_2^*, \\ -\Delta u_3 &= (a_{11} - s + a_{21}^*\delta)u_1 + (a_{12}^* + a_{22}\delta - s\delta)u_2^* + su_3 + f_1 + \delta f_2^*. \end{aligned} \tag{14}$$

Using part-II of Theorem 2.1 we get $u_1 \geq 0, u_2 \leq 0$ in $\Omega.$

Remark 2.1: The elliptic system (5) with $a_{12}(x) \leq 0, a_{21}(x) \leq 0,$ is a non-cooperative. But assuming $a_{12}^* = -a_{12}, a_{21}^* = -a_{21}, u_2^* = -u_2,\ f_2^* = -f_2,$

we get the following co-operative elliptic system

$$\begin{aligned} -\Delta u_1 &= a_{11}u_1 + a_{12}^* u_2^* + f_1, \\ -\Delta u_2^* &= a_{21}^* u_1 + a_{22}u_2^* + f_2^*, \end{aligned} \quad \text{in } \Omega. \tag{15}$$

In this case we get positivity-negativity theorem if $f_1(x) \geq 0,\ f_2(x) \leq 0,$ and

$$(\alpha_{13}): a_{11}(x) - a_{12}(x) < \lambda_1,$$
$$(\alpha_{14}): -a_{21}(x) + a_{22}(x) < \lambda_1.$$

Remark 2.2: If the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ of non-cooperative elliptic system (5) are constants with $a_{12} \leq 0, a_{21} \geq 0,$ Then the positivity Theorem hold under following conditions:

$(\alpha_{15}): a_{22}+2\sqrt{-a_{12}a_{21}}<a_{11},$

$(\alpha_{16}): a_{22}<\lambda_1, a_{11}<\lambda_1-\dfrac{a_{12}a_{21}}{\lambda_1-a_{22}}$

$(\alpha_{17}): \sqrt{-a_{12}a_{21}}<\lambda_1-a_{22},$

$(\alpha_{18}): f_1(x)\geq 0,\ f_2(x)\geq 0,\ f_1(x)-f_2(x)\delta_-\geq 0,\quad x\in\Omega,$

[1], Figueiredo and Mitidieri.

Theorem: 2.3 If $a_{11}, a_{12}, a_{21}, a_{22}$ are constants with $a_{12}>0, a_{21}\leq 0$ for a non-cooperative elliptic system (5) and if the conditions

$(\alpha_{19}): a_{22}+2\sqrt{a_{11}a_{12}}<a_{11},$

$(\alpha_{20}): a_{22}<\lambda_1, a_{11}<\lambda_1+\dfrac{a_{12}a_{21}}{\lambda_1-a_{22}}$

$(\alpha_{21}): \sqrt{a_{12}a_{21}}<\lambda_1-a_{22},$

$(\alpha_{22}): f_1(x)\geq 0,\ f_2(x)=0,\ in\quad\Omega,$

and condition (α_8) is satisfied, then $u_1\geq 0, u_2\leq 0$ in $\Omega.$

Proof: Define $u_2^*=-u_2, a_{12}^*=-a_{12}, a_{21}^*=-a_{21}, f_2^*=-f_2$. With this notations system (5) can be written as

$$\begin{aligned}-\Delta u_1 &= a_{11}u_1+a_{12}^*u_2^*+f_1,\\ -\Delta u_2^* &= a_{21}^*u_1+a_{22}u_2^*+f_2^*\quad in\qquad \Omega.\end{aligned}\tag{16}$$

System (16) is a non-cooperative elliptic system as $a_{12}^*\leq 0.$ It can be embedded into a co-operative elliptic system

$$\begin{aligned}-\Delta u_1 &= (a_{11}-r)u_1+(a_{12}^*-r\delta)u_2^*+ru_3+f_1,\\ -\Delta u_2^* &= a_{21}^*u_1+a_{22}u_2^*+f_2^*,\\ -\Delta u_3 &= (a_{11}-s+a_{21}^*\delta)u_1+(a_{12}^*+a_{22}\delta-s\delta)u_2^*+su_3+f_1+\delta f_2^*.(17)\end{aligned}$$

Using Remark 2.2 we get n $u_1\geq 0, u_2\leq 0$ in $\Omega.$

In next section we will extend the embedding Theorem by considering 3×3 non co-operative elliptic system and by embedding it into a 4×4 co-operative elliptic system, and hence derive the positivity Theorem.

3. EMBEDDING OF 3×3 NON-COOPERATIVE SYSTEM INTO A 4×4 CO-OPERATIVE ELLIPTIC SYSTEM AND POSITIVITY- NEGATIVITY THEOREM:

Here we consider the elliptic system (3), (4) of section-I with L as a self adjoint operator, in particular $L_k=-\Delta,$ the Laplace operator, $k=1,2,3\cdots,n,.$ Now we state positivity-negativity theorem of [5], Hiwarekar, Kasture.

Theorem: 3.1 (Positivity-Negativity Theorem) Consider system (3),(4) of section-I with $L=-\Delta.$ Assume that the conditions (α_1) to (α_3) are satisfied. Let

$$(\alpha_{23}): \max[\max_{k\in(1,2\cdots m)}[\sup_{x\in\Omega}[\sum_{j=1}^{m}a_{kj}(x)-\sum_{j=m+1}^{n}a_{kj}(x)]];$$

$$\max_{k\in(m+1,m+2\cdots n)}[\sup_{x\in\Omega}[-\sum_{j=1}^{m}a_{kj}(x)+\sum_{j=m+1}^{n}a_{kj}(x)]]]<\lambda_1,$$

where λ_1 is the first eigen value of $-\Delta.$ Then $f_k \geq 0, k = 1, 2, \cdots m, \quad f_k \leq 0, k = m+1, m+2, \cdots n,$ imply

$$u_k \geq 0, k = 1, 2, \cdots m,$$
$$u_k \leq 0, k = m+1, m+2, \cdots n, \quad in \quad \Omega.$$

Now we consider the following system for functions defined on Ω.

$$\begin{aligned} -\Delta u_1 &= a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + f_1, \\ -\Delta u_2 &= a_{21}u_1 + a_{22}u_2 + a_{23}u_3 + f_2, \\ -\Delta u_3 &= a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + f_3 \end{aligned} \tag{18}$$

$$u_1 = u_2 = u_3 = 0 \text{ on } \partial\Omega. \tag{19}$$

We assume that the following conditions hold:

(α_{24}): For $x \in \Omega, \quad a_{ij}(x)$ are such that
$a_{21}(x) \equiv 0, a_{31}(x) < 0, a_{23}(x) > 0, a_{32}(x) > 0, a_{12}(x) < 0, a_{13}(x) > 0.$

(α_{25}): $a_{11}(x) < \lambda_1, a_{22}(x) + a_{23}(x) < \lambda_1, -a_{31}(x) + a_{32}(x) + a_{33}(x) < \lambda_1,$

where λ_1 is the first eigen value of $-\Delta.$
(α_{26}): $f_1(x) \geq 0, f_2(x) \leq 0, f_3(x) \leq 0,$ and
(α_{27}): there exists constants $k_1 \leq 0$ *and* $k_3 \geq 0$ such that $f_1 + k_1 f_2 + k_3 f_3 \geq 0.$

Theorem 3.2: A solution (u_1, u_2, u_3) of the system (18),(19) under conditions $(\alpha_{24}), (\alpha_{25}), (\alpha_{26}), (\alpha_{27})$ is such that
$u_1(x) \geq 0, u_2(x) \leq 0, u_3(x) \leq 0$ in $\Omega.$

Proof: We prove this positivity-negativity Theorem by using the embedding technique. For this we assume

$$u_4 = u_1 + \frac{a_{12}}{r}u_2 + \frac{a_{13}}{r}u_3, \quad r \neq 0. \tag{20}$$

Let $$s - r = \nu = \frac{a_{13}a_{32} + a_{22}a_{12}}{a_{12}} = \frac{a_{13}a_{33} + a_{23}a_{12}}{a_{13}} \tag{21}$$

where $r(x)$ and $s(x)$ are chosen latter and $\frac{a_{12}}{r} = k_1 \leq 0, \frac{a_{13}}{r} = k_3 \geq 0.$ With this we have

$$-\Delta u_4 = \left[a_{11} + \frac{a_{13}a_{31}}{r} - s \right] u_1 + su_4 + f_1 + k_1 f_1 + k_3 f_3, \tag{22}$$

$$-\Delta u_1 = (a_{11} - r)u_1 + ru_4 + f_1. \tag{23}$$

Now we consider the following system

$$\begin{aligned} -\Delta u_1 &= (a_{11} - r)u_1 + ru_4 + f_1, \\ -\Delta u_2 &= a_{22}u_2 + a_{23}u_3 + f_2, \\ -\Delta u_3 &= a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + f_3 \\ -\Delta u_4 &= \left[a_{11} + \frac{a_{13}a_{31}}{r} - s \right] u_1 + su_4 + f_1 + k_1 f_2 + k_3 f_3, \end{aligned} \tag{24}$$

with $u_1 = u_2 = u_3 = u_4 = 0$ on $\partial\Omega.$ (25)

We define

$$u_1^* = u_1, u_2^* = -u_2, u_3^* = -u_3,$$
$$u_4^* = u_4, a_{11}^* = a_{11}, a_{22}^* = a_{22},$$
$$a_{31}^* = -a_{31}, a_{32}^* = a_{32}, a_{33}^* = a_{33},$$
$$a_{13}^* = a_{13}, a_{23}^* = a_{23}, a_{12}^* = a_{12},$$
$$f_1^* = f_1, f_2^* = -f_2, f_3^* = -f_3,$$

so that the Above system can be written as

$$\begin{aligned} -\Delta u_1^* &= \left(a_{11}^* - r\right)u_1^* + ru_4^* + f_1^*, \\ -\Delta u_2^* &= a_{22}^* u_2^* + a_{23}^* u_3^* + f_2^*, \\ -\Delta u_3^* &= a_{31}^* u_1^* + a_{32}^* u_2^* + a_{33}^* u_3^* + f_3^* \\ -\Delta u_4^* &= \left[a_{11}^* - \frac{a_{13}^* a_{31}^*}{r} - s\right]u_1^* + su_4^* + f_1^* - k_1 f_2^* - k_3 f_3^* \end{aligned} \tag{26}$$

with $u_1^* = u_2^* = u_3^* = u_4^* = 0$ on $\partial\Omega.$ (27)

above system will be a co-operative elliptic system if $r \geq 0$ and $a_{11} + \frac{a_{13}a_{31}}{r} - s \geq 0.$

This will be satisfied if we choose

$$s \leq \min_{x\in\Omega} a_{11} + \min_{x\in\Omega} k_3\, a_{31}.$$

For positivity of a solution we require that

$a_{11}^* < \lambda_1, a_{22}^* + a_{23}^* < \lambda_1, a_{31}^* + a_{32}^* + a_{33}^* < \lambda_1,$ and $a_{11}^* - \frac{a_{13}^* a_{31}^*}{r} < \lambda_1.$

These conditions are satisfied because of (α_{25}) and because $a_{31} < 0, r > 0, a_{13} > 0$ and $a_{11} < \lambda_1.$ Choose k_1 and k_3 so that $f_1 + k_1 f_2 + k_3 f_3 \geq 0.$

Hence applying positivity Theorem 3.1 we get

$u_1(x) \geq 0, u_2(x) \leq 0, u_3(x) \leq 0$ in $\Omega.$

Remark 3.1: If the condition $a_{13} \leq 0$ of [5], Hiwarekar, Kasture, Theorem 3.2 is not satisfied, then also the positivity-negativity Theorem holds.

Example 3.1: Let

$$\Omega : \left\{(x_1, x_2) / x_1^2 + x_2^2 < 1\right\},$$
$$\partial\Omega : \left\{(x_1, x_2) / x_1^2 + x_2^2 = 1\right\}.$$

We consider system (18), (19) with

$$a_{11} < 0, a_{12} = -\left(x_1^2 + x_2^2\right), a_{13} = x_1^2 + x_2^2,$$
$$a_{21} = 0, a_{22} = -10\left(x_1^2 + x_2^2\right), a_{23} = x_1^2 + x_2^2 + 5,$$
$$a_{31} = a_{13} < 0, a_{32} = 5, a_{33} = -9\left(x_1^2 + x_2^2\right),$$

$$f_1(x) = a_1 \left(e_1^{1-\left(x_1^2+x_2^2\right)} - 1\right) + \left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 1\right) - \left(x_1^2 + x_2^2\right)\left(e^{\left(x_1^2+x_2^2\right)} - e\right) + 4e^{1-\left(x_1^2+x_2^2\right)}\left(1 - \left(x_1^2 + x_2^2\right)\right),$$

$$f_2(x) = 10\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 1\right) - \left(x_1^2 + x_2^2 + 5\right) + \left(e^{\left(x_1^2+x_2^2\right)} - e\right) - 4,$$

$$f_3(x) = a_3 \left(e_1^{1-\left(x_1^2+x_2^2\right)} - 1\right) - 5\left(x_1^2 + x_2^2 - 1\right) + 9\left(x_1^2 + x_2^2\right)\left(e^{\left(x_1^2+x_2^2\right)} - e\right) - 4e^{\left(x_1^2+x_2^2\right)}\left(x_1^2 + x_2^2 + 1\right).$$

Here we can verify that all conditions of Theorem 3.1 are satisfied and hence we have the conclusion $u_1(x) \geq 0, u_2(x) \leq 0, u_3(x) \leq 0$ in Ω.

We can verify that

$$u_1(x) = e^{1-\left(x_1^2+x_2^2\right)} - 1 \geq 0, \quad u_2(x) = x_1^2 + x_2^2 - 1 \leq 0,$$
$$u_3(x) = e^{\left(x_1^2+x_2^2\right)} - e \geq 0.$$

Remark: 3.2 With the assumptions of the theorem 1.2 and with $\frac{a_{12}}{r} = k_1 \leq 0, \frac{a_{13}}{r} = k_3 \geq 0$ it can be shown that last equality of (21) hold for all $a_{ij}(x)$.

Remark: 3.2 Cardoulis [6], obtained embedding Theorem for elliptic system involving a Scrodinger operator assuming that $a_{ij} \in L^\infty(R^n)$. We obtained an embedding Theorem and positivity Theorem of a solution for embedding of 3×3 non co-operative elliptic system by embedding it into a 4×4 co-operative elliptic system without assuming $a_{ij} \in L^\infty(R^n)$. Our system, however, does not involve Scrodinger operator.

4. POSITIVITY OF A SOLUTION OF HIGHER ORDER EQUATIONS:

Figueiredo, Mitidieri, [1], obtained a positivity of a solution of fourth order elliptic equation by transforming it into a second order co-operative elliptic system. We extend this result to higher order elliptic equations using same technique. Such results are found useful for the problem of oscillation of a suspension bridge, Mitidieri and Sweers, [8], Mc Kanna and Walter, [7]. Thus positivity results of higher order elliptic equations are important.

We consider a sixth order elliptic boundary value problem.

$$(\Delta + a_3)(\Delta + a_2)(\Delta + a_1)u_1 = \mu u_1 + f_1 \text{ in } \Omega, \tag{28}$$

$$\text{with } u_1 = 0, \Delta u_1 = 0, \Delta^2 u_1 = 0 \text{ on } \partial\Omega, \tag{29}$$

where a_1, a_2, a_3 and μ are functions of x. We are assuming that solution of the problem exists. Also assume that the following conditions are satisfied:

$(\alpha_{28}): a_1 + 1 < \lambda_1, a_2 + 1 < \lambda_1, a_3 - \mu < \lambda_1$, where λ_1 is the first eigen value of $-\Delta$.

$(\alpha_{29}): \mu \leq 0,\ f_1 < 0.$

We state the following positivity Theorem.

Theorem: 4.1 A solution of (28),(29) is positive if conditions $(\alpha_{28}), (\alpha_{29})$ are satisfied.

Proof: Let $(\Delta + a_1)u_1 = u_2$, $(\Delta + a_2)(\Delta + a_1)u_1 = u_3$, so that the given equations (28),(29) can be written as a system

$$-\Delta u_1 = a_1 u_1 - u_2,$$
$$-\Delta u_2 = a_2 u_2 - u_3, \tag{30}$$
$$-\Delta u_3 = a_3 u_3 - \mu u_1 - f_1, \quad in \quad \Omega$$

with $u_1 = u_2 = u_3 = 0$ on $\partial\Omega.$ (31)

System (30) is a non-cooperative elliptic system, we can transform it into a co-operative elliptic system using the following substitution:

$u_2 = -u^*{}_2, u_3 = u^*{}_3, f_1 = -f^*{}_1, \mu = -\mu^*,$ so that we get

$$-\Delta u_1 = a_1 u_1 + u^*{}_2,$$
$$-\Delta u^*{}_2 = a_2 u^*{}_2 + u^*{}_3, \tag{32}$$
$$-\Delta u^*{}_3 = a_3 u^*{}_3 + \mu^* u_1 + f^*{}_1, \quad in \quad \Omega,$$

which is a co-operative elliptic system. Using Theorem 1.3 of [5], Hiwarekar, Kasture we get

$u_1(x) \geq 0, u^*{}_2(x) \geq 0, u^*{}_3(x) \geq 0$ in $\Omega.$

Hence $u_1 \geq 0$ in $\Omega.$

Following Theorem is a generalization for the positivity of a solution of a higher order elliptic boundary value problem.

Theorem 4.2: Consider a boundary value problem

$$(\Delta + a_n)(\Delta + a_{n-1})\cdots(\Delta + a_2)(\Delta + a_1)u_1 = \mu u_1 + f_1 \quad \text{in } \Omega \tag{33}$$

$\Delta^r u_1 = 0, r = 1, 2, 3\cdots, n-1,$ and $u_1 = 0$, on $\partial\Omega.$ (34)

If the conclusion

$(\alpha_{30}) : a_r + 1 < \lambda_1, \; r = 1, 2, 3..., n-1, \qquad a_n + (-\mu)^n < \lambda_1,$

$(\alpha_{31}) : (-\mu)^n \geq 0, \quad (-f)^n \geq 0,$ are satisfied, then $u_1 \geq 0$ in $\Omega.$

Remark 4.1: If $a_1, a_2, a_3, \ldots a_n$ are constants then the conditions of Theorem 4.2 hold

$$a_1 < \lambda_1, \; (\lambda_1 - a_1)(\lambda_1 - a_2) > 0,$$
$$(\lambda_1 - a_1)(\lambda_1 - a_2)(\lambda_1 - a_3) > 0,$$
$$\ldots\ldots.$$
$$(\lambda_1 - a_1)(\lambda_1 - a_2),\ldots,(\lambda_1 - a_n) - \mu > 0.$$

Remark 4.2: The extension of the results of the section for more general self adjoint elliptic operators $L_1, L_2, \ldots\ldots, L_n$ can be obtained on similar way.

Remark 4.3: Hsu T Ku Mci C Ku, Xin-Min Zhang, [3], obtained lower bound estimates of Dirichlet eigen value problems of higher order elliptic equations on bounded domains in R^n. They also obtained similar estimates for self adjoint operators. Such estimates may be useful in verifying conditions $(\alpha_{28}), (\alpha_{30})$, in our Theorems 4.1 and 4.2.

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