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PROBABILISTIC ANALYSIS OF INVENTORY SYSTEMS WITH SALES TIME DEPENDING ON INVENTORY THRESHOLD

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ABSTRACT

A production and sales system is considered. In model 1 stock arrives in accordance with Poisson process and the arrival starts sale immediately. In addition a machine produces one product at a time and the production time is general. The sale time for the production of the products start with a probability p on all production epochs and if the sale time does not begin the products are stored in the inventory till a threshold level is reached and after that the sale time starts. In model 2 there is no stock arrival process but a machine with exponential life time produces the products and the repair time of the machine is also exponential. Assuming that the products are sent for sales as in model 1, joint Laplace transforms of time to sales and sales time, their means and a numerical result are presented for the two models.

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Key words and phrases: *Storage system, Production and Sales time, Joint transform.*

1. INTRODUCTION

Storage systems of (s, S) type was studied by Arrow, Karlin and Scrat [1]. Such systems with random lead times and unit demand were treated by Danial and Ramanarayanan [2]. Models with bulk demands were analyzed by Ramanarayanan and Jacob [9]. Murthy and Ramanarayanan [4, 5, 6, 7] considered several (s, S) inventory systems. Kun- Shan Wu and Liang –Yuh Ouyang [3] have studied (Q, r, L) inventory model with defective items. In this paper two new models in this area are treated. In model 1, we consider the case of a company which procures stocks for sales from outside and the company has a machine which produces the products one by one. The sales start on the stock arrival epoch and if stock arrival is delayed, the sales can start either on production epochs with probability p or on reaching the inventory threshold level. So far no model in this area with the above natural combination for sales has been studied by researchers. When stock arrives from outside it is natural to sell immediately to make profits. When the products are produced one by one depending on the need for funds the sales may start on production epochs. If they are not sold on the production epochs they are stored in the inventory and the saletime starts when the inventory maximum threshold is reached for storing. In model 2, there is no arrival of stocks from outside but the machine producing products is subject to failure and repair. The joint Laplace transforms of time to sales and sale time; their means and a numerical example are also presented.

2. MODEL 1: CONSTANT STOCK ARRIVAL RATE WITH GENERAL PRODUCTION AND SALES

The main assumptions of the model are given below.

(1) The stock arrival process is Poisson with rate λ. On each arrival of the stock the sale time for it begins which is random with cumulative distribution function cdf G(y) and probability density function pdf g(y).

(2) Independently of the arrival process, the products are manufactured by a machine for sales one at a time. The inter production times of products are independent random variables with cdf F(x) and pdf f(x).

(3) At each production epoch along with the product produced, all the products in the inventory are sent for sales with probability p or the product produced is stored along with other products in the inventory with probability 1-p.

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(4) The waiting time is lengthened for sales of the products with the same probability law till the inventory level becomes equal to a random threshold level Z after which all the products in the inventory are sent for sales.

(5) The threshold Z has the probability function P(Z = n) = $\widetilde{p}_n$, n = 1, 2, 3, …..

Let $Q_k = P(Z > k), k = 0,1,2,3,...,$ where $Q_k = 1 - \widetilde{p}_0 - \widetilde{p}_1 - \widetilde{p}_2 - ...\widetilde{p}_k$

With $Q_0 = 1$ and $\widetilde{p}_0 = 0$ generating functions,

$\varphi(r) = \sum_{k=1}^{\infty} \widetilde{p}_k r^k$ and $\Phi(r) = \sum_{k=0}^{\infty} Q_k r^k$, $Q_k = 1 - \widetilde{p}_0 - \widetilde{p}_1 - \widetilde{p}_2 - ...\widetilde{p}_k$. with $Q_0 = 1$ and $\widetilde{p}_0 = 0$

(6) Sale time of each product produced is random with cdf H(y) and pdf h(y).

We may derive the joint distribution of T, the time to start sales and total sales time, S as follows. Noting that the stock arrival time has exponential distribution and production time is general, the joint probability density function of T and S is

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} P(T \le x, S \le y) = f(x,y)$$

$$= \sum_{0}^{\infty} \left[\lambda e^{-\lambda x} \left[F_n(x) - F_{n+1}(x) \right] Q_n (1-p)^n g \oplus h_n(y) \right]$$

$$+ \sum_{1}^{\infty} e^{-\lambda x} f_n(x)(1-p)^{n-1} pQ_n h_n(y) \;\; + \sum_{1}^{\infty} e^{-\lambda x} f_n(x)(1-p)^{n-1} p\widetilde{p}_n h_n(y)$$

$$+ \sum_{1}^{\infty} e^{-\lambda x} f_n(x)(1-p)^n \widetilde{p}_n h_n(y) \qquad (1)$$

We consider the following four different cases to write the above equation.

(i) The stock arrives and is sent for sales along with other products in the inventory.

(ii) The sales start at a production epoch with probability p before the arrival of stock and before the inventory level is reaching the threshold Z.

(iii) The sales start at a production epoch with probability p before the arrival of stock and the inventory level reaches also simultaneously the threshold Z.

(iv) The sales start at the inventory threshold level Z before the arrival of stock and before sale at any production epochs.

The first sum on the right side of equation (1) is the part of the pdf that the stock arrives at x from outside, only n productions are completed during (0, x), the products are not sent for sales at production epochs, the inventory threshold Z > n and the salestime is the sum of sales time of arrived products and the produced products. Here n = 0, 1, 2, 3… The symbol $\oplus$ means convolution and $h_n(x)$ is n-fold convolution of h(x) with itself. The second sum is the part of the pdf that there is no arrival from outside, the n^{th} production occurs at x, the products are sent for sales at the n^{th} production epoch with probability p, the threshold Z > n and the sales time is over at y. The third sum is the part of the pdf that there is no arrival from outside, the n^{th} production occurs at x, the products are sent for sales at the n^{th} production epoch with probability p, in addition the inventory reaches the threshold level Z = n and the sales time is over at y. The fourth sum is the part of the pdf that there is no arrival from outside, the n^{th} production occurs at x, it is sent to inventory with probability(1-p), the inventory level reaches the threshold level Z = n; the products are sent for sales due to that and the sales time is over at y. Let us define the joint Laplace transform as follows.

$$E(e^{-\xi T} e^{-\eta S}) = \int_0^{\infty}\int_0^{\infty} f(x,y) e^{-\xi x} e^{-\eta y} dxdy = f^*(\xi,\eta)$$

$$= \sum_{0}^{\infty} \left[\left(\frac{\lambda}{\lambda+\xi} \right) f^*(\xi+\lambda)(1 - f^{*n}(\xi+\lambda)) Q_n (1-p)^n g^*(\eta) h^{*n}(\eta) \right]$$

$$+ \sum_{1}^{\infty} f^*(\xi+\lambda)(1-p)^{n-1} pQ_n h^{*n}(\eta) + \sum_{1}^{\infty} f^{*n}(\xi+\lambda)(1-p)^{n-1} p\tilde{p}_n Q_n h^{*n}(\eta) \qquad (2)$$

$$+ \sum_{1}^{\infty} f^{*n}(\xi+\lambda)(1-p)^n \tilde{p}_n h^{*n}(\eta)$$

Which on simplification gives combining second and third terms

$$f^*(\xi,\eta)=\sum_0^\infty\left[\left(\frac{\lambda}{\lambda+\xi}\right)f^{*n}(\xi+\lambda)(1-f^*(\xi+\lambda))Q_n(1-p)^n g^*(\eta)h^{*n}(\eta)\right]$$
$$+\sum_1^\infty f^{*n}(\xi+\lambda)(1-p)^{n-1}p\ \ _{n-1}h^{*n}(\eta)+\sum_1^\infty Q f^{*n}(\xi+\lambda)(1-p)^n\tilde{p}_n h^{*n}(\eta)$$

Further simplification gives

$$f^*(\xi,\eta)=\left[\left(\frac{\lambda}{\lambda+\xi}\right)(1-f^*(\xi+\lambda))g^*(\eta)+pf^*(\xi+\lambda)h^*(\eta)\right] \quad (3)$$
$$\Phi[f^*(\xi+\lambda)h^*(\eta)(1-p)]+\varphi[f^*(\xi+\lambda)h^*(\eta)(1-p)].$$

We may note $\Phi(r)=\dfrac{1\oplus(r)}{1-r}$ which gives

$$f^*(\xi,\eta)=\varphi[f^*(\xi+\lambda)h^*(\eta)(1-p)]+\{1-\varphi[f^*(\xi+\lambda)h^*(\eta)(1-p)]\}$$
$$\cdot\left\{\frac{\left[\left(\frac{\lambda}{\lambda+\xi}\right)(1-f^*(\xi+\lambda))g^*(\eta)+pf^*(\xi+\lambda)h^*(\eta)\right]}{1-f^*(\xi+\lambda)h^*(\eta)(1-p)}\right\}$$

Then the Laplace transform of the time to sales is

$$E(e^{-\xi T})=\phi[f^*(\xi+\lambda)(1-p)]+\{1-\varphi[f^*(\xi+\lambda)(1-p)]\}\left\{\frac{\left[\left(\frac{\lambda}{\lambda+\xi}\right)(1-f^*(\xi+\lambda))+pf^*(\xi+\lambda)\right]}{1-f^*(\xi+\lambda)(1-p)}\right\} \quad (4)$$

This on differentiation gives

$$E(T)=\left(\frac{1}{\lambda}\right)\left(\frac{1-f^*(\lambda)}{1-(1-p)f^*(\lambda)}\right)\{1-\varphi[f^*(\lambda)(1-p)]\} \quad (5)$$

Then the Laplace transform of the sales time is

$$E(e^{-\eta}S)=\varphi[f^*(\lambda)h^*(\eta)(1-p)]+\{1-\varphi[f^*(\lambda)h^*(\eta)(1-p)]\}\left\{\frac{(1-f^*(\lambda))g^*(\eta)+pf^*(\lambda)h^*(\eta)}{1-f^*(\lambda)h^*(\eta)(1-p)}\right\}$$

This on differentiation gives

$$E(S)=\{1-\varphi[(1-p)f^*(\lambda)]\}\left\{\frac{(1-f^*(\lambda))E(G)+f^*(\lambda)E(H)}{1-f^*(\lambda)(1-p)}\right\} \quad (6)$$

Here E(G) is expected sale time of arriving stock and E(H) is the expected sale time of a product produced.

3. MODEL 2: MACHINE FAILURE AND REPAIR CASE WITH GENERAL PRODUCTION AND SALES

The main assumptions of the model are given below.

(1) The products are manufactured by a machine during its life time for sales one at a time. The inter-production times of products are independent random variables with cdf F(x) and pdf f(x).

(2) At each production epoch of the machine along with the product produced, all the products in the inventory are sent for sales with probability p or the product produced is stored along with other products in the inventory with probability 1 – p.

(3) The waiting time is lengthened for sales of the products with the above probability law till the inventory level becomes equal to a random threshold level Z, when all the products in the inventory are sent for sales.

(4) The inventory threshold level Z has the probability function

$$\mathrm{P}\left(\mathrm{Z} = \mathrm{n}\right) = \widetilde{p}_n, \ \mathrm{n} = 1,2,3,\ldots$$

Let $\mathrm{Q}_k = \mathrm{P}\left(\mathrm{Z} > \mathrm{k}\right)$, $\mathrm{k} = 0,1,2,3,....$ where $\mathrm{Q}_k = 1 - \widetilde{p_0} - \widetilde{p_1} - \widetilde{p_2} - ...\widetilde{p_k}$. with $\mathrm{Q}_0 = 1$ and $\widetilde{p}_0 = 0$.

Let the generating function be,

$$\varphi(r) = \sum_{k=1}^{\infty} \widetilde{p_k} r^k \ \text{ and } \ \Phi(r) = \sum_{k=0}^{\infty} Q_k r^k.$$

(5) The sale time of each product is random with cdf H(y) and pdf h(y) .

(6) The life time of the machine is exponential with parameter λ when the machine fails, it is sent for repair. The repair time is exponential random variable with parameter μ .

(7) During the repair time of the machine, the products if any already produced are sold one by one.

As in model 1 we may derive the joint distribution of T and S, the time to start sales and total sales time including repair time if any, as follows. The joint probability density function of T and S is, the time to start sales and total sales time including repair time if any as follows. The joint probability density function of T and S is,

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P\left(T \le x, S \le y\right) &= f\left(x,y\right)\\ &= \sum_{0}^{\infty}\left[\lambda e^{-\lambda x}\left[F_n\left(x\right) - F_{n+1}\left(x\right)\right]Q_n\left(1-p\right)^n\left(\left(1-e^{-\mu y}\right)h_n\left(y\right) + \mu e^{-\mu y}H_n\left(y\right)\right)\right]\\ &\quad + \sum_{1}^{\infty}e^{-\lambda x}f_n\left(x\right)\left(1-p\right)^{n-1}p_nQ_n\left(y\right) + \sum_{1}^{\infty}e^{-\lambda x}f_n\left(x\right)\left(1-p\right)^{n-1}p\widetilde{p_n}h_n\left(y\right)\\ &\quad + \sum_{1}^{\infty}e^{-\lambda x}f_n\left(x\right)\left(1-p\right)^n\widetilde{p_n}h_n\left(y\right).\end{aligned} \tag{7}$$

For writing the first term we may note that when the machine fails at x, all the products in the inventory if any, are sent for sales one by one and the repair also starts simultaneously. The other terms of the above equation is written using the arguments used for model 1.Taking the double Laplace transform we get

$$\begin{aligned}f^*\left(\xi,\eta\right) &= \sum_{0}^{\infty}\left[\left(\frac{\lambda}{\lambda+\xi}\right)f^{*^n}\left(\xi+\lambda\right)\left(1-f^*\left(\xi+\lambda\right)\right)Q_n\left(1-p\right)^n\left(h^{*^n}\left(\eta\right) - h^{*^n}\left(\eta+\mu\right) + \frac{\mu}{\mu+\eta}h^{*^n}\left(\eta+\mu\right)\right)\right]\\ &\quad + \sum_{1}^{\infty}f^{*^n}\left(\xi+\eta\right)\left(1-p\right)^{n-1}pQ_{n-1}h^{*^n}\left(\eta\right) + \sum_{1}^{\infty}f^{*^n}\left(\xi+\lambda\right)\left(1-p\right)^n\widetilde{p_n}h^{*n}\left(\eta\right)\end{aligned}$$

We obtain

$$\begin{aligned}f^*(\xi+\mu) &= \sum_{0}^{\infty}\left[\left(\frac{\lambda}{\lambda+\xi}\right)\left(1-f^*\left(\xi+\lambda\right)\right) + p^*\left(\xi+\lambda\right)fh^*\left(\eta\right)\right]\Phi\left[f^*\left(\xi+\lambda\right)h^*\left(\eta\right)\left(1-p\right)\right]\\ &\quad - \left[\left(\frac{\lambda}{\lambda+\xi}\right)\left(1-f^*\left(\xi+\lambda\right)\right)\right]\left(\frac{\eta}{\eta+\mu}\right)\Phi\left[f^*\left(\xi+\lambda\right)h^*\left(\eta+\mu\right)\left(1-p\right)\right]\\ &\quad + \varphi\left[f^*\left(\xi+\lambda\right)h^*\left(\eta\right)\left(1-p\right)\right].\end{aligned} \tag{8}$$

The Laplace transform of the time to sales is same as that of the one given for model 1 equation (4) and E(T) is also as given in equation (5). The Laplace transform of the sales time and repair time if any is given by

$$E(e^{-\eta s}) = [1 - f^*(\lambda) + p^*(\lambda) f h^*(\eta)]\Phi[f^*(\lambda)h^*(\eta)(1-p)]$$
$$-[(1-f^*(\lambda))]\left[\left(\frac{\eta}{\eta+\mu}\right)\Phi[f^*(\lambda)h^*(\eta+\mu)(1-p)\right] + \varphi[f^*(\lambda)h^*(\eta)(1-p)]$$

The expected sales time including repair time, if any, is

$$E(S) = (\frac{1}{\mu})(1-f^*(\lambda))\Phi[f^*(\lambda)h^*(\mu)(1-p)] + f^*(\lambda)E(H)\Phi[f^*(\lambda)(1-p)] \quad (9)$$

4. NUMERICAL EXAMPLE

We consider the inventory threshold Z has probability function $P(Z=n) = \frac{1}{2^n}$, n = 1, 2, 3… with probability generating function as $\varphi(r) = \sum_{1}^{\infty}\left(\frac{1}{2}\right)^n r^n$ and so that $\varphi(r) = \frac{r}{2-r}$ and $\Phi(r) = \frac{2}{2-r}$.

Consider the production time is exponential with parameter α so that We have $F(x) = 1 - e^{-\alpha x}$.

For model 1 from equations (5) and (6) we get

$$\Phi\left[f^*(\lambda)(1-p)\right] = \Phi\left[(1-p)(\frac{\alpha}{\alpha+\lambda})\right] = \frac{2(\alpha+\lambda)}{\alpha+\alpha p+2\lambda} \quad (10)$$

$$E(T) = \frac{2}{\alpha+\alpha p+2\lambda} \quad (11)$$

$$E(S) = 2\left(\frac{\lambda E(G)+\alpha E(H)}{\alpha+\alpha p+2\lambda}\right) \quad (12)$$

For model 2, E(T) remains as in equation (11). Let h(.) be exponential pdf with Parameter β. Then E(S) becomes

$$E(S) = \frac{2\lambda(\beta+\mu)}{\mu[2(\alpha+\lambda)(\beta+\mu)-(1-p)\alpha\beta]} + \frac{2\alpha}{\beta(\alpha+\alpha p+2\lambda)} \quad (13)$$

Fixing p = 0.1, α = 10, E(G) = 20, E(H) = 2, μ = .02, β = $\frac{1}{2}$ we vary λ.

We note,

$$E(T) = \frac{2}{11+2\lambda} \quad (14)$$

For model 1

$$E(T) = 2\left(\frac{20\lambda+20}{11+2\lambda}\right) \quad (15)$$

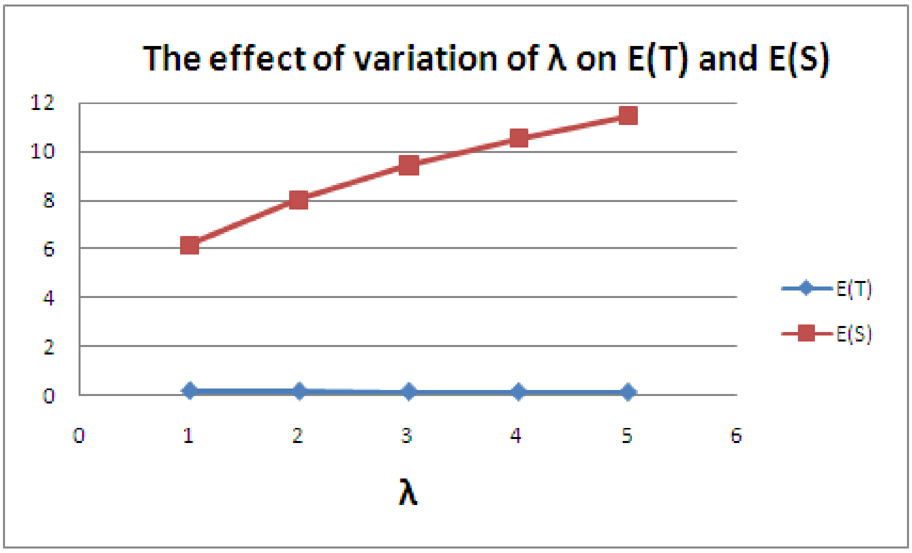
For model 2

$$E(s) = \frac{2\lambda(.52)}{.02[2(10+\lambda)(0.52)-4.5]} + \frac{40}{11+2\lambda} \quad (16)$$

The effect of variation of λ on E (T) and E(S):

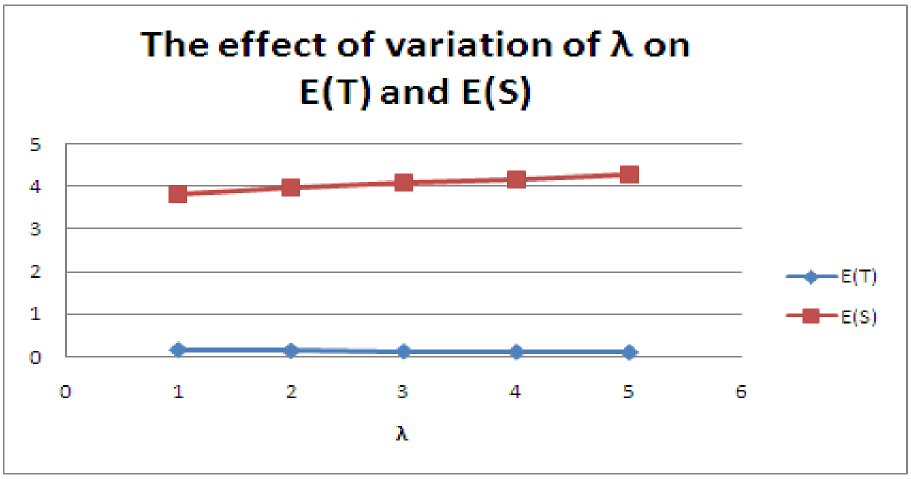
For model: 1

Expected value\ λ	1	2	3	4	5
E(T)	0.1539	0.1333	0.1176	0.1053	0.0952
E(S)	6.153	8.000	9.4117	10.5263	11.4285



For model: 2,

Expected value\ λ	1	2	3	4	5
E(T)	0.1539	0.1333	0.1176	0.1053	0.0952
E(S)	3.81897	3.9626	4.0794	4.1675	4.2742



In tables I and II, when λ increases E(T) decreases E(S) increases.

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