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# MASS TRANSFER EFFECT ON A FREE CONVECTIVE VISCO-ELASTIC FLUID OVER AN INFINITE VERTICAL POROUS PLATE WITH VISCOUS DISSIPATION

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***ABSTRACT***

*An unsteady, two-dimensional, hydromagnetic, laminar mixed convective boundary layer flow of an incompressible and electrically conducting visco-elastic fluid (Walters $B'$ fluid model) along an infinite vertical plate embedded in the porous medium with heat and mass transfer has been studied, by taking into account the effect of viscous dissipation. The dimensionless governing equations for this investigation are solved analytically using two-term harmonic and non-harmonic functions. The effects of the physical parameters of the problem on the analytical solution of the momentum, heat and concentration equations are discussed. Our observation is in good agreement with the previous work pursued by Poonia and Chaudhary [1] without elastic parameter ($R_c$) i.e in case of viscous fluid.*

***Key word:*** *MHD flow, Free convection, Heat and mass transfer, Porous medium, Viscous dissipation, Visco-elastic fluid.*

## 1. INTRODUCTION

The phenomenon of hydromagnetic flow with heat and mass transfer in an electrically conducting fluid past a porous plate embedded in a porous medium has attracted the attention of a good number of investigators because of its varied applications in many engineering problems such as MHD generators, plasma studies, nuclear reactors, oil exploration, geothermal energy extractions and in the boundary layer control in the field of aerodynamics.

Owing to their numerous applications in industrial manufacturing process, the problem of heat and mass transfer in the boundary layers of a continuously moving semi infinite flat plate has attracted the attention of researchers for the past 3 decades. Some of the application areas are hot rolling, study production, metal spinning, drawing plastic films, glass blowing, continuous casting of metals and spinning of fibers.

Flow through porous medium is very prevalent in nature and therefore this study has become of principal interest in many scientific and engineering applications. There is increasing interest in magnetohydrodynamic (MHD) flows within fluid saturated porous media, because of numerous applications in geophysics and energy related problems, such as thermal insulation of buildings, enhanced recovery of petroleum resources, geophysical flows, packed bed reactors and sensible heat storage beds.

Combined buoyancy-generated heat and mass transfer due to temperature and concentration variations, in fluid-saturated porous media, have several important applications in variety of engineering processes including heat exchanger devices, petroleum reservoirs, chemical catalytic reactors, solar energy porous wafer collector systems, ceramic materials, migration of moisture through air contained in fibrous insulations and grain storage installations and dispersion of chemical contaminants through water-saturated soil, super convicting geothermic etc. MHD free convection and mass transfer flow over an infinite vertical porous plate with viscous dissipation has been investigated by Poonia and Chaudhary [1]. The vertical free convection boundary layer flow in porous media owning to combined heat and mass transfer has been studied by Bejan and Khair [2]. Deka [3] studied transient free convective flow past an

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infinite vertical cylinder with heat and mass transfer. Eldable et al. [4] investigated unsteady motion of MHD viscous incompressible fluid with heat and mass transfer through porous medium near a moving vertical plate.

Effect of Hall current on MHD free convection flow with heat and mass transfer past a vertical porous plate has been studied by Agrawal et al. [5]. Soundalgekar [6] presented an exact solution to the flow of a viscous fluid past an impulsively started infinite isothermal vertical plate with mass transfer. Choudhary et al. [7] have considered the effect of radiation on MHD heat transfer past vertical plate.

A study of Hall effects over the heat and mass transfer flow of visco-elastic fluid is made by Choudhary et al. [8]. Rushi Kumar and Nagaranjan [9] studied the mass transfer effects of MHD free convection flow of incompressible viscous dissipative fluid past an infinite vertical plate. Kumar Rakesh and Chand Khem [10] have investigated effect of slip conditions and Hall current on unsteady MHD flow of a visco elastic fluid past an infinite vertical porous plate through porous medium. Rahamnn and Sarkar [11] investigated the unsteady MHD flow of a visco- elastic oldroyd fluid under time varying body forces through a rectangular channel.

Singh and Singh [12] analyzed MHD flow of a dusty visco-elastic (oldroyd B-liquid) through a porous medium between two parallel plates inclined to the horizon. Ibrahim et al. [13] discussed the flow of a visco-elastic fluid between coaxial rotating porous disks with uniform suction or injection. Biswal and Sahoo [14] studied Hall current effects on free convective hydromagnetic flow of visco-elastic fluid past an infinite vertical plate. Norouzi et al. [15] have investigated convective heat transfer of viscoelastic flow in a curved duct.

Hooman and Merrikh [16] analyzed the solution of forced convection in a duct of rectangular cross-section saturated by a porous medium. Cheng [17] investigated fully developed natural convection heat and mass transfer in a vertical annular porous medium with a symmetric wall temperatures and concentrations.

Many common liquids such as oils, certain paints, polymer solution, some organic liquids and many new material of industrial importance exhibit both viscous and elastic properties. Therefore, the above fluid called visco-elastic fluids, is being studied extensively. Many researchers have shown their interest in the fluctuating flow of a viscous incompressible fluid past an infinite or semi-infinite flat plate. Visco-elastic fluid flow through porous media has attracted the attention of scientists and engineers because of its importance in the flow of the oil through porous rocks, the extraction of energy from geothermal region, the filtration of solids from liquids and drug permeation through human skin. The flow through porous media occurs in the ground water hydrology, irrigation and drainage problems and also in absorption and filtration processes in chemical engineering, the scientific treatment of the problem of irrigation, soil erosion and tile drainage are the present developments of porous media.

The objective of the present study is to consider the effect of non-Newtonion parameter particularly elastic parameter on the flow characteristics. The influence of the heat and mass transfer effects on an unsteady hydromagnetic flow of a visco-elastic fluid along an infinite vertical porous plate through porous medium in the presence of transverse magnetic field taking into consideration the effects of the viscous dissipation has been studied. The equation of continuity, motion, energy and mass transfer, which govern the flow field are solved by using a regular perturbation method.

## 2. MATHEMATICAL FORMULATION

We consider the unsteady two dimensional hydromagnetic laminar mixed convective boundary layer flow of a viscous incompressible and electrically conducting visco-elastic fluid along an infinite vertical flat plate in a uniform porous medium, in the presence of thermal and concentration buoyancy effect. The $x'-$axis is assumed to be oriented vertically upward direction along the plate and $y'-$ axis is taken normal to the plane of the plate. A uniform magnetic field is applied in the direction perpendicular to the plate. Assume the suction velocity to be time dependent. Now, under the usual Boussinesq's approximation, the governing boundary layer equations are:

Equation of Continuity: $$\frac{\partial v'}{\partial y'}=0 \tag{1}$$

Equation of Motion:

$$\frac{\partial u'}{\partial t'}+v'\frac{\partial u'}{\partial y'}=\nu\frac{\partial^2 u'}{\partial y'^2}+g\beta\left(T'-T'_\infty\right)+g\beta^*\left(C'-C'_\infty\right)-\frac{\sigma B_0^2 u'}{\rho}-\frac{\nu u'}{K'}-\frac{k_o}{\rho}\left[\frac{\partial^3 u'}{\partial t'\partial y'^2}+v'\frac{\partial^3 u'}{\partial y'^3}\right] \tag{2}$$

Equation of Energy: $$\frac{\partial T'}{\partial t'}+v'\frac{\partial T'}{\partial y'}=a\frac{\partial^2 T'}{\partial y'^2}+\frac{\mu}{\rho C_p}\left(\frac{\partial u'}{\partial y'}\right)^2 \tag{3}$$

Equation of Mass Transfer: $$\frac{\partial C'}{\partial t'} + v'\frac{\partial C'}{\partial y'} = D\frac{\partial^2 C'}{\partial y'^2} \tag{4}$$

Where $u', v'$ denote the components of velocity in the boundary layer in $x'$ and $y'$ direction respectively ; $T'$ - the temperature in the boundary ; $T'_\infty$ - the temperature of the free stream; $t'$ - the time ; $\beta, \beta^*$ - the volumetric co-efficient of thermal and concentration expansion respectively ; $\rho -$ the density of the fluid ; $\mu -$ the coefficient of viscosity ; g – the acceleration due to the gravity ; $\nu$ – the kinematics viscosity ; $\sigma -$ The electrical conductivity ; $C_p -$ the specific heat at constant pressure ; $a = \frac{K'}{\rho C_p}$ (the thermal diffusivity) ; $K' -$ The co-efficient of thermal conductivity ; $k_0$ -the elasticity parameter ; $B_0$ – the magnetic induction ; $C'$ -the concentration in the boundary layer ; $C'_\infty$ - The concentration in the fluid for away from the plate ; $D$ – The mass diffusivity.

The boundary conditions for the velocity, temperature and concentration fields are:

$$\left.\begin{array}{l} y' = 0 : u' = 0, T' = T'_\infty + T_0(t)\left(T'_0 - T'_\infty\right) \\ C' = C'_\infty + C_0(t)\left(C'_0 - C'_\infty\right) \\ y' \to \infty :\ u' \to 0,\ T' \to T'_\infty,\ C' \to C'_\infty \end{array}\right\} \tag{5}$$

Non-dimensional quantities are defined as:

$$\begin{array}{l} u = \frac{u'}{v_0},\ y = \frac{v_0 y'}{\nu},\ t = \frac{v_0^2 t'}{4\nu}, S_c = \frac{\nu}{D}, K = \frac{K' v_0^2}{\nu^2}\ \omega = \frac{4\nu\omega'}{v_0^2},\ T_0(t) = 1 + \in e^{i\omega t},\ \theta = \frac{T' - T'_\infty}{T'_0 - T'_\infty} \\ \varphi = \frac{C' - C'_\infty}{C'_0 - C'_\infty},\ M = \frac{\sigma B_0^2 \nu}{\rho v_0^2},\ P_r = \frac{\mu C_p}{K} \\ G_r = \frac{g\beta\nu\left(T'_0 - T'_\infty\right)}{v_0^3},\ G_c = \frac{g\beta^*\nu\left(C'_0 - C'_\infty\right)}{v_0^3} \end{array} \tag{6}$$

From equation of continuity (1) it is clear that the suction velocity normal to the plate is either a constant or a function of the time.

Hence, it is assumed in the form $$v' = -v_0(1 + \in \alpha e^{i\omega t}) \tag{7}$$

Where $\alpha$ is a real positive constant, $\in$ and $\in \alpha$ are small less than unity and $v_0$ is a non-zero positive constant suction velocity, the negative sign indicates that the suction is towards the plate.

In terms of (6), equations (2), (3), (4) become

$$\frac{\partial^2 u}{\partial y^2} + \left(1 + \in \alpha e^{i\omega t}\right)\frac{\partial u}{\partial y} = \frac{1}{4}\frac{\partial u}{\partial t} - G_r\theta - G_c\varphi + \left(M + \frac{1}{K}\right)u + R_c\left[\frac{1}{4}\frac{\partial^3 u}{\partial t\partial y^2} - \left(1 + \in \alpha e^{i\omega t}\right)\frac{\partial^3 u}{\partial y^3}\right] \tag{8}$$

$$\frac{\partial^2 \theta}{\partial y^2} + P_r\left(1 + \in \alpha e^{i\omega t}\right)\frac{\partial \theta}{\partial y} = \frac{1}{4}P_r\frac{\partial \theta}{\partial t} - E\, P_r\left(\frac{\partial u}{\partial y}\right)^2 \tag{9}$$

$$\frac{\partial^2 \varphi}{\partial y^2} + S_c\left(1 + \in \alpha e^{i\omega t}\right)\frac{\partial \varphi}{\partial y} = \frac{1}{4}S_c\frac{\partial \varphi}{\partial t} \tag{10}$$

 

The boundary conditions are:

$$\left.\begin{array}{l} y=0:\ u=0,\ \theta=T_0(t),\ \varphi=C_0(t) \\ y\to\infty:\ u\to 0,\ \theta\to 0,\ \varphi\to 0 \end{array}\right\} \tag{11}$$

Where $T_0(t)$ – The temperature at the wall, $M$ – The magnetic parameter, $P_r$ – The Prandtl number, $K$ – The Porosity parameter, $G_r$ – The thermal Grashof number, $G_c$ – The solutal Grashof number, $E$ – The Eckert number, $\omega$ - The frequency of the suction velocity,

$S_c$ – The Schmidt number, $R_c=\dfrac{k_0 v_0^2}{\rho\nu^2}$ (The visco-elastic parameter),

$E=\dfrac{v_0^2}{C_P\left(T_0'-T_\infty'\right)}$ ( Eckert number).

## 3. SOLUTION OF THE PROBLEM

For the solution of equations (8), (9) and (10), we assume

$$\left.\begin{array}{l} u(y,t)=u_1(y)+\in e^{i\omega t}u_2(y) \\ \theta(y,t)=1+\in e^{i\omega t}-\theta_1(y)-\in e^{i\omega t}\theta_2(y) \\ \varphi(y,t)=1+\in e^{i\omega t}-\varphi_1(y)-\in e^{i\omega t}\varphi_2(y) \end{array}\right\} \tag{12}$$

Substituting equation (12) in equations (8), (9) and (10) , equating harmonic terms and neglecting co-efficient of $\in^2$, we get

$$R_c u_1'''(y)+u_1''(y)+u_1'(y)-\left(M+\frac{1}{K}\right)u_1(y)=-G_r\left[1-\theta_1(y)\right]-G_c\left[1-\varphi_1(y)\right] \tag{13}$$

$$\begin{aligned} R_c u_2'''(y)+\left(1-\frac{R_c i\omega}{4}\right)u_2''(y)+u_2'(y)-\left(M+\frac{1}{K}+\frac{i\omega}{4}\right)u_2(y) \\ =-G_r\left[1-\theta_2(y)\right]-G_c\left[1-\varphi_2(y)\right]-\alpha u_1'(y)-R_c\alpha u_1''(y) \end{aligned} \tag{14}$$

$$\theta_1''(y)+P_r\theta_1'(y)=E\,P_r\left\{u_1'(y)\right\}^2 \tag{15}$$

$$\theta_2''(y)+P_r\theta_2'(y)-\frac{i\omega}{4}P_r\theta_2(y)=-\frac{i\omega}{4}P_r+2EP_r u_1'(y)\,u_2'(y)-\alpha P_r\theta_1'(y) \tag{16}$$

$$\varphi_1''(y)+S_c\varphi_1'(y)=0 \tag{17}$$

$$\varphi_2''(y)+S_c\varphi_2'(y)-\frac{i\omega}{4}S_c\varphi_2(y)=-\frac{i\omega}{4}S_c-\alpha S_c\,\varphi_1'(y) \tag{18}$$

where, primes denote differentiation w.r.t. ‘ $y$ ’. The corresponding conditions are:

$$\left.\begin{array}{l} y=0: u_1=0,\ u_2=0,\ \theta_1=0,\ \theta_2=0,\ \varphi_1=0,\ \varphi_2=0 \\ y\to\infty: u_1\to 0,\ u_2\to 0,\ \theta_1\to 1,\ \theta_2\to 1,\ \varphi_1\to 1,\ \varphi_2\to 1 \end{array}\right\} \tag{19}$$

Solving equations (17) and (18), under the boundary conditions (19), we get $\varphi_1(y) = 1 - e^{-S_c y}$ (20)

$$\varphi_2(y) = (iI_0 - 1)e^{-\lambda_1 S_c y} + 1 - iI_0 e^{-S_c y} \tag{21}$$

where $I_0 = \dfrac{4\alpha S_c}{\omega}$ $\lambda_1 = \dfrac{1}{2}\left(1 + \sqrt{1 + \dfrac{i\omega}{S_c}}\right)$

The equations (13) to (16) are still coupled and non-linear, whose exact solution are not possible, so we can expand $u_1, u_2, \theta_1, \theta_2$ in terms of E (Eckert no.) in the following form, as the Eckert number is very small for incompressible flows.

$$\left.\begin{aligned} u_1(y) &= u_{11}(y) + Eu_{12}(y) \\ u_2(y) &= u_{21}(y) + Eu_{22}(y) \\ \theta_1(y) &= \theta_{11}(y) + E\theta_{12}(y) \\ \theta_2(y) &= \theta_{21}(y) + E\theta_{22}(y) \end{aligned}\right\} \tag{22}$$

Introducing equations (22) into (13) to (16), we obtain the following system of equations.

$$R_c u_{11}'''(y) + u_{11}''(y) + u_{11}'(y) - \left(M + \frac{1}{K}\right)u_{11}(y) = -G_r\left[1 - \theta_{11}(y)\right] - G_c\left[1 - \varphi_1(y)\right] \tag{23}$$

$$R_c u_{12}'''(y) + u_{12}''(y) + u_{12}'(y) - \left(M + \frac{1}{K}\right)u_{12}(y) = G_r\theta_{12}(y) \tag{24}$$

$$\begin{aligned} R_c u_{21}'''(y) + \left(1 - \frac{R_c i\omega}{4}\right)u_{21}''(y) + u_{21}'(y) - \left(M + \frac{1}{K} + \frac{i\omega}{4}\right)u_{21} \\ = -G_r\left[1 - \theta_{21}(y)\right] - G_c\left[1 - \varphi_2(y)\right] - \alpha u_{11}'(y) - R_c \alpha u_{11}'''(y) \end{aligned} \tag{25}$$

$$R_c u_{22}'''(y) + \left(1 - \frac{R_c i\omega}{4}\right)u_{22}''(y) + u_{22}'(y) - \left(M + \frac{1}{K} + \frac{i\omega}{4}\right)u_{22}(y) = G_r\theta_{22}(y) - \alpha u_{12}'(y) - R_c\alpha u_{12}'''(y) \tag{26}$$

$$\theta_{11}''(y) + P_r\,\theta_{11}'(y) = 0 \tag{27}$$

$$\theta_{12}''(y) + P_r\,\theta_{12}'(y) = P_r\left[u_{11}'(y)\right]^2 \tag{28}$$

$$\theta_{21}''(y) + P_r\,\theta_{21}'(y) - \frac{i\omega}{4}P_r\,\theta_{21}(y) = -\frac{i\omega}{4}P_r - \alpha\,P_r\,\theta_{11}'(y) \tag{29}$$

$$\theta_{22}''(y) + P_r\,\theta_{22}'(y) - \frac{i\omega}{4}P_r\,\theta_{22}(y) = -\alpha\,P_r\,\theta_{12}'(y) + 2P_r\,u_{11}'(y)u_{21}'(y) \tag{30}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} y = 0 : u_{11} = 0, u_{12} = 0,\ u_{21} = 0, u_{22} = 0 \\ \theta_{11} = 0, \theta_{12} = 0,\ \theta_{21} = 0,\ \theta_{22} = 0 \\ y \to \infty : u_{11} \to 0, u_{12} \to 0, u_{21} \to 0, u_{22} \to 0 \\ \theta_{11} \to 1, \theta_{12} \to 0, \theta_{21} \to 1, \theta_{22} \to 0 \end{aligned}\right\} \tag{31}$$

Solving equations (27) and (29) under the boundary conditions (31), we get

$$\theta_{11}(y) = 1 - e^{-P_r y} \tag{32}$$

$$\theta_{21} = 1 - e^{-b_1 P_r y} + iI_{26}\left(e^{-b_1 P_r y} - e^{-P_r y}\right) \tag{33}$$

Where $b_1 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{i\omega}{P_r}}\right)$ & $I_{26} = \frac{4\alpha P_r}{\omega}$

The equations (23) to (26) and the equations (28) & (30) are still coupled and non-linear, whose exact solution are not possible. So we can expand $u_{11}$, $u_{12}$, $u_{21}$, $u_{22}$ in terms of elastic parameter ( $R_c$ ) in the following form

$$\left.\begin{aligned} u_{11}(y) &= u_{111}(y) + R_c u_{112}(y) \\ u_{12}(y) &= u_{121}(y) + R_c u_{122}(y) \\ u_{21}(y) &= u_{211}(y) + R_c u_{212}(y) \\ u_{22}(y) &= u_{221}(y) + R_c u_{222}(y) \end{aligned}\right\} \tag{34}$$

The corresponding boundary conditions are:

$$\left.\begin{aligned} & y = 0 : u_{111} = 0, u_{112} = 0, u_{121} = 0, u_{122} = 0, \\ & u_{211} = 0, u_{212} = 0, u_{221} = 0, u_{222} = 0 \\ & \theta_{11} = 0, \theta_{12} = 0, \theta_{21} = 0, \theta_{22} = 0, \\ & y \to \infty : u_{111} = 0, u_{112} = 0, u_{121} = 0, u_{122} = 0, \\ & u_{211} = 0, u_{212} = 0, u_{221} = 0, u_{222} = 0 \\ & \theta_{11} \to 1, \theta_{12} \to 0, \theta_{21} \to 1, \theta_{22} \to 0 \end{aligned}\right\} \tag{35}$$

Introducing equations (35) into equations (23) to (26), we obtain the following system of equations

$$u''_{111}(y) + u'_{111}(y) - \left(M + \frac{1}{K}\right)u_{111}(y) = -G_r\left[1 - \theta_{11}(y)\right] - G_c\left[1 - \varphi_1(y)\right] \tag{36}$$

$$u''_{112}(y) + u'_{112}(y) - \left(M + \frac{1}{K}\right)u_{112}(y) = -u'''_{111}(y) \tag{37}$$

$$u''_{121}(y) + u'_{121}(y) - \left(M + \frac{1}{K}\right)u_{121}(y) = G_r\theta_{12} \tag{38}$$

$$u''_{122}(y) + u'_{122}(y) - \left(M + \frac{1}{K}\right)u_{122}(y) = -u'''_{121}(y) \tag{39}$$

$$u''_{211}(y) + u'_{211}(y) - \left(M + \frac{1}{K} + \frac{i\omega}{4}\right)u_{211}(y) = -G_r\left[1 - \theta_{21}(y)\right] - G_c\left[1 - \varphi_2(y)\right] - \alpha u'_{111}(y) \tag{40}$$

$$u''_{212}(y) + u'_{212}(y) - \left(M + \frac{1}{K} + \frac{i\omega}{4}\right)u_{212}(y) = -\alpha u'_{112}(y) - \alpha u'''_{111} - u'''_{211} + \frac{i\omega}{4}u''_{211}(y) \tag{41}$$

$$u''_{221}(y)+u'_{221}(y)-\left(M+\frac{1}{K}+\frac{i\omega}{4}\right)u_{221}(y)=G_r\theta_{22}(y)-\alpha u'_{121}(y) \quad (42)$$

$$u''_{222}(y)+u'_{222}(y)-\left(M+\frac{1}{K}+\frac{i\omega}{4}\right)u_{222}(y)=-\alpha u'_{122}(y)-\alpha u'''_{121}(y)-u'''_{221}(y)+\frac{i\omega}{4}u''_{221}(y) \quad (43)$$

Using the boundary condition (35) the solution of the equations (36) to (43) and the equations (28) & (30) are given by

$$u_{111}(y)=\left(I_3+I_4\right)e^{-a_1 y}-I_3 e^{-P_r y}-I_4 e^{-S_c y} \quad (44)$$

$$u_{112}(y)=\left(J_1+J_4\right)e^{-a_1 y}-J_2 e^{-P_r y}-J_3 e^{-S_c y} \quad (45)$$

$$\theta_{12}(y)=I_{11}e^{-P_r y}+I_5 e^{-2P_r y}+I_6 e^{-2S_c y}+I_7 e^{-2a_1 y}+I_8 e^{-(P_r+S_c)y}-I_9 e^{-(a_1+S_c)y}-I_{10}e^{-(a_1+P_r)y} \quad (46)$$

$$u_{121}(y)=I_{25}e^{-a_1 y}+I_{18}e^{-P_r y}+I_{19}e^{-2P_r y}+I_{20}e^{-2S_c y}+I_{21}e^{-2a_1 y}+I_{22}e^{-(P_r+S_c)y}-I_{23}e^{-(a_1+S_c)y}-I_{24}e^{-(a_1+P_r)y} \quad (47)$$

$$u_{122}(y)=\left(J_{13}+J_5\right)e^{-a_1 y}+J_6 e^{-P_r y}+J_7 e^{-2P_r y}+J_8 e^{-2S_c y}+J_9 e^{-2a_1 y}+J_{10}e^{-(P_r+S_c)y}-J_{11}e^{-(a_1+S_c)y}-J_{12}e^{-(a_1+P_r)y} \quad (48)$$

$$u_{211}(y)=I_{30}e^{-d_1 y}-A_{22}e^{-b_1 P_r y}-A_{23}e^{-P_r y}-A_{24}e^{-\lambda_1 S_c y}-A_{25}e^{-S_c y}+A_{26}e^{-a_1 y} \quad (49)$$

$$u_{212}(y)=\left(I_{32}+A_{48}\right)e^{-d_1 y}+A_{45}e^{-a_1 y}-A_{46}e^{-P_r y}-A_{47}e^{-S_c y}-A_{49}e^{-b_1 P_r y}-A_{50}e^{-\lambda_1 S_c y} \quad (50)$$

$$\begin{aligned}\theta_{22}(y)=\;&I_{61}e^{-b_1 P_r y}+iI_{26}I_{11}e^{-P_r y}+A_{78}e^{-2P_r y}+A_{79}e^{-2S_c y}+A_{80}e^{-(P_r+S_c)y}\\&+A_{81}e^{-(a_1+S_c)y}-A_{82}e^{-(a_1+P_r)y}+A_{60}e^{-(a_1+d_1)y}+A^{*}_{81}e^{-2a_1 y}-A_{64}e^{-(a_1+b_1 P_r)y}\\&-A_{65}e^{-(a_1+\lambda_1 S_c)y}+A_{66}e^{-(d_1+P_r)y}-A_{70}e^{-(1+b_1)P_r y}-A_{71}e^{-(P_r+\lambda_1 S_c)y}-A_{72}e^{-(d_1+S_c)y}\\&+A_{76}e^{-(S_c+b_1 P_r)y}+A_{77}e^{-(1+\lambda_1)S_c y}\end{aligned} \quad (51)$$

$$\begin{aligned}u_{221}(y)=\;&I_{71}e^{-d_1 y}+A_{83}e^{-b_1 P_r y}+A_{108}e^{-P_r y}+A_{109}e^{-2P_r y}+A_{110}e^{-2S_c y}\\&+A_{111}e^{(P_r+S_c)y}+A_{112}e^{-(a_1+S_c)y}-A_{113}e^{-(a_1+P_r)y}+A_{90}e^{-(a_1+d_1 y)}+A_{114}e^{-2a_1 y}\\&-A_{92}e^{-(a_1+b_1 P_r)y}-A_{93}e^{-(a_1+\lambda_1 S_c)y}+A_{94}e^{-(d_1+P_r)y}-A_{95}e^{-(1+b_1)P_r y}-A_{96}e^{-(P_r+\lambda_1 S_c)y}\\&-A_{97}e^{-(d_1+S_c)y}+A_{98}e^{-(S_c+b_1 P_r)y}+A_{99}e^{-(1+\lambda_1)S_c y}-A_{100}e^{-a_1 y}\end{aligned} \quad (52)$$

$$\begin{aligned}u_{222}(y)=\;&\left(I_{72}+A_{177}\right)e^{-d_1 y}+A_{169}e^{-a_1 y}+A_{170}e^{-P_r y}+A_{171}e^{-2P_r y}+A_{172}e^{-2S_c y}\\&+A_{173}e^{-2a_1 y}+A_{174}e^{-(P_r+S_c)y}+A_{175}e^{-(a_1+S_c)y}-A_{176}e^{-(a_1+P_r)y}+A_{178}e^{-b_1 P_r y}\\&+A_{179}e^{-(a_1+d_1)y}-A_{180}e^{-(a_1+b_1 Pr)y}-A_{181}e^{-(a_1+\lambda_1 S_c)y}-A_{182}e^{-(d_1+P_r)y}+A_{183}e^{-(1+b_1)P_r y}\\&-A_{184}e^{-(P_r+\lambda_1 S_c)y}-A_{185}e^{-(d_1 S_c)y}+A_{186}e^{-(S_c+b_1 P_r)y}+A_{187}e^{-(1+\lambda_1)S_c y}\end{aligned} \quad (53)$$

$$\begin{aligned}u_{11}(y)&=u_{111}(y)+R_c u_{112}(y)\\&=I_{34}e^{-a_1 y}-I_{35}e^{-P_r y}-I_{36}e^{-S_c y}\end{aligned} \quad (54)$$

$$u_{12}(y)=u_{121}(y)+R_c u_{122}(y)$$

$$
\begin{aligned}
&= I_{37}e^{-a_1 y} + I_{38}e^{-P_r y} + I_{39}e^{-2P_r y} + I_{40}e^{-2S_c y} + I_{41}e^{-2a_1 y} + I_{42}e^{-(P_r+S_c)y} - I_{43}e^{-(a_1+S_c)y} \\
&\quad -I_{44}e^{-(a_1+P_r)y}
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
u_{21}(y) &= u_{211}(y) + R_c u_{212}(y) \\
&= A_{51}e^{-d_1 y} + A_{52}e^{-a_1 y} - A_{53}e^{-P_r y} - A_{54}e^{-S_c y} - A_{55}e^{-b_1 P_r y} - A_{56}e^{-\lambda_1 S_c y}
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
u_{22}(y) &= u_{221}(y) + R_c u_{222}(y) \\
&= A_{188}e^{-d_1 y} + A_{189}e^{-b_1 P_r y} + A_{190}e^{-P_r y} + A_{191}e^{-P_r y} + A_{192}e^{-2S_c y} + A_{193}e^{-(P_r+S_c)y} \\
&\quad + A_{194}e^{-(a_1+S_c)y} - A_{195}e^{-(a_1+P_r)y} + A_{196}e^{-(a_1+d_1)y} + A_{197}e^{-2a_1 y} - A_{198}e^{-(a_1+b_1 P_r)y} \\
&\quad - A_{199}e^{-(a_1+\lambda_1 S_c)y} + A_{200}e^{-(d_1+P_r)y} - A_{201}e^{-(1+b_1)P_r y} - A_{202}e^{-(P_r+\lambda_1 S_c)y} \\
&\quad - A_{203}e^{-(d_1+S_c)y} + A_{204}e^{-(S_c+b_1 P_r)y} + A_{205}e^{-(1+\lambda_1)S_c y} - A_{206}e^{-a_1 y}
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
u_1(y) &= u_{11}(y) + Eu_{12}(y) \\
&= \left(I_{34} + EI_{37}\right)e^{-a_1 y} + \left(I_{35} + EI_{38}\right)e^{-P_r y} - I_{36}e^{-S_c y} + EI_{39}e^{-2P_r y} + EI_{40}e^{-2S_c y} \\
&\quad + EI_{41}e^{-2a_1 y} + EI_{42}e^{-(P_r+S_c)y} - EI_{43}e^{-(a_1+S_c)y} - EI_{44}e^{-(a_1+P_r)y}
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
u_2(y) &= u_{21}(y) + Eu_{22}(y) \\
&= \left(A_{51} + EA_{188}\right)e^{-d_1 y} + \left(A_{52} - A_{206}\right)e^{-a_1 y} - \left(A_{53} - EA_{190}\right)e^{-P_r y} - A_{54}e^{-S_c y} \\
&\quad - \left(A_{55} - EA_{189}\right)e^{-b_1 P_r y} - A_{56}e^{-\lambda_1 S_c y} + EA_{191}e^{-2P_r y} + EA_{192}e^{-2S_c y} + E_{193}e^{-(P_r+S_c)y} \\
&\quad + EA_{194}e^{-(a_1+S_c)y} - EA_{195}e^{-(a_1+P_r)y} + EA_{196}e^{-(a_1+d_1)y} + EA_{197}e^{-2a_1 y} \\
&\quad - EA_{198}e^{-(a_1+b_1 P_r)y} - EA_{199}e^{-(a_1+\lambda_1 S_c)y} + EA_{200}e^{-(d_1+P_r)y} - EA_{201}e^{-(1+b_1)P_r y} \\
&\quad - EA_{202}e^{-(P_r+\lambda_1 S_c)y} - EA_{203}e^{-(d_1+S_c)y} + EA_{204}e^{-(S_c+b_1 P_r)y} + EA_{205}e^{-(1+\lambda_1)S_c y}
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
\theta_1(y) &= \theta_{11}(y) + E\theta_{12}(y) \\
&= \left(1 - e^{-P_r y}\right) + E\Big[I_{11}e^{-P_r y} + I_5 e^{-P_r y} + I_6 e^{-2S_c y} + I_7 e^{-2a_1 y} \\
&\qquad + I_8 e^{-(P_r+S_c)y} - I_9 e^{-(a_1+S_c)y} - I_{10}e^{-(a_1+P_r)y}\Big]
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\theta_2(y) &= \theta_{21}(y) + E\theta_{22}(y) = \left[1 - e^{-b_1 P_r y} + iI_{26}\left(e^{-b_1 P_r y} - e^{-P_r y}\right)\right] \\
&\quad + E\Big[I_{61}e^{-b_1 P_r y} + iI_{26}e^{-P_r y} + A_{78}e^{-P_r y} + A_{79}e^{-S_c y} + A_{80}e^{-(P_r+S_c)y} \\
&\quad + A_{81}e^{-(a_1+S_c)y} - A_{82}e^{-(a_1+P_r)y} + A_{60}e^{-(a_1+d_1)y} + A_{61}e^{-2a_1 y} - A_{70}e^{-(1+b_1)P_r y} \\
&\quad - A_{71}e^{-(P_r+\lambda_1 S_c)y} - A_{72}e^{-(d_1+S_c)y} + A_{76}e^{-(Sc+b_1 P_r)y} + A_{77}e^{-(a_1+\lambda_1 S_c)y} \\
&\quad - A_{64}e^{-(a_1+b_1 P_r)y} - A_{65}e^{-(a_1+\lambda_1 S_c)y} + A_{66}e^{-(d_1+P_r)y}
\end{aligned} \tag{61}
$$

## 4. RESULTS AND DISCUSSION

Graphical representation of results is very useful to discuss the physical features presented by the solutions. In order to get a physical insight into the problem, factors such as velocity, temperature, concentration have been discussed by assigning numerical values to various parameters obtained in the mathematical formulation of the problem and results are graphically shown.

Fig.1 exhibits the effects of different pertinent parameters the thermal Grashof number ($G_r$), the solutal Grashof number ($G_c$), the Prandtl number ($P_r$), the Schmidt number ($S_c$), the Eckert number ($E$), The magnetic parameter ($M$), the Porosity parameter ($K$) and the Elastic parameter ($R_c$) on the fluid flow.

It is observed that under the influence of magnetic parameter ($M$) in the case of air ($P_r = 0.7$) and water ($P_r = 7.0$), the velocity increases rapidly near to the wall of the porous plate, reaches a maximum and then decays to the free stream value. It is concluded that in the case of air ($P_r = 0.7$) and water ($P_r = 7.0$), the fluid velocity decreases with increasing magnetic parameter in the presence of elastic elements. The impression of a transverse magnetic field to an electrically conducting fluid give rise to a resistive type force is called as Lorentz force. This force has tendency to slow down the motion of the fluid. This result qualitatively agrees with the expectations, since magnetic field exerts retarding force on the mixed convection flow. It is interesting to note that the Lorentz force slowdown the motion of the fluid in the boundary layer.

Further, it is observed that with an increase of the viscous dissipation parameter i.e Eckert number ($E$) and porosity parameter ($K$) the fluid velocity increases sharply and a peak value near to the plate and decays continuously with increasing free stream value in the presence of elastic elements ($R_c$). Again, in the presence of elastic elements the Schmidt number leads to decreases the fluid velocity. The same observation was made by Poonia and Chaudhury [1] in the absence of elastic elements.

Further, it is observed that an increase in the thermal Grashof number ($G_r$), the solutal Grashof number ($G_c$) leads to rise the fluid velocity due to enhancement in buoyancy force sharply and attains distinctive maximum value near to the wall of the porous plate and then decays to the free stream value in the presence of elastic elements. Physically $G_r > 0$ means heating of the fluid or cooling of the boundary surface, $G_r < 0$ means cooling of the fluid or heating of the boundary surface and $G_r = 0$ corresponds to the absence of free convection current.

Table.1 depicts the effect of visco-elastic parameter on the fluid flow. On carefully observation, it is concluded that the fluid velocity decreases with increasing of elastic elements which is the objective of our problem.

Fig.2 exhibits the temperature distribution for various pertinent parameters, the Prandtl number ($P_r$), the Eckert number ($E$), the Elastic parameter ($R_c$) and the frequency of the suction velocity ($\omega$) on the fluid flow. The Prandtl number defines the ratio of momentum diffusivity to thermal diffusivity. It is observed that an increase in the Prandtl number results a decrease of the thermal boundary layer thickness and in general lower average temperature within the boundary layer. The reason is that increasing values of Prandtl number equivalent to increase the thermal conductivities and therefore heat is able to diffuse away from the heated plate more rapidly. Thus, it is concluded that in case of smaller Prandtl number as the thermal boundary layer is thicker, the rate of heat transfer is reduced in the presence of elastic elements.

Table.2 shows the effect of elastic parameter on the temperature distribution. It is found that with an increase of elastic property the greater viscous dissipative heat causes a rise in the temperature. The reverse effect is marked in case of smaller viscous dissipation parameter i.e the Eckert number.

Fig.3 shows the effect of Schmidt number on the concentration. It is observed that as the Schmidt number increases, the concentration decreases.

## 5. CONCLUSION

- In presence or absence of elastic elements, the magnetic interaction parameter, Prandtl number, Schmidt number leads to decrease the fluid velocity.
- With greater viscous dissipative heat causes a rise in the velocity as well as the temperature.
- Fluid velocity increases with increasing the porosity parameter.
- Fluid velocity and concentration decreases with increasing the Schmidt number.
- Temperature and concentration remain same for increasing the frequency of the suction velocity.
- Fluid velocity increased with increasing of thermal Grashof number and solutal Grashof number which leads to enhance the thermal and concentration buoyancy effects.

 

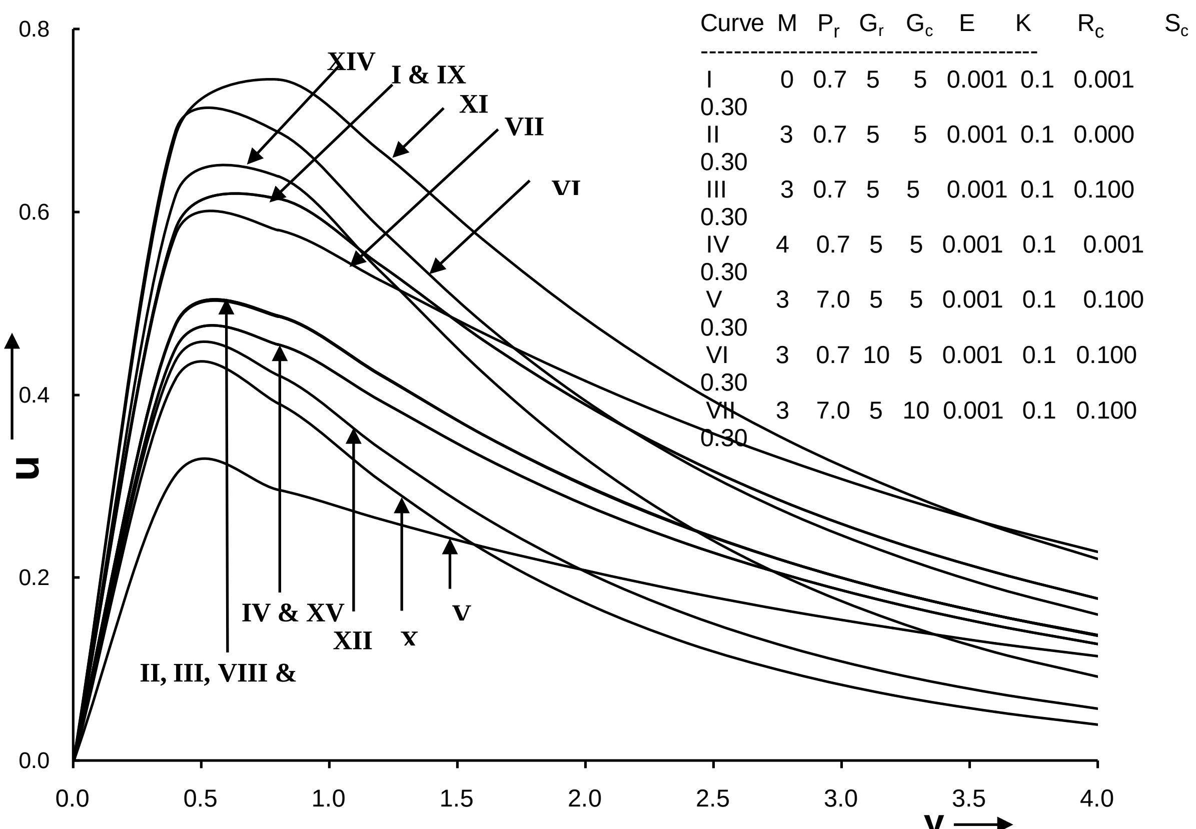


**Fig-1 Velocity distribution for $\omega = 10$, $\alpha = 0.5$ & $\varepsilon = 0.01$**

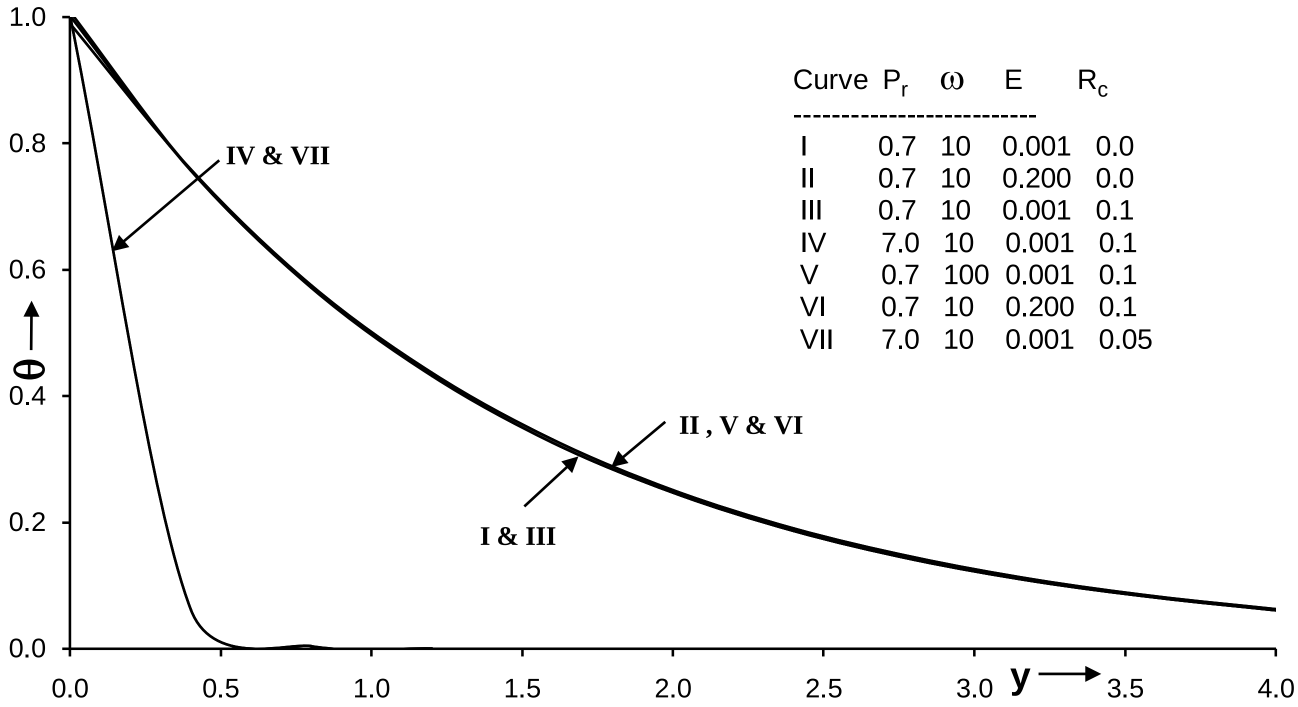


**Fig-2Temperature distribution for K = 0.1, M = 2, $G_r = 5$, $\alpha = 0.5$, $\varepsilon = 0.01$, $G_c = 5$ & $S_c = 0.30$**

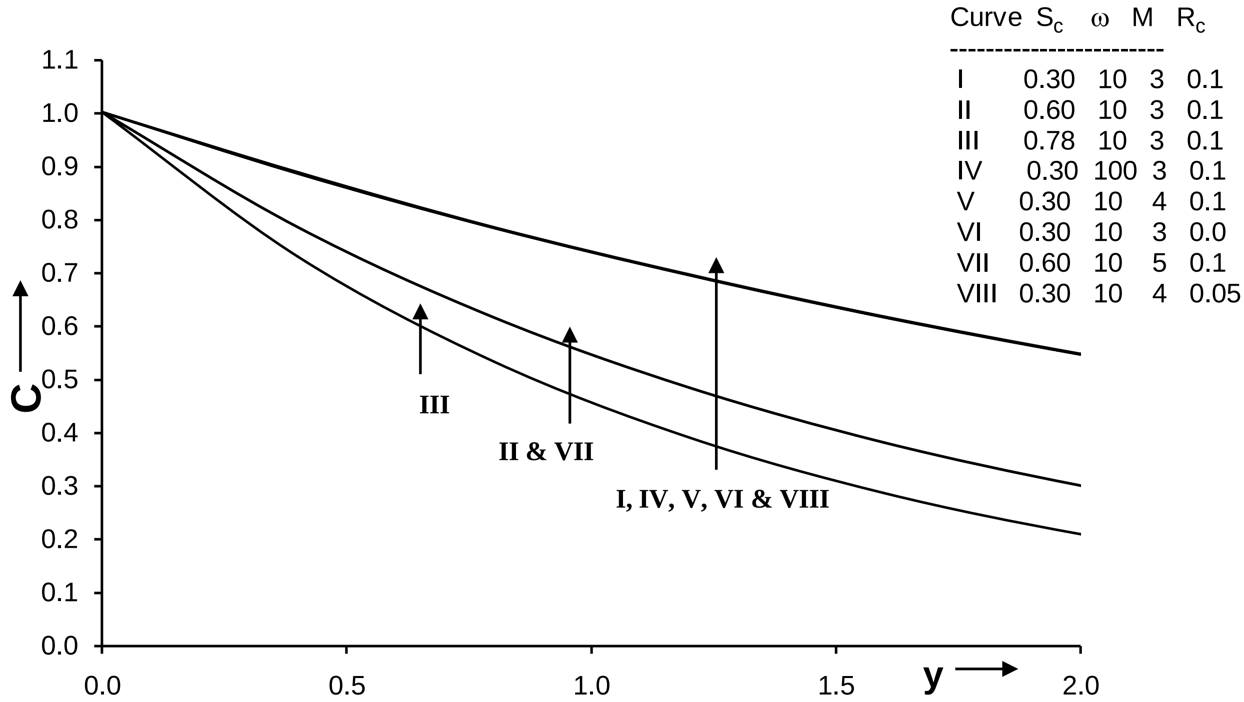


**Fig-3 Concentration distribution for K = 0.1, E = 0.001, $G_r$ = 5, α = 0.5, ε = 0.01, $G_c$ = 5 & $P_r$ = 0.7**

**TABLE.1 EFFECT OF VISCO-ELASTIC PARAMETER ON VELOCITY DISTRIBUTION**

$$(\omega=10,\alpha=0.5,\varepsilon=0.01)$$

| $Y$ | $M$ | $P_r$ | $G_r$ | $G_c$ | $E$ | $K$ | $S_c$ | $U$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | $R_c=0.000$ | $R_c=0.005$ | $R_c=0.100$ |
| 0 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.00001 | -0.00004 | -0.00105 |
| 0.4 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.47726 | 0.47722 | 0.47645 |
| 0.8 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.48636 | 0.48633 | 0.48573 |
| 1.2 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.42221 | 0.42218 | 0.42173 |
| 1.6 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.35670 | 0.35668 | 0.35633 |
| 2 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.30087 | 0.30086 | 0.30059 |
| 2.4 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.25474 | 0.25473 | 0.25452 |
| 2.8 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.21671 | 0.21670 | 0.21654 |
| 3.2 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.18522 | 0.18521 | 0.18508 |
| 3.6 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.15899 | 0.15899 | 0.15889 |
| 4 | 3 | 0.7 | 5 | 5 | 0.001 | 0.1 | 0.3 | 0.13702 | 0.13701 | 0.13694 |

**TABLE.2 EFFECT OF VISCO-ELASTIC PARAMETER ON TEMPERATURE DISTRIBUTION**

$$(K=0.1, M=2, G_r=5, \alpha=0.5, \varepsilon=0.01, G_c=5, S_c=0.30)$$

| $Y$ | $\omega$ | $\theta$ $(P_r=0.7, E=0.2)$ | | $\theta$ $(P_r=7.0, E=0.001)$ | |
|---|---|---|---|---|---|
| | | $R_c=0.0$ | $R_c=0.1$ | $R_c=0.05$ | $R_c=0.1$ |
| 0 | 10 | 0.99051 | 0.99060 | 1.00360 | 1.00349 |
| 0.4 | 10 | 0.75859 | 0.75863 | 0.06115 | 0.06114 |
| 0.8 | 10 | 0.57546 | 0.57547 | 0.00373 | 0.00373 |
| 1.2 | 10 | 0.43680 | 0.43680 | 0.00024 | 0.00024 |

 

| | | | | | |
|---|---|---|---|---|---|
| 1.6 | 10 | 0.33152 | 0.33151 | 0.00002 | 0.00002 |
| 2 | 10 | 0.25153 | 0.25152 | 0.00001 | 0.00001 |
| 2.4 | 10 | 0.19078 | 0.19077 | 0.00001 | 0.00001 |
| 2.8 | 10 | 0.14466 | 0.14466 | 0.00000 | 0.00000 |
| 3.2 | 10 | 0.10968 | 0.10967 | 0.00000 | 0.00000 |
| 3.6 | 10 | 0.08314 | 0.08313 | 0.00000 | 0.00000 |
| 4 | 10 | 0.06301 | 0.06301 | 0.00000 | 0.00000 |

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**Appendix:**

$$\lambda_1 = \frac{1}{2}\left(1+\sqrt{1+\frac{i\omega}{S_c}}\right),\ n = M+\frac{1}{K},\ a_1 = \frac{1}{2}\left(1+\sqrt{1+4n}\right),$$

$$b_1 = \frac{1}{2}\left(1+\sqrt{1+\frac{i\omega}{P_r}}\right),\ d_1 = \frac{1}{2}\left(1+\sqrt{1+4\left(n+\frac{iw}{4}\right)}\right)$$

$$I_0 = \frac{4\alpha S_c}{\omega},\ I_1 = P_r^{\,2}-P_r-n,\ I_2 = S_c^{\,2}-S_c-n,\ I_3 = \frac{G_r}{I_1},\ I_4 = \frac{G_c}{I_2}$$

$$J_1 = \frac{a_1^3\left(I_3+I_4\right)}{I_{29}},\ J_2 = \frac{P_r^{\,3} I_3}{I_1},\ J_3 = \frac{S_c^{\,3} I_4}{I_2},\ J_4 = J_3+J_2-J_1\ \ J_5 = \frac{a_1^3 I_{25}}{I_{29}},$$

$$J_6 = \frac{P_r^{\,3} I_{18}}{I_1},\ J_7 = \frac{8P_r^{\,3} I_{19}}{I_{12}},\ J_8 = \frac{8S_c^{\,3} I_{20}}{I_{13}},\ J_9 = \frac{8a_1^3 I_{21}}{I_{14}},\ J_{10} = \frac{\left(P_r+S_c\right)^3 I_{22}}{I_{15}}$$

$$J_{11} = \frac{\left(a_1+S_c\right)^3 I_{23}}{I_{16}},\ J_{12} = \frac{\left(a_1+P_r\right)^3 I_{24}}{I_{17}},\ J_{13} = J_{11}+J_{12}-J_5-J_6-J_7-J_8-J_9-J_{10}$$

$$I_5 = \frac{P_r\left(I_3^2+2R_c I_3 J_2\right)}{2},\ I_6 = \frac{P_r S_c (I_4^2+2R_c I_4 J_3)}{4S_c-2P_r}\ I_7 = \frac{P_r a_1\left\{\left(I_3+I_4\right)^2+2R_c\left(I_3+I_4\right)\left(J_1+J_4\right)\right\}}{4a_1-2P_r},$$

$$I_8 = \frac{2P_r^{\,2}\left\{\left(I_3 I_4+R_c(I_3 J_3+I_4 J_2\right)\right\}}{S_c+P_r},\ I_9 = \frac{2a_1 P_r S_c\left\{I_4\left(I_3+I_4\right)+R_c J_3\left(I_3+I_4\right)+R_c I_4\left(J_1+J_4\right)\right\}}{\left(a_1+S_c\right)^2-P_r\left(a_1+S_c\right)},$$

$$I_{10} = \frac{2P_r^{\,2}\left\{I_3\left(I_3+I_4\right)+R_c J_2\left(I_3+I_4\right)+R_c I_3\left(J_1+J_4\right)\right\}}{a_1+P_r}$$

$$I_{11} = I_9+I_{10}-I_5-I_6-I_7-I_8,$$

$$I_{12} = 4P_r^{\,2}-2P_r-n,\ I_{13} = 4S_c^{\,2}-2S_c-n,\ I_{14} = 4a_1^2-2a_1-n,$$

$$I_{15} = \left(P_r+S_c\right)^2-\left(Pr+S_c\right)-n,\ I_{16} = \left(a_1+S_c\right)^2-\left(a_1+S_c\right)-n,$$

$$I_{17} = \left(a_1+P_r\right)^2-\left(a_1+P_r\right)-n,\ I_{18} = \frac{GrI_{11}}{I_1} = I_3 I_{11}\quad \left[Q\ \ I_3 = \frac{Gr}{I_1}\right]$$

$$I_{19} = \frac{G_r I_5}{I_{12}},\ I_{20} = \frac{G_r I_6}{I_{13}},\ I_{21} = \frac{G_r I_7}{I_{14}},\ I_{22} = \frac{G_r I_8}{I_{15}},\ I_{23} = \frac{G_r I_9}{I_{16}},\ I_{24} = \frac{G_r I_{10}}{I_{17}},$$

$$I_{25} = I_{23} + I_{24} - I_{18} - I_{19} - I_{20} - I_{21} - I_{22}, \; I_{26} = \frac{4\alpha P_r}{\omega},$$

$$I_{27} = b_1^2 P_r^2 - b_1 P_r - n, I_{28} = \lambda_1^2 S_c^2 - \lambda_1 S_c - n, I_{29} = a_1^2 - a_1 - n,$$

$$I_{30} = A_{22} + A_{23} + A_{24} + A_{25} - A_{26}, \quad I_{31} = d_1^2 - d_1 - n,$$

$$I_{32} = A_{46} + A_{47} + A_{49} + A_{50} - A_{45} - A_{48}, I_{33} = \frac{1}{2P_r - \frac{iw}{4}},$$

$$I_{34} = (I_3 + I_4) + R_c(J_1 + J_4), \; I_{35} = I_3 + R_c J_2, I_{36} = I_4 + R_c J_3,$$

$$I_{37} = I_{25} + R_c(J_{13} + J_5), \; I_{38} = I_{18} + R_c J_6, I_{39} = I_{19} + R_c J_7, I_{40} = I_{20} + R_c J_8,$$

$$I_{41} = I_{21} + R_c J_9, \quad I_{42} = I_{22} + R_c J_{10}, \quad I_{43} = I_{23} + R_c J_{11}, \quad I_{44} = I_{24} + R_c J_{12}, \quad I_{45} = 2\alpha P_r I_5 I_{33},$$

$$I_{46} = \frac{2P_r S_c I_6}{2S_c^2 - \frac{iw}{4}P_r}, I_{47} = (P_r + S_c)^2 - P_r(P_r + S_c) - \frac{iw}{4}P_r, \quad I_{48} = (a_1 + S_c)^2 - P_r(a_1 + S_c) - \frac{iw}{4}P_r$$

$$I_{49} = (a_1 + P_r)^2 - P_r(a_1 + P_r) - \frac{iw}{4}P_r$$

$$I_{50} = (a_1 + d_1)^2 - P_r(a_1 + d_1) - \frac{iw}{4}P_r \; I_{51} = 4a_1^2 - 2a_1 P_r - \frac{iw}{4}P_r$$

$$I_{52} = (a_1 + b_1 P_r)^2 - P_r(a_1 + b_1 P_r) - \frac{iw}{4}P_r, I_{53} = (a_1 + \lambda_1 S_c)^2 - P_r(a_1 + \lambda_1 S_c) - \frac{iw}{4}P_r,$$

$$I_{54} = (d_1 + P_r)^2 - P_r(d_1 + P_r) - \frac{iw}{4}P_r, I_{55} = (1 + b_1)^2 P_r^2 - P_r^2(1 + b_1) - \frac{iw}{4}P_r,$$

$$I_{56} = (P_r + \lambda_1 S_c)^2 - P_r(P_r + \lambda_1 S_c) - \frac{iw}{4}P_r, I_{57} = (d_1 + S_c)^2 - P_r(d_1 + S_c) - \frac{iw}{4}P_r,$$

$$I_{58} = 4S_c^2 - 2P_r S_c - \frac{iw}{4}P_r, I_{59} = (S_c + b_1 P_r)^2 - P_r(S_c + b_1 P_r) - \frac{iw}{4}P_r,$$

$$I_{60} = (1 + \lambda_1)^2 S_c^2 - P_r(1 + \lambda_1)S_c - \frac{iw}{4}P_r$$

$$I_{61} = A_{82} + A_{64} + A_{65} + A_{70} + A_{71} + A_{72} - iI_{26}I_{11} - A_{78} - A_{79} - A_{80} - A_{81} - A_{60} - A^*_{81} - A_{66} - A_{76} - A_{77}$$

$$I_{62} = (a_1 + d_1)^2 - (a_1 + d_1) - n, \; I_{63} = (a_1 + b_1 P_r)^2 - (a_1 + b_1 P_r) - n$$

$$I_{64} = (a_1 + b_1 S_c)^2 - (a_1 + \lambda_1 S_c) - n, \; I_{65} = (d_1 + P_r)^2 - (d + P_r) - n$$

$$I_{66} = (1 + b_1)^2 P_r^2 - (1 + b_1)P_r - n, \; I_{67} = (P_r + \lambda_1 S_c)^2 - (P_r + \lambda_1 S_c) - n$$

$$I_{68} = (d_1 + S_c)^2 - (d_1 + S_c) - n, \; I_{69} = (S_c + b_1 P_r)^2 - (S_c + b_1 P_r) - n$$

$$I_{70} = (1 + \lambda_1)^2 S_c^2 - (1 + \lambda_1)S_c - n$$

$$I_{71} = A_{113} + A_{92} + A_{93} + A_{95} + A_{96} + A_{97} + A_{100} - A_{83} - A_{108} - A_{109} - A_{110} - A_{111} - A_{112} - A_{90} - A_{114} - A_{94} - A_{98} - A_{99}$$

$$I_{72} = A_{176} + A_{180} + A_{181} + A_{184} + A_{185} - A_{169} - A_{170} - A_{171} - A_{172} - A_{173} - A_{174} - A_{175} - A_{177} - A_{178} - A_{179} - A_{182} - A_{183} - A_{186} - A_{187}$$

 

$$A_1=\frac{1}{I_1-\frac{iw}{4}},\ A_2=\frac{1}{I_2-\frac{iw}{4}},A_3=\frac{1}{I_{12}-\frac{i\omega}{4}},\ A_4=\frac{1}{I_{13}-\frac{i\omega}{4}}\quad A_5=\frac{1}{I_{15}-\frac{i\omega}{4}},\ A_6=\frac{1}{I_{16}-\frac{i\omega}{4}},$$

$$A_7=\frac{1}{I_{17}-\frac{i\omega}{4}}\quad ,A_8=\frac{1}{I_{62}-\frac{i\omega}{4}}$$

$$A_9=\frac{1}{I_{27}-\frac{iw}{4}},A_{10}=\frac{1}{I_{28}-\frac{iw}{4}},\ A_{11}=\frac{1}{I_{29}-\frac{iw}{4}}\ ,A_{12}=\frac{1}{I_{31}-\frac{iw}{4}}$$

$$A_{13}=\frac{1}{I_{14}-\frac{i\omega}{4}},\ A_{14}=\frac{1}{I_{63}-\frac{i\omega}{4}},\ A_{15}=\frac{1}{I_{64}-\frac{i\omega}{4}},\ A_{16}=\frac{1}{I_{65}-\frac{i\omega}{4}}$$

$$A_{17}=\frac{1}{I_{66}-\frac{i\omega}{4}},\ A_{18}=\frac{1}{I_{67}-\frac{i\omega}{4}},A_{19}=\frac{1}{I_{68}-\frac{i\omega}{4}},\ A_{20}=\frac{1}{I_{69}-\frac{i\omega}{4}},A_{21}=\frac{1}{I_{70}-\frac{i\omega}{4}}$$

$$A_{22}=G_rA_9(1-iI_{26}),\ A_{23}=A_1\left(iI_{26}G_r+\alpha P_r I_3\right)$$

$$A_{24}=A_{10}G_c\left(1-iI_0\right),\ A_{25}=A_2\left(iI_0G_c+\alpha I_4S_c\right),A_{26}=\alpha_1a_1\left(I_3+I_4\right)A_{11}$$

$$A_{27}=\alpha_1a_1\left(J_1+J_4\right)A_{11},\ A_{28}=\alpha J_2 P_r A_1,\quad A_{29}=\alpha J_3S_cA_2,\quad A_{30}=\alpha a_1^3\left(I_3+I_4\right)A_{11}$$

$$A_{31}=\alpha I_3 P_r{}^3 A_1,\ A_{32}=\alpha I_4S_c{}^3A_2,\ A_{33}=d_1^3I_{30}A_{12},\ A_{34}=A_{22}b_1^3 P_r{}^3,\ A_{35}=A_{23} P_r{}^3 A_1,$$

$$A_{36}=A_{24}\lambda_1^3S_c{}^3A_{10},A_{37}=A_{25}S_c{}^3A_2,A_{38}=a_1^3A_{26}A_{11},A_{39}=d_1^2I_{30}A_{12},$$

$$A_{40}=A_{22}b_1^2 P_r{}^2 A_9,\ A_{41}=A_{23} P_r{}^2 A_1,\ A_{42}=A_{24}\lambda_1^2S_c{}^2A_{10},\ A_{43}=A_{25}S_c{}^2A_2,\ A_{44}=a_1^2A_{26}A_{11},$$

$$A_{45}=A_{27}+A_{30}+A_{38}-\frac{iw}{4}A_{44}\ ,A_{46}=A_{28}+A_{31}+A_{35}+\frac{iw}{4}A_{41},$$

$$A_{47}=A_{29}+A_{32}+A_{37}+\frac{iw}{4}A_{43},A_{48}=A_{33}+\frac{iw}{4}A_{39}\ A_{49}=A_{34}+\frac{iw}{4}A_{40},\ A_{50}=A_{36}+\frac{iw}{4}A_{42}$$

$$A_{51}=I_{30}+R_c\left(I_{32}+A_{48}\right),\ A_{52}=A_{31}+R_cA_{45}\ A_{53}=A_{28}+R_cA_{46},$$

$$A_{54}=A_{30}+R_cA_{47},\ A_{55}=A_{27}+R_cA_{49},\ A_{56}=A_{29}+R_cA_{50}$$

$$A_{57}=\frac{\alpha\left(P_r+S_c\right)P_r I_8}{I_{47}},\ A_{58}=\frac{\alpha P_r\left(a_1+S_c\right)I_9}{I_{48}},\ A^*_{58}=\frac{2a_1\alpha P_r I_7}{I_{51}}\ A_{59}=\frac{\alpha P_r(a_1+P_r)I_{10}}{I_{49}}$$

$$A_{60}=\frac{2P_r\ a_1d_1I_{34}A_{51}}{I_{50}},\ A_{61}=\frac{2P_r\ a_1^2I_{34}A_{52}}{I_{51}}\ \ A_{62}=\frac{2P_r{}^2\ a_1I_{34}A_{53}}{I_{49}}\ A_{63}=\frac{2a_1\ P_r\ S_cI_{34}A_{54}}{I_{48}},$$

$$A_{64}=\frac{2a_1b_1\ P_r{}^2\ I_{34}A_{55}}{I_{52}},A_{65}=\frac{2a_1\lambda_1\ P_r\ S_cI_{34}A_{56}}{I_{53}},\ A_{66}=\frac{2d_1\ P_r{}^2\ I_{35}A_{51}}{I_{54}}\ ,A_{67}=\frac{2a_1\ P_r{}^2\ I_{35}A_{52}}{I_{49}},$$

$$A_{68}=2P_r{}^2\ I_{35}A_{53}I_{33},\ A_{69}=\frac{2P_r{}^2\ S_cI_{35}A_{54}}{I_{47}}\ A_{70}=\frac{2P_r{}^3\ b_1I_{35}A_{55}}{I_{55}},A_{71}=\frac{2\lambda_1\ P_r{}^2\ S_cI_{35}A_{56}}{I_{56}},$$

$$A_{72}=\frac{2d_1\ P_r\ S_cI_{36}A_{51}}{I_{57}},A_{73}=\frac{2a_1\ P_r\ S_cI_{36}A_{52}}{I_{48}},A_{74}=\frac{2P_r{}^2\ S_cI_{36}A_{53}}{I_{47}},A_{75}=\frac{2P_r\ S_c{}^2I_{36}A_{54}}{I_{58}},$$

$$A_{76}=\frac{2b_1\ P_r{}^2\ I_{36}A_{55}}{I_{59}},\ A_{77}=\frac{2\lambda_1\ P_r\ S_c{}^2I_{36}A_{56}}{I_{60}},A_{78}=I_{45}-I_{68},\ A_{79}=I_{46}+A_{75}$$

$$A_{80}=A_{57}-A_{69}+A_{74},\quad A_{81}=A_{58}-A_{63}-A_{73},\ A^*_{81}=A_{61}+A^*_{58}$$

 

$A_{82} = A_{59} + A_{62} - A_{67}$

$A_{83} = G_r I_{61} A_9,\ A_{84} = iI_{26} G_r A_1,\ A_{85} = A_{78} G_r A_3,\ A_{86} = A_{79} G_r A_4$

$A_{87} = A_{80} G_r A_5,\ A_{88} = A_{81} G_r A_6,\ A_{89} = A_{82} G_r A_7,\ A_{90} = A_{60} G_r A_8$

$A_{91} = A_{61} G_r A_{13},\ A_{92} = A_{64} G_r A_{14},\ A_{93} = A_{65} G_r A_{15},\ A_{94} = A_{66} G_r A_{16}$

$A_{95} = A_{70} G_r A_{17},\ A_{96} = A_{71} G_r A_{18},\ A_{97} = A_{72} G_r A_{19},\ A_{98} = A_{76} G_r A_{20}$

$A_{99} = A_{77} G_r A_{21},\ A_{100} = \alpha a_1 I_{25} A_{11},\ A_{101} = \alpha P_r I_{18} A_1,\ A_{102} = 2\alpha P_r I_{19} A_3$

$A_{103} = 2\alpha S_c I_{20} A_4,\ A_{104} = 2\alpha a_1 I_{21} A_{13},\ A_{105} = \alpha(P_r + S_c) I_{22} A_5,$

$A_{106} = \alpha(a_1 + S_c) I_{23} A_6$

$A_{107} = \alpha(a_1 + P_r) I_{24} A_7,\ A_{108} = A_{84} + A_{101},\ A_{109} = A_{85} + A_{102},\ A_{110} = A_{86} + A_{103},$

$A_{111} = A_{87} + A_{105},\ A_{112} = A_{88} - A_{106},\ A_{113} = A_{89} + A_{107},\ A_{114} = A_{90} + A_{104}$

$A_{115} = \alpha a_1 (J_{13} + J_5) A_{11},\quad A_{116} = \alpha P_r J_6 A_1,\quad A_{117} = 2\alpha P_r J_7 A_3,\ A_{118} = 2\alpha S_c J_8 A_4,$

$A_{119} = 2a_1 \alpha J_9 A_{13},\ A_{120} = (P_r + S_c) J_{10} A_5,\ A_{121} = \alpha(a_1 + S_c) J_{11} A_6$

$A_{122} = \alpha(a_1 + P_r) J_{12} A_7,\ A_{123} = \alpha a_1^3 I_{25} A_{11},\ A_{124} = \alpha P_r^{\ 3} I_{18} A_1,\ A_{125} = 8\alpha P_r^{\ 3} I_{19} A_3$

$A_{126} = 8\alpha S_c^{\ 3} I_{20} A_4,\ A_{127} = 8\alpha a_1^3 I_{21} A_{13},\ A_{128} = \alpha\left(P_r + S_c\right)^3 I_{22} A_5,\ A_{129} = \alpha\left(a_1 + S_c\right)^3 I_{23} A_6$

$A_{130} = \alpha\left(a_1 + P_r\right)^3 I_{24} A_7,\ A_{131} = d_1^3 I_{71} A_{12},\ A_{132} = b_1^3 P_r^{\ 3} A_{83} A_9,\ A_{133} = P_r^{\ 3} A_{108} A_1$

$A_{134} = 8 P_r^{\ 3} A_{109} A_3,\ A_{135} = 8 S_c^{\ 3} A_{110} A_4,\ A_{136} = \left(P_r + S_c\right)^3 A_{111} A_5,\ A_{137} = (a_1 + S_c)^3 A_{112} A_6$

$A_{138} = (a_1 + P_r)^3 A_{113} A_7,\ A_{139} = \left(a_1 + d_1\right)^3 A_{90} A_8,\ A_{140} = 8a_1^3 A_{114} A_{13}$

$A_{141} = \left(a_1 + b_1 P_r\right)^3 A_{92} A_{14},\ A_{142} = \left(a_1 + \lambda_1 S_c\right)^3 A_{93} A_{15},\ A_{143} = \left(d_1 + P_r\right)^3 A_{94} A_{16}$

$A_{144} = (1 + b_1)^3 P_r^{\ 3} A_{95} A_{17},\quad A_{145} = (P_r + \lambda_1 S_c)^3 A_{96} A_{18},\ A_{146} = (d_1 + S_c)^3 A_{97} A_{16}$

$A_{147} = \left(S_c + b_1 P_r\right)^3 A_{98} A_{20},\ A_{148} = \left(1 + \lambda_1\right)^3 S_c^{\ 3} A_{99} A_{21},\ A_{149} = a_1^3 A_{100} A_{11}$

$A_{150} = d_1^2 I_{71} A_{12},\ A_{151} = b_1^2 P_r^{\ 2} A_{83} A_9,\ A_{152} = P_r^{\ 2} A_{108} A_1,\ A_{153} = 4P_r^{\ 2} A_{109} A_3$

$A_{154} = 4S_c^{\ 2} A_{110} A_4,\ A_{155} = (P_r + S_c)^2 A_{111} A_5,\ A_{156} = (a_1 + S_c)^2 A_{112} A_6$

$A_{157} = (a_1 + P_r)^2 A_{113} A_7,\ A_{158} = (a_1 + d_1)^2 A_{90} A_8,\ A_{159} = 4a_1^2 A_{114} A_{13}$

$A_{160} = (a_1 + b_1 P_r)^2 A_{92} A_{16},\ A_{161} = (a_1 + \lambda_1 S_c)^2 A_{93} A_{15},\ A_{162} = (d_1 + P_r)^2 A_{94} A_{16}$

$A_{163} = (1 + b_1)^2 P_r^{\ 2} A_{95} A_{17},\ A_{164} = (P_r + \lambda_1 S_c)^2 A_{96} A_{18},\ A_{165} = (d_1 + S_c)^2 A_{97} A_{16}$

$A_{166} = (S_c + b_1 P_r)^2 A_{98} A_{20},\ A_{167} = (1 + \lambda_1)^2 S_c^{\ 2} A_{99} A_{21},\ A_{168} = a_1^2 A_{100} A_{11}$

$$A_{169} = A_{115} + A_{123} - A_{149} - \frac{i\omega}{4} A_{168},\ A_{170} = A_{116} + A_{124} + A_{133} + \frac{i\omega}{4} A_{152}$$

$$A_{171} = A_{117} + A_{125} + A_{134} + \frac{i\omega}{4} A_{153},\ A_{172} = A_{118} + A_{126} + A_{135} + \frac{i\omega}{4} A_{154}$$

$$A_{173} = A_{119} + A_{127} + A_{140} + \frac{i\omega}{4} A_{159},\ A_{174} = A_{120} + A_{128} + A_{136} + \frac{i\omega}{4} A_{155}$$

$$A_{175} = -A_{121} - A_{129} + A_{137} + \frac{i\omega}{4} A_{156},\ A_{176} = A_{122} + A_{130} + A_{138} + \frac{i\omega}{4} A_{157}$$

$$A_{177} = A_{131} + \frac{i\omega}{4} A_{150},\ A_{178} = A_{132} + \frac{i\omega}{4} A_{151},\ A_{179} = A_{139} + \frac{i\omega}{4} A_{158}$$

$$A_{180} = A_{141} + \frac{i\omega}{4} A_{160},\ A_{181} = A_{142} + \frac{i\omega}{4} A_{161},\ A_{182} = A_{143} + \frac{i\omega}{4} A_{162}$$

 

$$A_{183} = -A_{144} + \frac{i\omega}{4}A_{163},\ A_{184} = A_{145} + \frac{i\omega}{4}A_{164},\ A_{185} = A_{146} + \frac{i\omega}{4}A_{165}$$

$$A_{186} = -A_{147} + \frac{i\omega}{4}A_{166},\ A_{187} = A_{148} + \frac{i\omega}{4}A_{167}$$

$$A_{188} = I_{71} + R_c(I_{72} + A_{177}),\ A_{189} = A_{83} + R_cA_{178},\ A_{190} = A_{108} + R_cA_{170}, A_{191} = A_{109} + R_cA_{171}$$

$$A_{192} = A_{110} + R_cA_{172},\ A_{193} = A_{111} + R_cA_{174}, A_{194} = A_{112} + R_cA_{175}, A_{195} = A_{113} + R_cA_{176}$$

$$A_{196} = A_{90} + R_cA_{179},\ A_{197} = A_{114} + R_cA_{173},\ A_{198} = A_{92} + R_cA_{180}$$

$$A_{199} = A_{93} + R_cA_{181},\ A_{200} = A_{94} + R_cA_{182},\ A_{201} = A_{95} - R_cA_{183}, A_{202} = A_{96} + R_cA_{184}$$

$$A_{203} = A_{97} + R_cA_{185},\ A_{204} = A_{98} + R_cA_{186}, A_{205} = A_{99} + R_cA_{187}, A_{206} = A_{100} - R_cA_{169}$$

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