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DOMINATOR CHROMATIC NUMBER ON SHADOW GRAPHS AND PATH UNION OF GRAPHS

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ABSTRACT

Let G = (V, E) be a simple, finite and undirected graph. A dominator coloring of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G′ and G″ and joining each vertex v′ in G′ to the neighbours of the corresponding vertex v″ in G″. In this paper, we obtain dominator chromatic number of shadow graphs of some interesting classes of graphs and also find the dominator chromatic number of path union of graphs.

Key Words: *proper coloring, dominator coloring, shadow graph and path union.*

AMS Subject Classification: *05C15, 05C69.*

1. PRELIMINARIES

All graphs considered are simple, finite and undirected graphs. The order and size of G are denoted by n and m respectively. For graph theoretic terminology, we refer to Harary [4].

A subset D of V is called a *dominating set* of G if every vertex in V – D is adjacent to at least one vertex in D. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a minimal dominating set in G.

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$, is the minimum number of colors required for a proper coloring of G. A c*olor class* is the set of vertices, having the same color. The color class corresponding to the color i is denoted by C_i

A *dominator coloring* of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The convention is that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number χ_d (G) is the minimum number of colors required for a dominator coloring of G [1, 3].

The shadow graph D_2 (G) of a connected graph G is constructed by taking two copies of G say G′ and G″ and joining each vertex v′ in G′ to the neighbours of the corresponding vertex v″ in G″ [2, 6].

Let $G_1, G_2 \dots G_n$, $n \geq 2$ be n copies of a graph G. The graph G′ obtained by adding an edge between G_i and G_{i+1} for i = 1, 2… n-1 is called path union of G [5].

2. DOMINATOR COLORING ON SHADOW GRAPHS OF SOME CLASSES OF GRAPHS

In this section, dominator coloring on shadow graphs of path, cycle, complete, wheel, star graph and bistar graphs are consider and their corresponding chromatic numbers are obtained.

Theorem 2.1: For path P_n of order $n \geq 2$, $\chi_d[D_2(P_n)] = \begin{cases} \lceil 2n/3 \rceil & \text{when } n \leq 11 \\ \lceil n/2 \rceil + 3 & \text{when } n = 4k+2, k \geq 3 \\ \lceil n/2 \rceil + 2 & \text{otherwise.} \end{cases}$

Proof: Let G = $D_2(P_n)$ be the shadow graph of path P_n. Let $v_1', v_2', \dots, v_n'$ be the vertices of P_n', the first copy of P_n and let $v_1'', v_2'', \dots, v_n''$ be the vertices of P_n'', second copy of P_n.

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Let $G = D_2(P_n)$. Here we note that $|V(G)| = 2n$ and $|E(G)| = 4n-4$.

We construct a dominator coloring $f: V(G) \rightarrow \{1, 2, \ldots, \chi_d[D_2(P_n)]\}$ as follows:

Case 1: When $n \le 11$

When $n = 3k$, $k \ge 2$, for each i, $i = 3j+1$, where $0 \le j \le k-1$, $f(v_i') = f(v_i'') = f(v_{i+2}') = f(v_{i+2}'') = \lceil 2i/3 \rceil$ and let $f(v_i') = f(v_i'') = \lceil 2i/3 \rceil$, for each i, $i = 3j-1$, $1 \le j \le k$. When $n = 3k+1$, $k \ge 2$, for each i, $i = 3j+1$, where $0 \le j \le k-2$, $f(v_i') = f(v_i'') = f(v_{i+2}') = f(v_{i+2}'') = \lceil 2i/3 \rceil$; for each i, $i = 3j-1$, $1 \le j \le k$, $f(v_i') = f(v_i'') = \lceil 2i/3 \rceil$ and let $f(v_{n-3}') = f(v_{n-3}'') = f(v_{n-4}') + 2 = f(v_n') = f(v_n'')$ and $f(v_{n-1}') = f(v_{n-1}'') = f(v_{n-2}')+1$. When $n = 3k+2$, $k \ge 1$, for each i, $i = 3j+1$, where $0 \le j \le k-1$, $f(v_i') = f(v_i'') = f(v_{i+2}') = f(v_{i+2}'') = \lceil 2i/3 \rceil$ and let $f(v_i') = f(v_i'') = \lceil 2i/3 \rceil$; for each i, $i = 3j-1$, $1 \le j \le k$ and let $f(v_{n-1}') = f(v_{n-1}'') = f(v_{n-3}') + 1$ and $f(v_n') = f(v_n'') = f(v_{n-1}') + 1$.

It can be easily verified that $\chi_d[D_2(P_n)] = \lceil 2n/3 \rceil$ for n = 2, 3 or 4. Hence $\chi_d[D_2(P_n)] = \lceil 2n/3 \rceil$ for $n \le 11$.

Case 2: When $n \ge 12$

Case 2a: When $n = 4k$, $k \ge 3$

Let $f(v_1') = f(v_1'') = 1$. For each i, $i = 4j$, where $1 \le j \le k$, $f(v_i') = f(v_i'') = 1$; for each i, $i = 4j+1$, where $1 \le j \le k-1$, $f(v_i') = f(v_i'') = 2$ and let $f(v_i') = f(v_i'') = (i/2) + 2$ and for each i, $i = 4j - 2$, where $1 \le j \le k$, $f(v_{i+1}') = f(v_{i+1}'') = (i/2) + 3$.

Case 2b: When $n = 4k+1$, $k \ge 3$

Let $f(v_1') = f(v_1'') = 1 = f(v_n') = f(v_n'')$. For each i, $i = 4j$, where $1 \le j \le k-1$, $f(v_i') = f(v_i'') = 1$; for each i, $i = 4j+1$, where $1 \le j \le k-1$, $f(v_i') = f(v_i'') = 2$ and let $f(v_i') = f(v_i'') = (i/2) + 2$ and for each i, $i = 4j - 2$, where $1 \le j \le k$, $f(v_{i+1}') = f(v_{i+1}'') = (i/2) + 3$ and let $f(v_{n-1}') = f(v_{n-1}'') = f(v_{n-2}')+1$.

Case 2c: When $n = 4k+2$, $k \ge 3$

Let $f(v_1') = f(v_1'') = 1$. For each i, $i = 4j$, where $1 \le j \le k$, $f(v_i') = f(v_i'') = 1$; for each i, $i = 4j+1$, where $1 \le j \le k-1$, $f(v_i') = f(v_i'') = 2$ and let $f(v_i') = f(v_i'') = (i/2) + 2$ and for each i, $i = 4j - 2$, where $1 \le j \le k$, $f(v_{i+1}') = f(v_{i+1}'') = (i/2) + 3$ and let $f(v_{n-1}') = f(v_{n-1}'') = f(v_{n-3}')+1$ and $f(v_n') = f(v_n'') = f(v_{n-1}') + 1$.

Case 2d: When $n = 4k+3$, $k \ge 3$

Let $f(v_1') = f(v_1'') = 1$. For each i, $i = 4j$, where $1 \le j \le k$, $f(v_i') = f(v_i'') = 1$; for each i, $i = 4j+1$, where $1 \le j \le k$, $f(v_i') = f(v_i'') = 2$ and for each i, $i = 4j - 2$, where $1 \le j \le k+1$, $f(v_i') = f(v_i'') = (i/2) + 2$ and $f(v_{i+1}') = f(v_{i+1}'') = (i/2) + 3$.

Each vertex labeled 1 or 2 dominates some uniquely colored neighbor and each vertex colored k for $3 \le k \le 3+\lceil n/2 \rceil$ dominates its own color class.

Hence $$\chi_d[D_2(P_n)] = \begin{cases} \lceil 2n/3 \rceil & \text{when } n \le 11 \\ \lceil n/2 \rceil + 3 & \text{when } n = 4k+2, k \ge 3 \\ \lceil n/2 \rceil + 2 & \text{otherwise.} \end{cases}$$

Theorem 2.2: The cycle C_n, $n \ge 3$ has $$\chi_d[D_2(C_n)] = \begin{cases} n & \text{when } n = 3 \\ \lceil n/3 \rceil & \text{when } n = 4 \\ \lceil 2n/3 \rceil & \text{when } n = 5, 6 \text{ or } 7 \\ \lceil 2n/3 \rceil + 1 & \text{when } n = 3k+1, k \ge 3 \\ \lceil 2n/3 \rceil + 2 & \text{otherwise.} \end{cases}$$

Proof: The verification of cases $3 \le n \le 8$ is straightforward. We construct a dominator coloring $f: V[D_2(C_n)] \rightarrow \{1, 2, \ldots, \chi_d[D_2(C_n)]\}$ as follows.

Case (i): When $n = 3k$, $k \geq 3$

For each i, $0 \leq i \leq k-1$, $f(v_{3i+1}') = f(v_{3i+1}'') = 1$; for each i, $0 \leq i \leq k-1$, $f(v_{3i+2}') = f(v_{3i+2}'') = 2$; for each i, $1 \leq i \leq k$, $f(v_{3i}') = i+2$, and for each i, $1 \leq i \leq k$, $f(v_{3i}'') = f(v_n') + i$.

Case (ii): When $n = 3k+1$, $k \geq 3$

For each i, $0 \leq i \leq k-1$, $f(v_{3i+1}') = f(v_{3i+1}'') = 1$; for each i, $0 \leq i \leq k-1$, $f(v_{3i+2}') = f(v_{3i+2}'') = 2$; for each i, $1 \leq i \leq k$, $f(v_{3i}') = i+2$, let $f(v_{n-1}'') = f(v_{n-1}')$, let $f(v_n') = f(v_{n-1}')+1$ and let $f(v_n') = f(v_n'')$. Let $f(v_{3i}'') = f(v_n') + i$, for each i, $1 \leq i \leq k-1$.

Case (iii): When $n = 3k+2$, $k \geq 3$

For each i, $0 \leq i \leq k$, $f(v_{3i+1}') = f(v_{3i+1}'') = 1$; for each i, $0 \leq i \leq k-1$, $f(v_{3i+2}') = f(v_{3i+2}'') = 2$; for each i, $1 \leq i \leq k$, $f(v_{3i}') = i+2$ and let $f(v_n') = f(v_{n-2}')+1$ and $f(v_{n-1}') = f(v_{n-1}'')$ and for each i, $1 \leq i \leq k$, $f(v_{3i}'') = f(v_n') + i$ and let $f(v_n'') = f(v_{n-2}'')+1$.

For $n \geq 8$, each vertex labeled 1 or 2 dominates some uniquely colored neighbor and each vertex colored k for $3 \leq k \leq 2+\lceil 2n/3 \rceil$ dominates its own color class.

Hence $$\chi_d[D_2(C_n)] = \begin{cases} n & \text{when } n = 3 \\ \lceil n/3 \rceil & \text{when } n = 4 \\ \lceil 2n/3 \rceil & \text{when } n = 5, 6 \text{ or } 7 \\ \lceil 2n/3 \rceil + 1 & \text{when } n = 3k+1, k \geq 3 \\ \lceil 2n/3 \rceil + 2 & \text{otherwise.} \end{cases}$$

Example: 2.3 The dominator coloring of $D_2(C_8)$ is shown in the following figure 1.

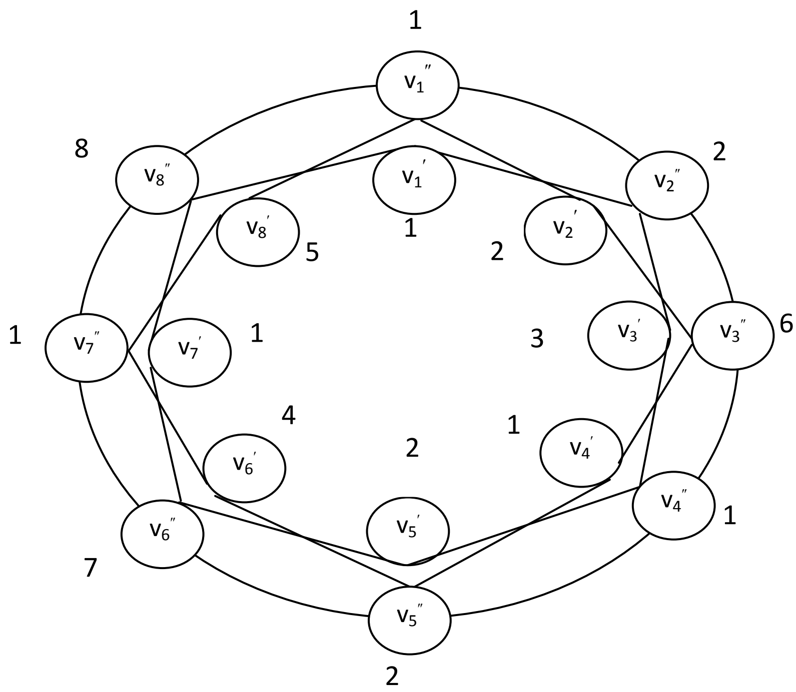

Figure: 1

The color classes of $D_2(C_8)$ are $C_1 = \{v_1', v_4', v_7', v_1'', v_4'', v_7''\}$, $C_2 = \{v_2', v_5', v_2'', v_5''\}$, $C_3 = \{v_3'\}$, $C_4 = \{v_6'\}$, $C_5 = \{v_8'\}$, $C_6 = \{v_3''\}$, $C_7 = \{v_6''\}$ and $C_8 = \{v_8''\}$. Therefore $\chi_d[D_2(C_8)] = 8$.

Theorem: 2.4 The wheel graph $W_{1,n}$, $n \geq 3$ has $\chi_d[D_2(W_{1,n})] = \begin{cases} 4 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases}$

Proof: Let $D_2(W_{1,n})$ be a shadow graph of wheel. Let the vertices of $D_2(W_{1,n})$, $n \geq 3$ be labeled as follows. Let v_1', v_2', …, v_n' be the vertices of $W_{1,n}'$, the first copy of $W_{1,n}$ and let v_1'', v_2'', …, v_n'' be the vertices of $W_{1,n}''$, second copy of $W_{1,n}$, where the vertices at the centre of two copies of $W_{1,n}$ are labeled by v_1' and v_1''. The vertices on the rim of $W_{1,n}'$ and $W_{1,n}''$ be labeled consecutively by v_2', …, v_n' and v_2'', …, v_n''.

A dominator coloring of $D_2(W_{1,n})$ is by coloring the centre vertices v_1' and v_1'' by color 1 and coloring the vertices in the rim alternatively by 2 and 3 from the vertex v_2' and v_2''. When n is odd, the vertices v_{n-2}' and v_{n-2}'' receive color 2, the vertices v_{n-1}' and v_{n-1}'' receive color 3 and the vertices v_n' and v_n'' receive color 4 respectively. When n is even, the vertices v_{n-1}' and v_{n-1}'' receive color 2 and the vertices v_n' and v_n'' receive color 3 respectively. The centre vertices v_1' and v_1'' dominate themselves, the remaining vertices on the rim of $W_{1,n}'$ and $W_{1,n}''$ dominate the color class 1. Hence

$$\chi_d[D_2(W_{1,n})] = \begin{cases} 4 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

Theorem: 2.5

(1) The star $K_{1,n}$ has $\chi_d[D_2(K_{1,n})] = 2$ for all $n \geq 2$.
(2) The bistar $B_{m,n}$ has $\chi_d[D_2(B_{m,n})] = 3$ for all $m, n \geq 1$.
(3) The complete graph K_n has $\chi_d[D_2(K_n)] = n$ for all $n \geq 2$.

Proof:

(1) Since $\chi_d[D_2(K_{1,n})] \geq \chi[D_2(K_{1,n})]$, the result follows.

(2) Let $G = D_2(B_{m,n})$ be a shadow graph of bistar. Let $V(G) = X_1 \cup X_2 \cup Y_1 \cup Y_2$, where $X_1 = \{u_1, u_1'\}$, $Y_1 = \{u_2, u_2'\}$,

$X_2 = \{v_1, v_2, …, v_m, v_1', v_2', …, v_m'\}$ and $Y_2 = \{w_1, w_2, …, w_n, w_1', w_2', …, w_n'\}$. Consider a proper coloring of G in which $V_1 = X_2 \cup Y_2$, $V_2 = X_1$ and $V_3 = Y_1$. Each vertex in the set X_2 dominates the color class V_2, each vertex in the set Y_2 dominate the color class V_3, each vertex in the set X_1 dominate the color class V_3 and each vertex in the set Y_1 dominate the color class V_2. Therefore this is a dominator coloring and $\chi_d[D_2(B_{m,n})] \leq 3$.

Now we construct a different proper coloring in which $V_1 = Y_1 \cup X_2$ and $V_2 = X_1 \cup Y_2$. This is not a dominator coloring, since, for example, the vertex x_i, $1 \leq i \leq m$ does not dominate a color class. But this is the only proper coloring of G using two colors. Thus $\chi_d[D_2(B_{m,n})] > 2$, hence the result.

(3) A dominator coloring of $D_2(K_n)$ is by coloring its vertices by colors 1, 2,, n respectively.Therefore $\chi_d[D_2(K_n)] = n$.

3. DOMINATOR COLORING ON PATH UNION OF GRAPHS

In this section, dominator chromatic number of path union of graphs is obtained.

Theorem: 3.1 Let G be a connected graph of order n. Then $\chi_d[P(mG)] = m+n-1$ if and only if $G = K_n$ for $n \in N$, where P(mG) denotes the path union of m copies of G.

Proof: Let G be a connected graph of order n with $\chi_d[P(mG)] = m+n-1$. Assume the contrary that $G \neq K_n$. Thus, there are at least two non-adjacent vertices, say x and y. Define a coloring of G such that x and y receive the same color, and each of the remaining vertices receive unique color. This is a dominator coloring, and so $\chi_d[P(G)] < m+n-1$, which is a contradiction.

Thus $G = K_n$.

The converse part is obvious.

Example: 3.2 In figure 3, Path union of 3 copies of K_5 is depicted with a dominator coloring.

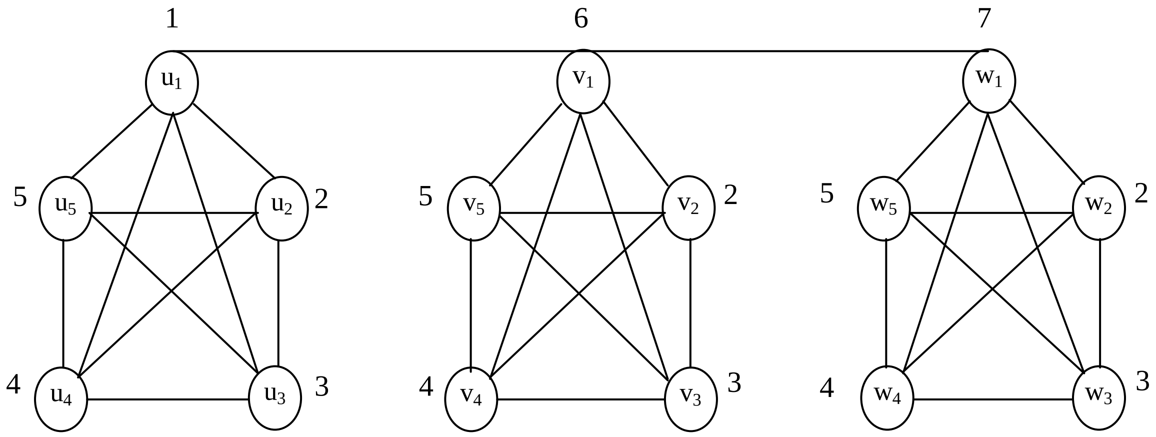

Figure: 3

The color classes of $P(3K_5)$ are $C_1 = \{u_1\}$, $C_2 = \{u_2, v_2, w_2\}$, $C_3 = \{u_3, v_3, w_3\}$, $C_4 = \{ u_4, v_4, w_4\}$, $C_5 = \{ u_5, v_5, w_5\}$, $C_6 = \{v_1\}$ and $C_7 = \{w_1\}$.Therefore $\chi_d[P(3K_5)] = 7$.

Theorem 3.3: Let G be a connected graph of order n. Then $\chi_d[P(mG)] \leq m[\chi_d(G)]$, where $G \neq K_{n,}$ $n \in N$ and P(mG) denotes the path union of m copies of G.

Proof: Let G be a connected graph with χ_d- colors. In P(mG), let the m copies of G be colored with disjoint χ_d- colors. This coloring in P(mG) with $m(\chi_d)$ is already a dominator coloring. Therefore dominator chromatic number of P(mG), which is the minimum number of colors required dominator coloring, is less than or equal to $m(\chi_d)$.

The bound in the above Theorem is sharp, because:

(i) $\chi_d[P(mC_4)] = m[\chi_d(C_4)] = 2m$.

(ii) $\chi_d[P(m(K_n\text{-}e))] < m\chi_d(K_n\text{-}e)$, $n \geq 2$.

OBSERVATION

For a connected graph G, $n \geq 2$, $\chi_d[D_2(G)] \geq \chi_d(G) \geq \chi(G)$.

Strict inequality as well as equality in observation is possible. As by Theorem 2.2, $\chi_d[D_2(C_n)] > \chi_d(C_n) > \chi(C_n)$, where $n \geq 8$ and $\chi_d[D_2(K_{1,\,n})] = \chi_d(K_{1,\,n}) = \chi(K_{1,\,n})$, $n \geq 2$. Therefore the bound in the observation is sharp.

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