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SOME PROPERTIES OF SUBMANIFOLDS IN CONFORMAL MANIFOLD

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ABSTRACT

In this paper we study the certain properties of submanifold in conformal manifolds. We find some system of quantities which are constitute geometric objects of the submanifold and connected with differential neighborhood of submanifold S^m which is known as fundamental geometric object of submanifold. We prove that the fundamental object of second order of a submanifold S^m allow us to construct the invariant family of central m-spheres and the bundle of tensors. We study the submanifolds carrying a net of curvature lines and also we find the condition that a net of curvature lines on a submanifold S^m is totally holonomic, only m-sphere and hypersurfaces of a conformal manifold can possess an irreducible net of curvature lines.

Key Words: *Conformal, Submanifolds, Geometric Object, Curvature Lines.*

1. INTRODUCTION

Let S^m be a smooth, connected and simply connected submanifold of dimension, $m \leq n-1$, in the conformal manifold M .

Any point $x \in S^m$, we associated a family of conformal frames consisting of the point $X_0 = x$, hyperspheres X_a that are tangent to S^m at the point X_0, hypersphere X_p that are orthogonal to S^m at X_0, and a point X_{n+1} that is the second common point of the hyperspheres X_a and X_p. The condition of moving frame for the frame element given by

$$(X_0, X_0) = 0, (X_{n+1}, X_{n+1}) = 0, (X_p, X_0) = 0, (X_a, X_0) = 0,$$

$$(X_p, X_{n+1}) = (X_a, X_{n+1}) = 0, (X_p, X_a) = 0. \tag{1.1}$$

in addition we denote

$$(X_p, X_q) = g_{pq}, \ (X_a, X_b) = g_{ab}, \tag{1.2}$$

and introduce the following normalization condition:

$$(X_0, X_{n+1}) = -1. \tag{1.3}$$

where (,) denotes the scalar product.

The matrix of scalar products of frame elements can be written as

$$(X_\alpha, X_\beta) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & g_{pq} & 0 & 0 \\ 0 & 0 & g_{ab} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

where $\quad 0 \leq \alpha, \beta \leq n+1.$

Since the manifold M is proper conformal, the matrices (g_{pq}) and (g_{ab}) are non singular positive definite matrices. We denote by (g^{pq}) and (g^{ab}) their inverse matrices.

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Since the hyperspheres X_a are tangent to S^m , we have

$$dX_0 = \omega_0^0 X_0 + \omega^p X_p, \tag{1.5}$$

where $\omega^p = \omega_0^p$ are basis forms on S^m. This equation shows that the submanifoldis S^m determined by the following system of pfaffian equation

$$\omega^a = 0, \tag{1.6}$$

where $\omega^a = \omega_0^a$. It follows from (1.5) that

$$g = (dX_0, dX_0) = g_{pq}\,\omega^p \omega^q, \tag{1.7}$$

and this form determines a conformal structure on S^m .

The equation of infinitesimal displacement of first order frame of the submanifold S^m has the form

$$dX_\alpha = \omega_\alpha^\beta X_\beta \tag{1.8}$$

where 1-form ω_α^β satisfy structure equation $d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$ of the manifold M. We can prove that on the submanifold S^m the matrix (ω_α^β) of components of infinitesimal displacement of a first order frame of S^m takes the following form

$$\left(\omega_\alpha^\beta\right) = \begin{pmatrix} \omega_0^0 & \omega^p & 0 & 0 \\ \omega_q^0 & \omega_q^p & \omega_q^a & g_{qr}\,\omega^r \\ \omega_b^0 & \omega_b^p & \omega_b^a & 0 \\ 0 & g^{pr}\,\omega_r^0 & g^{ab}\,\omega_b^0 & -\omega_0^0 \end{pmatrix}, \tag{1.9}$$

and the forms $\omega_q^a, \omega_b^p, \omega_q^p$ and ω_b^a are connected by the condition

$$g_{pq}\,\omega_a^q + g_{ab}\,\omega_p^b = 0, \tag{1.10}$$

$$dg_{pq} = g_{pr}\,\omega_q^r + g_{rq}\,\omega_p^r, \tag{1.11}$$

$$dg_{ab} = g_{ac}\,\omega_b^c + g_{cb}\,\omega_a^c. \tag{1.12}$$

for a fixed point $x \in S^m$ (i.e. for $\omega^p = 0$), the forms ω_p^a and ω_a^p vanish, since the normal and tangent bundles of hyperspheres are transformed into themselves. By virtue of this, for $\omega^p = 0$ the matrix takes the form

$$\left(\pi_\beta^\alpha\right) = \begin{pmatrix} \pi_0^0 & 0 & 0 & 0 \\ \pi_p^0 & \pi_p^q & 0 & 0 \\ \pi_b^0 & 0 & \pi_b^a & 0 \\ 0 & g^{pr}\,\pi_r^0 & g^{ab}\,\pi_b^0 & \pi_0^0 \end{pmatrix}, \tag{1.13}$$

Equation (1.11) and (1.12) become

$$\nabla_\delta g_{pq} = 2\pi_0^0 g_{pq}, \ \nabla_\delta g_{ab} = 2\pi_0^0 g_{ab}, \tag{1.14}$$

The set of frames of first order associated with S^m is a fiber bundle $F^1(S^m)$ whose base is the submanifold S^m itself and whose fiber is the collection of frames associated with a point$x = X_0 \in S^m$ .The forms ω^p are base forms of this bundle and the forms π_α^β are its fiber forms.

Let us consider the subgroup $H_x^1(S^m)$ of the group PO$(n+2,1)$ of conformal transformation thet leaves invariant the point A_0 and the bundle of tangent hypersphere with the basis X_0 and X_a. This subgroup will act transitively on the subfamily of orthonormal frames defined by the conditions

$$g_{pq} = \delta_{pq}, \ g_{ab} = \delta_{ab}, \tag{1.15}$$

by means of which equation takes form

$$\omega_a^p + \omega_p^a = 0, \ \omega_p^q + \omega_q^p = 0, \ \omega_a^b + \omega_b^a = 0,$$

2. FUNDAMENTAL GEOMETRIC OBJECT OF SUBMANIFOLDS

If we take exterior derivative of equation $\omega^a = 0$, we obtain the following exterior quadratic equation

$$\omega_p^a \wedge \omega^p = 0. \tag{2.1}$$

Applying cartan's lemma to above equation, $\omega_p^a = \eta_{pq}^a \omega^q$

Further we take exterior differentiation of equation (2.2), we get

$$\left(\nabla\eta_{pq}^a + g_{pq}\,\omega_{n+1}^a\right) \wedge \omega^q = 0, \tag{2.3}$$

by cartan's lemma

$$\nabla\eta_{pq}^a + g_{pq}\,\omega_{n+1}^a = \eta_{pqr}^a\,\omega^r, \tag{2.4}$$

where η_{pqr}^a are symmetric with respect to all lower indices and

$$\nabla\eta_{pq}^a = d\eta_{pq}^a - \eta_{pr}^a\,\varphi_{q\emptyset}^r - \eta_{rq}^a\,\varphi_p^r + \eta_{pq}^b\,\varphi_b^a. \tag{2.5}$$

where $\varphi_q^p = \omega_q^p - \delta_q^p\,\omega_0^0$ and $\varphi_b^a = \omega_b^a - \delta_b^a\,\omega_0^0$.

Applying exterior differentiation to equation (2.4), we obtain the following exterior quadratic equation:

$$\left[\nabla\eta_{pqr}^a + 3\left(\eta_{(pq\,g_r)l}^a - g_{(pq\,\eta_r^a)l}\right)\omega_{n+1}^l - 3\eta_{(pq\,\eta_r^a)l}^b\,\omega_b^l\right]\Lambda\omega^k = 0, \tag{2.6}$$

where $\nabla\eta_{pqr}^a = d\eta_{pqr}^a - \eta_{sqr}^a\,\varphi_p^s - \eta_{psr}^a\,\varphi_q^s - \eta_{pqs}^a\,\varphi_r^s + \eta_{pqr}^b\,\varphi_b^a$.

by cartan's le mma

$$\nabla\eta_{pqr}^a + 3\left(\eta_{(pq\,g_r)l}^a - g_{(pq\,\eta_r^a)l}\right)\omega_{n+1}^l - 3\eta_{(pq\,\eta_r^a)l}^b\,\omega_b^l = \eta_{pqrs}^a\,\omega^s, \tag{2.7}$$

where the quantities η_{pqrs}^a are symmetric with respect to the all lower indices.

Equation (2.2) constitute the first differential prolongation of equation (1.6); equation (2.2) and (2.4) constitute the second differential prolongation of (1.6); equation (2.2),(2.4) and (2.7) constitute the third differential prolongation of equation (1.6) etc. each of these prolongation introduces a new system of quantities η_{pq}^a, η_{pqr}^a, η_{pqrs}^a, etc.

If $\omega^p = 0$, then the system of quantities η_{pq}^a that arose in the first differential prolongation of equation (1.6), satisfies the equation

$$\nabla_\delta\eta_{pq}^a = -g_{pq}\,\pi_{n+!}^a, \tag{2.8}$$

and thus this system itself not a geometric object. If we consider the system η_{pq}^a jointly with the tensor g_{pq}, which by

$$\nabla g_{pq} = 2g_{pq}\,\omega_0^0, \qquad \nabla g_{ab} = 2g_{ab}\,\omega_0^0 \tag{2.9}$$

Satisfies the system of equation

$$\nabla_\delta g_{pq} = 2g_{pq}\,\pi_0^0, \tag{2.10}$$

then we obtain a geometric object with $\frac{1}{2}m(m+!)(n-m+1)$ components. This geometric objects called the fundamental geometric object of second order.

If we continue the process of exterior differentiation and application cartan's lemma $k-1$ times, we obtain a system of quantities $\eta_{pq}^a, \eta_{pqr}^a$, $\eta_{p_{1,}p_{2,}p_{3}\ldots\ldots p_r}^a$. This system jointly with the tensor g_{pq} constitute the fundamental geometric objects of order k.

Thus the following theorem is valid.

Theorem 2.1: *The system of quantities* $\{g_{pq}, \eta_{pq}^a\}$, $\left\{g_{pq}, \eta_{pq}^a, \eta_{pqr}^a\right\}$, $\{g_{pq}, \eta_{pq}^a, \eta_{pqr}^a, \eta_{pqrs}^a\}$ *etc. constitute geometric objects of the submanifold* S^m *, which are connected with differential neighborhoods of* S^m *of order* $2, 3, 4 \ldots\ldots$ *etc.*

The sequence of geometric object is called the fundamental sequence of geometric object of S^m. It was proved by Laptev [13] that for a smooth submanifold S^m embedded into any n- dimensional homogeneous space S, the fundamental sequence of geometric object of S^m entirely exhausted the differential geometry of S^m. In the same paper Laptev proved that there exists a lowest number p such that the fundamental objects of order p determines the submanifold S^m up to a motion in the space S. this fundamental object is called complete.

We can see from the above considerations, the structure of fundamental geometric objects of S^m is very complex, so it is difficult to find their direct geometric meaning. We will construct simpler geometric objects through fundamental geometric objects whose geometric meaning is easier to establish, and they entirely exhaust the differential geometry of submanifold S^m.

We will single out some submanifold from the frame bundle $F^1(S^m)$ that are defined in differential neighborhood of second and third order of a point $X_0 \in S^m$ and connected more closely with S^m.

First we construct the quantities

$$\eta^a = \frac{1}{m}\eta^a_{pq}\, g^{pq}, \tag{2.11}$$

where g^{pq} is the inverse tensor of the tensor g_{pq} . by equation (2.9)

$$\nabla g^{pq} = -2\omega_0^0 g^{pq}. \tag{2.12}$$

Similarly

$$\nabla g^{ab} = -2\omega_0^0 g^{ab}, \tag{2.13}$$

differentiate equation (2.10) with respect to the tensor parameters we get

$$\nabla_\delta \eta^a + 2\eta^a \pi_0^0 + \pi^a_{n+1} = 0. \tag{2.14}$$

above equation shows that quantities η^a constitute geometric object of the submanifold S^m.

Let us define a new object

$$\eta_a = g_{ab}\eta^b \tag{2.15}$$

By equation (2.9) and (2.14) the object η_a satisfy the equation

$$\nabla\eta_a = -\omega_a^0 + \eta_{ar}\,\omega^r. \tag{2.16}$$

If the point X_0 is fixed (i.e. , if $\omega^p = 0$), then the equation take form

$$\nabla_\delta \eta_a = -\pi_a^0. \tag{2.17}$$

Now we consider the bundle of tangent hypersphere with the basis

$$A_a = X_a + x_a X_0. \tag{2.18}$$

and find what condition this bundle is invariant. Since

$$\delta A_a = (\nabla_\delta x_a + \pi_a^0)X_0 + \pi_b^a A_b.$$

the desired invariance condition has the form

$$\nabla_\delta x_a + \pi_a^0 = 0. \tag{2.19}$$

(2.19) The quantities x_a constitute a geometric object that is called the normalizing object of first kind. Comparing equation (2.17) and (2.19) we see that $x_a = \eta_a$ is a solution of this equation

Thus the hyperspheres

$$C_a = X_a + \eta_a X_0 \tag{2.20}$$

produce a basis of an invariant bundle of tangent hyperspheres. The hyperspheres of this bundle are called the central hyperspheres of the submanifold S^m at the point $x = X_0$. The intersection of these hyperspheres is an invariant tangent of m-sphere that is intrinsically connected with the point $x \in S^m$. This m- sphere is called the central m-sphere of the submanifold S^m.

Let us clarify the geometric meaning of central m-sphere. If we set

$$l_{pq}^a = \eta_{pq}^a - \eta^a g_{pq}, \tag{2.21}$$

We find that

$$\nabla_\delta l_{pq}^a = 0. \tag{2.22}$$

Above equation shows that quantities l_{pq}^a form an invariant bundle of (0, 2)-tensors. It is easy to verify that this bundle of tensors and the tensor g_{pq} satisfy the apolarity condition:

$$l_{pq}^a g_{pq} = 0. \tag{2.23}$$

The tensors define a bundle of invariant quadratic forms

$$\Phi_{(2)}^a = l_{pq}^a \omega^p \omega^q, \tag{2.24}$$

Let us find the geometric meaning of the tensors l_{pq}^a and the quadratic forms $\Phi_{(2)}^a$. consider the central hyperspheres

$$S = x^a C_a \tag{2.25}$$

and find the direction along which it has a second order tangency with submanifold S^m. Since

$$d^2 X_0 \equiv \omega^p \omega_p^a X_a + \omega^p \omega_p^{n+1} X_{n+1} \qquad (mod\ X_0, X_p), \tag{2.26}$$

by (2.20) and (2.21), we have

$$(d^2 X_0, S) = x_a l_{pq}^a \omega^p \omega^q, \tag{2.27}$$

where $x_a = g_{ab} x^b$. thus the direction along which the hyperspheres S has a second order tangency with the submanifold S^m at the point $x = X_0$ satisfy the equation

$$x_a l_{pq}^a \omega^p \omega^q = 0 \tag{2.28}$$

and form a cone of second order which is called the characteristic cone of the hypersphere S. system of characteristic cones at the point x forms the bundle defined by the cones

$$l_{pq}^a \omega^p \omega^q = 0 \tag{2.29}$$

which in turn are determined by the quadratic form $\Phi_{(2)}^a$.

Since the coefficients of equation (2.27) form a tensor that is apolar to the tensor g^{ab},

$$x_a l_{pq}^a g^{pq} = 0,$$

It is possible to inscribe an orthogonal m-hedron of vectors of the tangent space $T_x(S^m)$ into the characteristic cone.

Thus we arrive at the following result:

Theorem 2.2: *The fundamental object of second order of a submanifold S^m allow us to construct the invariant family of central m-spheres and the bundle of tensors l_{pq}^a. the invariant bundle of the second fundamental forms $\Phi_{(2)}^a$ at a point $X_0 \in S^m$, defined by this bundle of tensors , determines a cone of directions along which the submanifold S^m has a second order tangency with the central m- sphere. If the number of independent forms among these forms is greater than or equal to m, then this cone could degenerate to the point X_0.*

We consider the symmetric tensor

$$l^{ab} = g^{pr} g^{qs} l^a_{pq} l^b_{rs}. \tag{2.30}$$

The rank of the tensor l^{ab} is equal to the number m_1 of linearly independent tensors in the bundle $\{l^a_{pq}\}$.if we differentiate equation (2.30) with respect to the fiber parameters and apply equations (2.22), we get

$$\nabla_\delta l^{ab} = -4\, \pi^0_0 l^{ab}. \tag{2.31}$$

the weight of the tensor $l^{ab} = -4$.

Now we construct the another tensor

$$l^b_a = g_{ac} l^{cb}. \tag{2.32}$$

whose rank is the same as the rank of the tensor l^{ab}. differentiating equation (2.32) and applying equation (2.9) and (2.32) , we get

$$\nabla l^b_a = -2\, \omega^0_0 l^b_a + \mu^b_{ah} \omega^h, \tag{2.33}$$

where $$\mu^b_{ah} = g_{ac} g^{pr} g^{qs} \left(l^b_{pq} \eta^c_{rsh} + l^c_{pq} \eta^b_{rsh} \right) \tag{2.34}$$

Since the number of linearly independent tensors in the bundle of tensors l^a_{pq} is equal to m_1, the rank of the tensor l^{ab} and consequently the rank of the tensor l^b_a is equal to m_1. thus the relative invariant

$$l = l^{a_1}_{[a_1} l^{a_2}_{a_2} \dots \dots .. l^{a_{m\,1}}_{a_{m\,1}]}, \tag{2.35}$$

which is the sum of the diagonal minors of order m_1 of the tensor l^b_a, is different from 0 provided that $m_1 \neq 0$.

If at a point $x \in S^m$, the number $m_1 = 0$, then all tensos $l^a_{pq} = 0$. then, it is easy to see, the central hypersphere has a second order tangency with S^m at the point x .this types of points are called umbilical.

This proves the following results:

Theorem 2.3: *If the tensors $l^a_{pq} = 0$ at all points of the submanifold S^m, then S^m is an m-sphere or its open subset and at any non umbilical point $X_0 \in S^m$, one can construct a non vanishing relative invariant l which can be expressed in terms of the fundamental object of second order of the submanifold S^m.*

Suppose that the number m_1 is different from 0 and is constant on the submanifold S^m. By equation (2.34) the invariant l defined by equation (2.35) satisfies the equation

$$\nabla l = -2 m_1 l \left(\omega^0_0 + \mu_p \omega^p \right), \tag{2.36}$$

where $$\mu_p = -\frac{1}{2 m_1 l} \sum_{r=1}^{m_1} l^{a_1}_{[a_1} \dots \dots \dots \mu^{a_r}_{a_r|p|} \dots \dots .. l^{a_{m\,1}}_{a_{m\,1}]} \tag{2.37}$$

and μ^b_{ap} are the quantities defined in equation (2.34).

In the bundle of hyperspheres with the basis X_0 and X_p, we single out a subbundle with the basis

$$A_p = X_p - x_p X_0. \tag{2.38}$$

Differentiating above equation with respect to the fiber parameters

$$\delta A_p = \left(\nabla_\delta x_p + \pi^o_p \right) X_0 + \pi^q_p A_q. \tag{2.39}$$

Comparing this equation to the equation of condition for subbundle to be invariant under transformation of the stationary subgroup $\left(\delta A_p = \varphi^q_p A_q \right)$, we get

$$\nabla_\delta x_p = -\pi^0_p. \tag{2.40}$$

the quantities x_p form a geometric object that is called the normalizing object of second kind.

In fact the differential equation that such an invariant satisfies can be written in the form:

$$dI = I(\omega_0^0 + \mu_p \omega^p), \tag{2.41}$$

Exterior differentiation of equation (2.42) and application of cartan's lemma leads to the following system of equations that the quantities μ_p satisfy:

$$\nabla \mu_p = \omega_p^0 + \mu_{pq}\, \omega^q, \tag{2.42}$$

where $\mu_{pq} = \mu_{qp}$. the system (2.42) proves that the quantities $x_p = -\mu_p$ produce a solution of equation (2.40). Therefore the hyperspheres

$$C_p = X_p - \mu_p X_0 \tag{2.43}$$

form a basis of the bundle of normal hyperspheres, and their intersection , the $(n-m)$-sphere $C_1 \Lambda \ldots \ldots . . \Lambda C_m$, is an invariant normal $(n-m)$-sphere that is intrinsically connected with the point $x \in S^m$.

Since the invariant l is determined by a second –order differential neighborhood of the submanifold S^m , the geometric object μ_p is expressed in terms of the components of the fundamental objects of S^m of third order. Thus the invariant bundle of normal hypersphere with the basis C_p is determined by a third order neighborhood of a point $X_0 \in S^m$.

We have proved the following result:

Theorem 2.4: *At any point $X_0 \in S^m$, the fundamental object of third order of a submanifold S^m allow us to construct the normal m-sphere $C_1 \Lambda \ldots \ldots \ldots \ldots \ldots . \Lambda C_m$, which is intrinsically connected with the submanifold S^m.*

3. SUBMANIFOLDS CARRYING A NET OF CURVATURE LINES

We have a first order differential neighborhood of a point $x \in S^m$, there is associated the quadratic form (1.7) and with a second order differential neighborhood of a point $x \in S^m$, there is associated the invariant bundle of second fundamental forms (2.24):

$$g = g_{pq}\, \omega^p \omega^q\,, \qquad \Phi_{(2)}^a = l_{pq}^a\, \omega^p \omega^q. \tag{3.1}$$

If $m < n-1$, then these quadratic form (3.1) cannot be reduced simultaneously to sums of squares and it is not possible to define the principle directions and the curvature line on a general submanifold S^m.

However it is possible to consider a special class of submanifold S^m at each point of which the forms g and $\Phi_{(2)}^a$ can be reduced simultaneously to sums of squares. Such submanifolds S^m are called submanifolds carrying a net of curvature lines.

Suppose that the submanifold S^m carries a net of curvature lines. If we choose the hypersphere X_p of our conformal frame so that they are orthogonal to the corresponding curvature lines of this submanifold, its quadratic form g and $\Phi_{(2)}^a$ become

$$g = (\omega^1)^2 + \ldots \ldots \ldots + (\omega^m)^2,$$

$$\Phi_{(2)}^a = l_1^a (\omega^1)^2 + \ldots \ldots \ldots + l_m^a (\omega^m)^2, \tag{3.2}$$

If the submanifold S^m carries at least onet of curvature lines, and this net is taken as a coordinate net, then decompositions $(\omega_p^q = \sum_r t_{pr}^q \omega^r, \quad p \neq q)$ hold. If we substitute for the forms ω_p^q in equation $(\Delta \eta_p^a \Lambda \omega^p - \sum_{q \neq p} (l_p^a - l_q^a) \omega_p^q \Lambda \omega^q = 0)$ their expression $\left(\omega_p^q = \sum_r t_{pr}^q \omega^r, \quad p \neq q\right)$ and equate to zero the coefficient in the independent exterior products $\omega^q \Lambda \omega^r, \quad p, r \neq p,$ then we obtain the relations

$$(l_q^a - l_p^a) t_{pr}^q - (l_r^a - l_p^a) t_{pq}^r = 0, \tag{3.3}$$

where $p \neq q, r \; and \; q \neq r$.

By the frobenius theorem [12], equation $\omega^p = 0$ is compeletly integrable if and only if the condition $d\omega^p \Lambda \omega^p = 0$ holds. But by $(\omega_p^q = \sum_r t_{pr}^q \omega^r, \quad p \neq q)$, this condition written as

$$t_{qr}^p \omega^q \Lambda \omega^r \Lambda \omega^p = 0. \quad q, r \neq p,$$

The coefficients t_{qr}^p are symmetric with respect to the lower indices.

Moreover the coefficients t_{qr}^p satisfy condition $\left(t_{pr}^q + t_{qr}^p = 0\right)$. we have

$$t_{qr}^p = -t_{pr}^q = -t_{rp}^q = t_{qp}^r = t_{pq}^r = -t_{rq}^p = -t_{qr}^p,$$

these entire coefficient are equal to zero. Thus we have the following result:

Theorem 3.1: *a net of curvature lines on a subamanifold S^m is tatally holonomic if and only if the condition hold for mutually distinct indices p,q, and r.*

Theorem 3.2: *A submanifold S^m carrying a net of curvature lines for any point x , any three hypersphere B_p associated with the point x do not belong to a pencil, and then the net of curvature lines on S^m is totally holonomic.*

Proof: consider the system of equation (3.3). The determinant of the matrix of coefficient of each pair of these equations corresponding to different values of a has the form

$$\Delta = \begin{vmatrix} l_q^a - l_p^a & l_r^a - l_p^a \\ j_q^b - l_p^b & l_r^b - l_p^b \end{vmatrix}$$

And can also be written in the form

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ l_p^a & l_q^a & l_r^a \\ l_p^b & l_q^b & l_r^b \end{vmatrix}$$

System (3.3) has only the trivial solution with respect to the coefficient t_{pr}^q if and only if, for any three mutually distinct indices p, q, and r, at least one of the determinants Δ does not vanish. However since the columns of Δ are coordinates of the hypersphere B_p, B_q, and B_r, this condition is equivalent to the fact that any three such hypersphere do not belong to a pencil.

Theorem 3.3: *If the distribution Δ_p is involute, and among hyperspheres B_q, $q \neq p$, no two hyperspheres coincide, then the net of lines induced on integral submanifolds S_p^{m-1} of Δ_p by the net of curvature lines on S_p^{m-1}.*

Proof: since the distribution is $\omega^p = 0$ involutive, for fixed indices p and $q, r \neq p$, the condition $t_{qr}^p = t_{rq}^p$ hold. It can be easily cheaked by considering the second differential of the point X_0 on the submanifold S_p^{m-1} , that the coefficients t_r^p determine S_p^{m-1} one more invariant quadrant form

$$\Phi_{(2)}^p = \sum_{q,r \neq p} t_{qr}^p \omega^q \omega^r.$$

Permuting indices in equation (3.3), we can write the m as follows

$$\left(l_r^a - l_q^a\right) t_{qp}^r - \left(l_p^a - l_q^a\right) t_{qr}^p = 0$$

Transposing the indices q and r in the above equation, we obtain

$$\left(l_q^a - l_r^a\right) t_{rp}^q - \left(l_p^a - l_r^a\right) t_{rq}^p = 0.$$

Subtracting this equation from the above equation and taking into account $\left(t_{pr}^q + t_{qr}^p = 0\right)$ and the symmetry of coefficient t_{qr}^p with respect to the lower indices, we get

$$\left(l_q^a - l_r^a\right) t_{qr}^p = 0.$$

If the hypersphere B_q and B_r do not coincide, then $l_q^a \neq l_r^a$ at least for one value of a. By virtue of this, we have t_{qr}^p.

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