International Journal of Mathematical Archive-4(1), 2013, 271-275
Available online through www.ijma.info ISSN 2229 – 5046

A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON A FUZZY METRIC SPACE UNDER S-B PROPERTY

K. P. R. Sastry[1], G. A. Naidu[2] and N. Umadevi[3*]

[1]8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530017, India
[2]Department of Mathematics, Andhra University, Visakhapatnam-530003, India
[3]Department of Mathematics, Raghu Engineering College, Visakhapatnam-531 162, India

(Received on: 20-12-12; Revised & Accepted on: 23-01-13)

ABSTRACT

In this paper the concept of weak compatibility in a fuzzy metric space with S-B property has been applied to obtain a common fixed point theorem for four self maps on a fuzzy metric space.

Keywords: *Fuzzy metric space, weak compatible maps and S-B property.*

Mathematical Subject Classification (2010): *47H10, 54H25.*

---

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [11] in 1965. Since then, to use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and its applications. Kramosil and Michalek [5] have introduced the concept of fuzzy metric space in different ways. In 1988, Grabiec [4] extended the fixed point theorem of Banach [1] to fuzzy metric space. George and Veeramani [3] have modified the concept of fuzzy metric space introduced by Kramosil and Michalek [5]. They have also shown that every metric induces a fuzzy metric. Singh et. al. [8] proved various fixed point theorems using the concepts of semi-compatibility, compatibility and implicit relations in Fuzzy metric space.In this paper we prove a common fixed point theorem for four self maps under S-B property [9] and obtain Rajinder Sharma's [6] result as a corollary.

Definition 1.1: (Zadeh.L.A. [11]) A fuzzy set A in a nonempty set X is a function with domain X and values in $[0,1]$.

Definition 1.2: (Schweizer.B. and Sklar. A. [7]) A function $*: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $*$ satisfies the following conditions:
For $a, b, c, d \in [0,1]$,
(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a \; for \; all \; a \in [0,1]$
(iv) $a * b \leq c * d \; whenever \; a \leq c \; and \; b \leq d$

We observe that $a * b = \min\{a, b\}$ is a t-norm.

Definition 1.3: (Kramosil. I. and Michalek. J. [5]) A triple $(X, M,*)$ is said to be a fuzzy metric space (FM space, briefly) if X is a nonempty set, $*$ is a continuous t −norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

For $x, y, z \in X$ and $s, t > 0$.
(i) $M(x, y, 0) = 0$
(ii) $M(x, y, t) = 1 \; if \; and \; only \; if \; x = y$.
(iii) $M(x, y, t) = M(y, x, t)$
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(v) $M(x, y, \cdot) : [0, \infty) \longrightarrow [0,1]$ is left continuous.

Then M is called a fuzzy metric on X.

Corresponding author: *N. Umadevi[3*]*
[3]Department of Mathematics, Raghu Engineering College, Visakhapatnam-531 162, India

The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 1.4: (George.A. and Veeramani.P. [3]) Let $(X, M,*)$ be a fuzzy metric space. Then,
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1 \ \forall\, t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M\big(x_{n+p}, x_n, t\big) = 1 \ \forall\, t > 0 \text{ and } p = 1,2, \dots$$

A FM –space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.5: (Singh,B. and Jain,S. [8]) Two self maps S and T of a fuzzy metric space $(X, M,*)$ are said to be weakly compatible if they commute at coincidence points,that is, $Sx = Tx$ implies $STx = TSx$.

Definition 1.6: ([9], [6]) Let S and T be two self mappings of a fuzzy metric space$(X, M,*)$.We say that S and T satisfy the property S-B if there exists a sequence $\{x_n\}$ in Xsuch that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Lemma 1.7: ([2], [9]) If for all $x, y \in X, t > 0$ and for a number $k \in (0,1)$,

$$M(x, y, kt) \geq M(x, y, t) \text{ then } x = y.$$

In the rest of the paper, we assume that a fuzzy metric space $(X, M,*)$ satisfies the following condition:

$$\lim_{t\to\infty} M(x, y, t) = 1 \text{ for all } x, y \in X. \tag{I}$$

Rajinder Sharma [6] proved the following:

Theorem 1.8 [6]: Let $(X, M,*)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and condition (I). Let A, B, S $and\ T$ be mappings of X into itself such that
(1.8.1)$A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(1.8.2)(A, S) or (B, T) satisfies the property $(S - B)$,
(1.8.3) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} M^{2p}(Ax, By, kt) \geq min\,\{&M^{2p}(Sx, Ty, t), M^{q}(Sx, Ax, t), M^{q'}(Ty, By, t), M^{r}(Sx, By, t), M^{r'}(Ty, Ax, (2-\alpha)t), \\ &M^{s}(Sx, Ax, t), M^{s'}(Ty, Ax,, (2-\alpha)t),\ M^{l}(Sx, By, t), M^{l'}(Ty, By, t)\,\} \end{aligned}$$

for all $x, y \in X$, $\alpha \geq 0, \alpha \in (0,2), t > 0$ and $0 < p, q, q', r, r', s, s', l, l' \leq 1$ such that

$$2p = q + q' = r + r' = s + s' = l + l'.$$

(1.8.4) the pairs (A, S) and (B, T) are weakly compatible
(1.8.5) one of $A(X), B(X), S(X)$ or $T(X)$ is a closed subset of X.

Then A, B, S and T have a unique common fixed point in X.

2. MAIN RESULT

In this section we present our main result and obtain theorem 1.8 as a corollary**.**

Theorem 2.1:Let $(X, M,*)$ be a fuzzy metric space and $*$ is min $t - norm$ with condition (I). Let A, B, S and T be appings of X into itself such that

(2.1.1)$A(X) \subset T(X)$ and $B(X) \subset S(X)$, $T(X)$ is a closed subset of X.

(2.1.2) (B, T) satisfies the property $(S - B)$,

(2.1.3) there exists a constant $k \in (0,1)$ and $\alpha \in (0,2)$, such that $k < \alpha$, $k + \alpha < 2$ and satisfies

$$M(Ax, By, kt) \geq min\,\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, \alpha t), M(Ty, Ax, (2-\alpha)t)\} \quad \forall\, t > 0.$$

(2.1.4) (A, S) and (B, T) are weakly compatible.

Then A, B, S and T have a unique common fixed point in X.

Proof: Without loss of generality we suppose that (B,T)satisfies the S-B property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z \text{ for some } z \in X. \tag{J}$$

Since $B(X) \subset S(X)$ there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$.

Hence $\lim_{n\to\infty} Sy_n = z$. Now we prove that $\lim_{n\to\infty} Ay_n = z$. By (2.1.3),

$$M(Ay_n, Bx_n, kt) \geq min\{M(Sy_n, Tx_n, t), M(Sy_n, Ay_n, t), M(Tx_n, Bx_n, t), M(Sy_n, Bx_n, \alpha t), M(Tx_n, Ay_n, (2-\alpha)t)\}$$

$$= min\{M(Bx_n, Tx_n, t), M(Bx_n, Ay_n, t), M(Tx_n, Bx_n, t), M(Bx_n, Bx_n, \alpha t), M(Tx_n, Ay_n, (2-\alpha)t)\}$$

$$= min\{M(Tx_n, Bx_n, t), M(Bx_n, Ay_n, t), 1, M(Tx_n, Ay_n, (2-\alpha)t)\}$$

$$\lim_{n\to\infty} \inf M(Ay_n, Bx_n, kt) \geq min\{\lim_{n\to\infty}\inf M(Tx_n, Bx_n, t), \lim_{n\to\infty}\inf M(Bx_n, Ay_n, t)$$
$$\lim_{n\to\infty}\inf M(Tx_n, Ay_n, (2-\alpha)t)\} \text{ by (J)}$$
$$\geq \lim_{n\to\infty}\inf M(Ay_n, Bx_n, \lambda t) \text{ where } \lambda = \min\{1, (2-\alpha)\}$$

$$\geq \lim_{n\to\infty}\inf M(Ay_n, Bx_n, kt) \text{ (since } k < \lambda)$$

$$\lim_{n\to\infty}\inf M(Ay_n, Bx_n, t) \geq \inf M\left(Bx_n, Ay_n, (\tfrac{\lambda}{k})t\right) \geq \ldots \geq \lim_{n\to\infty}\inf M\left(Bx_n, Ay_n, (\tfrac{\lambda}{k})^m t\right)$$

$$\lim_{n\to\infty}\inf M\left(Bx_n, Ay_n, (\tfrac{\lambda}{k})^m t\right) \to 1 \text{ as } m \to \infty$$

$$\Rightarrow \lim_{n\to\infty}\inf M(Ay_n, Bx_n, t) \geq 1$$

$$\Rightarrow \lim_{n\to\infty}\inf M(Ay_n, Bx_n, t) = 1$$

So $Ay_n \to z$ as $Bx_n \to z$.

Since $T(X)$ is a closed subset of X, $\exists v \in X \ni Tv = z \in X$.

We have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = Tv$.

By (2.1.3),

$$M(Ay_n, Bv, kt) \geq min\{M(Sy_n, Tv, t), M(Sy_n, Ay_n, t), M(Tv, Bv, t), M(Sy_n, Bv, \alpha t), M(Tv, Ay_n, (2-\alpha)t)\}$$

$\therefore$ On letting $n \to \infty$, we get

$$M(z, Bv, kt) \geq min\{1,1, M(z, Bv, t), M(z, Bv, \alpha t), 1\}$$

$$= min\{M(z, Bv, t), M(z, Bv, \alpha t)\}$$

$$\geq M(z, Bv, \lambda t) \text{ where } \lambda = min\{1, \alpha\}$$

$$\Rightarrow M(z, Bv, kt) \geq M(z, Bv, t) \ \forall\, t > 0$$

By lemma (1.7), $z = Bv$. $\therefore Tv = Bv$.

Since (B,T) is weakly compatible, $BTv = TBv \Rightarrow Bz = Tz$.

Since$B(X) \subset S(X), \exists u \in X \ni Su = Bv$. By (2.1.3)

$$M(Au, Bv, kt) \geq min\{M(Su, Tv, t), M(Su, Au, t), M(Tv, Bv, t), M(Su, Bv, \alpha t), M(Tv, Au, (2-\alpha)t)\}$$

$$M(Au, Tv, kt) \geq min\{M(Bv, Tv, t), M(Tv, Au, t), M(Tv, Bv, t), M(Tv, Bv, \alpha t), M(Tv, Au, (2-\alpha)t)\}$$

$$\geq min\{M(Tv, Au, t), M(Tv, Au, (2-\alpha)t)\}$$

$$\geq M(Tv, Au, \lambda t) \text{ where } \lambda = min\{1, (2-\alpha)\}$$

$\Rightarrow M(Au, Tv, kt) \geq M(Au, Tv, t)\ \forall\ t > 0$

$\therefore$ By lemma (1.7) $Au = Tv$.

Since (A, S) is weakly compatible, we have $ASu = SAu \Rightarrow Az = Sz$.

By (2.1.3),
$M(Au, Bz, kt) \geq min\{M(Su, Tz, t), M(Su, Au, t), M(Tz, Bz, t), M(Su, Bz, \alpha t), M(Tz, Au, (2-\alpha)t)\}$

$\therefore M(z, Tz, kt) \geq min\{M(z, Tz, t), M(Au, Au, t), M(Tz, Tz, t), M(z, Tz, \alpha t), M(Tv, Au, (2-\alpha)t)\}$

$= min\{M(z, Tz, t), 1, 1, M(z, Tz, \alpha t), M(Tz, z, (2-\alpha)t)\}$

$= min\{M(z, Tz, t), M(z, Tz, \alpha t), M(Tz, z, (2-\alpha)t)\}$

$\geq M(z, Tz, \lambda t)$ where $\lambda = min\{1, \alpha, (2-\alpha)\}$

$\therefore M(z, Tz, kt) \geq M(z, Tz, t)\ \forall\ t > 0$

$\therefore$ By lemma (1.7), $z = Tz$.

By (2.1.3),

$M(Az, Bv, kt) \geq min\{M(Sz, Tv, t), M(Sz, Az, t), M(Tv, Bv, t), M(Sz, Bv, \alpha t), M(Tv, Az, (2-\alpha)t)\}$

$M(Az, z, kt) \geq min\{M(Az, z, t), M(Az, Az, t), M(Tv, Tv, t), M(Az, z, \alpha t), M(z, Az, (2-\alpha)t)\}$

$\geq min\{M(Az, z, t), 1, 1, M(Az, z, \alpha t), M(z, Az, (2-\alpha)t)\}$

$\geq M(Az, z, \lambda t)$ where $\lambda = min\{1, \alpha, (2-\alpha)\}$

$\Rightarrow M(Az, z, kt) \geq M(Az, z, t)\ \forall\ t > 0$

$\therefore$ By lemma (1.7) $z = Az$

$\Rightarrow z = Tz = Bz = Sz = Az$.

$\Rightarrow z$ is a common fixed point of A, B, S and T.

Uniqueness: Let p, q be two common fixed points of A, B, S and T. Then by (2.1.3),

$M(Ap, Bq, kt) \geq min\{M(Sp, Tq, t), M(Sp, Ap, t), M(Tq, Bq, t), M(Sp, Bq, \alpha t), M(Tq, Ap, (2-\alpha)t)\}$

$\geq min\{M(p, q, t), M(p, p, t), M(q, q, t), M(p, q, \alpha t), M(q, p, (2-\alpha)t)\}$

$\geq min\{M(p, q, t), 1, 1, M(p, q, \alpha t), M(q, p, (2-\alpha)t)\}$

$\geq M(p, q, \lambda t)$ where $\lambda = \min\{1, \alpha, (2-\alpha)\}$

$\Rightarrow M(p, q, kt) \geq M(p, q, \lambda t)$

$\Rightarrow M(p, q, kt) \geq M(p, q, t)$

$\therefore$ By lemma (1.7), $p = q$. This completes the proof of the theorem.

Corollary: 2.2 Let $(X, M, *)$ be a fuzzy metric space and $*$ is min $t-$norm with condition (I). Let A, B, S and T be mappings of X into itself such that

(2.2.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $T(X)$ is a closed subset of X.

(2.2.2) (B, T) satisfies the property $(S-B)$, (2.2.3) there exists a constant $k \in (0,1)$, $\mu > 0$ and $\alpha \in (0,2)$, such that $k < \alpha, k + \alpha < 2$

$$M^{\mu}(Ax, By, kt) \geq \min\{M^{\mu}(Sx, Ty, t), M^{\mu}(Sx, Ax, t), M^{\mu}(Ty, By, t)\ M^{\mu}(Sx, By, \alpha t), M^{\mu}(Ty, Ax, (2-\alpha)t)\}$$

(2.2.4) (A, S) *and* (B, T) are weakly compatible.

Then A, B, S *and* T have a unique common fixed point in X.

Proof:

$$M^{\mu}(Ax, By, kt) \geq \min\{M^{\mu}(Sx, Ty, t), M^{\mu}(Sx, Ax, t), M^{\mu}(Ty, By, t) M^{\mu}(Sx, By, \alpha t), M^{\mu}(Ty, Ax, (2-\alpha)t)\}$$

$$\Rightarrow M(Ax, By, kt) \geq \min\{M(Sx, Ty, t), M(Sx, Ax, t). M(Ty, By, t),\ M(Sx, By, \alpha t), M(Ty, Ax, (2-\alpha)t)\}$$

Hence the corollary follows.

Note: Theorem 1.8 follows as a corollary to the corollary (2.2), if $k < \alpha < 1$ *and* $k + \alpha < 2$,since in this case ,(1.8.3) ⇒ (2.2.3).

Also $t * t \geq t$ for all $t > 0 \Rightarrow *$ is the $\min t - norm$.

We conclude the paper with an open problem.

Open problem: If the $\min t - norm$ is replaced by any continuous $t - norm$, is theorem 2.1 still true?

ACKNOWLEDGEMENT

The third author (N.U) is grateful to Raghu Engineering College authorities for giving permission and the management of SITAM for giving facilities to carry on this research.

REFERENCES

[1] Banach.S.Theoriedesoperations Lineaires, Monografie Mathematyczne, Warsawa, Poland, 1932.

[2] Cho,Y.J., Fixed points in fuzzy metric spaces, J. Fuzzy Math.,5(4) (1997), 949-962.

[3] George.A and Veeramani.P, On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64(1994), 395-399.

[4] Grabiec. M, Fixed points in fuzzy metric space, fuzzy sets and systems, 27(1988), 385-389.

[5] Kramosil.I and Michalek.J, Fuzzy metric and statistical metric spaces, Ky-bernetika, 11(1975), 336-344. .

[6] Rajinder Sharma, Common fixed point of weakly compatible mappings under a new property in fuzzy metric spaces, vol2, no4 (2012), Network and Complex Systems.

[7] Schweizer.B and Sklar.A, Probabilistic Metric Spaces, North Holland, Amsterdam, 1983.

[8] Singh.B and Jain.S. Weak compatibility and fixed point theorems in fuzzy metric spaces, Ganita, 56(2) (2005), 167-176.

[9] Sharma, Sushil and Bamboria, D. (2006). Some new common fixed point theorems in fuzzy metric space under strict contractive conditions, *J. Fuzzy Math*., 14, No.2, 1-11.

[10] Mishra, S. N., Sharma, N. and Singh, S. L. (1994). Common fixed points of maps in fuzzy metric spaces. *Internat. J. Math. Math. Sci*., 17, 253-258 .

[11] Zadeh.L.A, Fuzzy sets, Infor. and Control, 8(1965), 338-353.

Source of support: Nil, Conflict of interest: None Declared